GRÖBNER BASES AND INVOLUTIVE METHODS FOR ALGEBRAIC AND DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we consider and illustrate by examples some recently developed computer algebra methods for analysis and for solving nonlinear algebraic and differential equations. The foundation of those methods is either transformation of initial equations to an equivalent, often called standard, form or their reduction to a finite set of standard form subsystems. As a standard form we consider here one or another Gröbner basis with special emphasis on its involutive extension. Applications to symmetry and integrability analysis of partial differential equations and solving polynomial equation systems are singled out as possible applications.

1. Introduction

Presently it becomes apparent that to solve currently central mathematical problems with the use of a computer, one should more often than not combine its symbolic, numeric, and graphics facilities. Many such problems are reduced to solving nonlinear differential or polynomial equation systems. The desirable answer can usually be obtained only numerically. Nevertheless, a computer algebra preprocessing of the problem may often help to extract valuable knowledge of the equation system and solutions even without its explicit solving. One may try, in particular, to: (i) verify compatibility of the initial equations; (ii) compute the dimension of the solution space; (iii) eliminate some subset of variables; (iv) reduce the system to a finite number of "simpler" subsystems; (v) rewrite it into another form more appropriate for numerical analysis and solving. For differential equations one can extend this list with: (vi) symmetry analysis; (vii) recognition of some special internal algebraic properties, such as, for example, integrability of partial differential equations (PDEs) by the inverse spectral transform.

It is notable that within computer algebra analysis such different mathematical objects as polynomial-nonlinear PDEs and polynomial equations can be analyzed to a great extent by essentially the same algorithmic ideas. Namely, that is just the construction of some appropriate Gröbner basis for the polynomial case[1] and its differential analog for PDEs[2,3]. The history of realizing that fact goes back to papers[4,5] where the foundations of the involutive approach to the analysis of algebraic PDEs was laid. One modern-day mathematical treatment of the PDE involutivity can be found in[6]. Recently the analogy between the Gröbner bases concepts in commutative and differential algebra was not only neatly underlined[7], but also started to be intensively exploited for designing related differential algorithms and software[8,9,10,11].

One should note, however, that there is still a very large gap between practical efficiency of algebraic and differential Gröbner basis software. One of the objectives is that a rather rich experience which had been collected in optimization of the pure algebraic Gröbner basis computation procedure has not been properly used yet in the differential case. Apart from non-termination of the algorithm in the general nonlinear case, the central problem here is to reveal constructive criteria for avoiding non-necessary reductions, like the ones of Buchberger[1], and searching for optimal computational strategies. While it is important for both Buchberger's differential algorithm and the involutive technique, the latter one as discovered in[4,5,6] is intrinsically natural just for differential equations.

On the other side, in the pioneering paper[12] it was argued that the involutive technique can also be useful in the pure algebraic case, resulting, in particular, in one more efficient algorithm for the Gröbner basis construction. Based on the definition of multiplicative and non-multiplicative variables as given in[6], the involutive algorithm of papers[12,13], unfortunately, doesn't terminate in the general case. However, the termination takes place, for example, for the degree compatible term ordering and for polynomial systems with finitely many solutions, that is, for zero-dimensional polynomial ideals[13]. Apart from that, the involutive form of Gröbner bases for zero-dimensional polynomial ideals reveals a number of rather attractive features[14]. It is important to note that any algorithmic improvement of the involutive technique at the algebraic level results in an analogous optimization at the differential one, at least for linear PDEs.

In this paper we briefly describe both the Gröbner basis technique and the involutive one as well as their mutual interconnection at the differential and algebraic levels. For the further mathematical details, improved algorithms and their implementation see[1,2,3,5,6,8,9,11,12,13,16]. To illustrate those techniques we use a reasonably simple, but non-trivial (from the computational point of view) example of a system of linear PDEs.

Then we discuss one of the most important applications, namely, Lie symmetry analysis of nonlinear differential equations[8,9,10,11,15]. Here the most non-trivial computational step is, generally, the transformation of the determining linear PDEs for the symmetry generators to a standard form which is just the involutive one. The latter not only makes integration of the determining equations much easier, but also allows one to find the size of the symmetry group[11] and even the Lie symmetry algebra without their explicit integration[9]. As an example, we construct the infinitesimal symmetry operators for one of the diffusion nonlinear evolution equations. In addition, we shortly consider some related computational aspects of the use of generalized, or Lie-Bäcklund, symmetries in the integrability analysis of nonlinear evolution equations[17].

2. Algebraic Gröbner Bases

Let $R = \{K[x] \mid x = (x_1, \ldots, x_n)\}$ be a polynomial ring over the coefficient field K of characteristic zero and $F = \{(f_1, \ldots, f_m) \mid f_i \in R\}$ be a set of polynomials. They form a **basis** of the **ideal** $Ideal(F) = \{\sum_{i=1}^{n} p_i f_i \mid p_i \in R\}$. Let $\succ$ be an **admissible ordering** on the set of monomials $\{x^\alpha \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$, where $\mathbb{N} = (0, 1, 2, \ldots)$. It means that $\succ$ is a linear ordering which satisfies the following two properties:

(a) $x^\alpha \succ 1$ for any $\alpha \neq 0$.

(b) If $x^\alpha \succ x^\beta$ then $x^\alpha x^\gamma \succ x^\beta x^\gamma$ for any $\gamma \in \mathbb{N}^n$.

Throughout this paper we use two different admissible orderings:

- The **lexicographical (Lex) ordering**:

  $x^\alpha \succ_{Lex} x^\beta$ if, for the vector difference $\alpha - \beta \in \mathbb{Z}^n$, there is $1 \leq k \leq n$ such that $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$, $\alpha_k > \beta_k$.

- The **degree reverse lexicographical (DegRevLex) ordering**:

  $x^\alpha \succ_{DegRevLex} x^\beta$ if $\sum_{i=1}^{n} \alpha_i > \sum_{i=1}^{n} \beta_i$ or, if $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ then there is $1 \leq k \leq n$ such that $\alpha_n = \beta_n, \ldots, \alpha_{k+1} = \beta_{k+1}$, $\alpha_k < \beta_k$.

Let $lt(f) = lc(f)lm(f)$ be the **leading term** of $f \in F$ with the **leading coefficient** $lc(f) \in K$ and the **leading monomial** $lm(f)$.

A polynomial p is said to be **reduced** by another polynomial q to p' if $p' = p - uq$ where $u \in R$ is a term such that $lt(uq)$ just coincides with one of the terms of p. If p' is again **reducible** by q, one can reduce it further to $p'' = p' - u'q$. This process may be continued until we obtain a polynomial $\bar{p}$ **irreducible** by q. If so, we say that $\bar{p}$ is the **normal form** of p modulo q and we write $\bar{p} = NF(p, q)$.

A reduction by a finite set of polynomials $F \subseteq R$ is defined as a successive reduction by all the elements of F. Reducing p modulo F as long as possible we obtain its normal form $\bar{p} = NF(p, F)$.

As a **normal form algorithm** one can use, for example, the following one:

Algorithm NormalForm:

Input: g, F
Output: $h = NF(g, F)$
begin
    $h := g$
    while exist $f \in F$ such that $lm(f)$ divides any term u of h **do**
        choose the first such $f \in F$
        $h := h - (u/lt(f))f$
    end
end

A set of polynomials $G = (g_1, \ldots, g_k) \subseteq Ideal(F)$ is called **Gröbner basis** of $Ideal(F)$, if $NF(p, G) = 0$ for any polynomial $p \in Ideal(F)$.

The **S-polynomial** of $p, q \in R$ is the combination

$$Spol(p, q) = \frac{lcm(lc(p), lc(q))}{lt(p)} p - \frac{lcm(lc(p), lc(q)}{lt(q))} q\,,$$

where lcm is the least common multiple of its arguments.

Buchberger's algorithm[1] for the construction of a Gröbner basis is based on the fact that G is a Gröbner basis if and only if

$$NF(Spol(g_1, g_2), G) = 0$$

for any $g_1, g_2 \in G$ and can be written as follows:

Algorithm GröbnerBasis:

Input: F
Output: G, a Gröbner basis for $Ideal(F)$
begin
  G:=**Autoreduce**(F)
  $H := \{(p, q) \mid p, q \in G, p \neq q\}$
  while $H \neq \emptyset$ **for each** $(p, q) \in H$ **do**
    $H := H \setminus (p, q)$
    $s := NF(Spol(p, q), G)$
    if $s \neq 0$ **then**
      $H := H \cup \{(g, s\} \mid g \in G\}$
      $G := G \cup \{s\}$
  end
  G:=**Autoreduce**(G)
end

Given polynomial set F, the subalgorithm **Autoreduce** performs all the interreductions of its elements and returns the set F'=**Autoreduce**(F) such that $f = NF(f, F' \setminus \{f\})$ for any $f \in F'$. The termination of Buchberger's algorithm **GröbnerBasis** is provided by Noetherity of the polynomial ring R[1].

The Gröbner basis method is just the proper algorithmic tool[1] for analysis of those general algebraic properties of polynomial systems as they are listed in the introduction. The pure lexicographical Gröbner basis has a triangular form with the successive elimination of variables. Usually it is the most appropriate form for the root computation by numerical methods. Being computed in any ordering, a Gröbner basis yields information on the compatibility of the system and the dimension of its solution space. It is, therefore, not surprising that a Gröbner basis can be considered as the **standard form** of a polynomial system. The splitting of a polynomial system can also be rather often achieved in practice either by factorization of the intermediate bases in Buchberger's algorithm[18] or by the homogeneity reduction[19].

3. Differential Gröbner bases

The basic concepts and algorithms of the Gröbner bases method admit a certain extension to systems of algebraic PDEs which have the form

$$f_l(x, u, D^1 u, \ldots, D^\alpha u) = 0 \quad (l = 1, \ldots, k), \tag{1}$$

where

$$x = (x_1, \ldots, x_n),\ u = (u_1, \ldots, u_m),\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,\ D^\alpha u_j = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u_j}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

and the left hand sides of system (1) are polynomials in u and its derivatives. Hence, we consider the ring of **differential polynomials** $R = K < x > \{u\}$ in **indeterminates** $u = (u_1, \ldots, u_m)$ over the **differential field** $K < x >$. The latter is a field of rational functions in $x = (x_1, \ldots, x_n)$ extended with the derivative operations $(\partial x_i / \partial x_j = \delta_{ij})$. Then, a finite set $F = (f_1, \ldots, f_k)$ of differential polynomials $f_i \in R$ in (1) will generate the **differential ideal**

$$DiffIdeal(F) = \{\sum_{i,a} h_{i,\alpha} D^\alpha f_i \mid h_{i,\alpha} \in R\}$$

with all possible finite sums over $\alpha \in \mathbb{N}^n$ in the right hand side.

An admissible ordering $\prec$ for the derivative terms $D^\alpha u_j$ must be compatible with differentiation:

(i) If $D^\alpha u_i \succ D^\beta u_j$ then $D^\gamma D^\alpha u_i \succ D^\gamma D^\beta u_j$ for any $\gamma \in \mathbb{N}^n$.
(ii) $D^\beta D^\alpha u_i \succ D^\alpha u_i$ for any $\beta \neq (0, \ldots, 0)$.

The differential term ordering induces that for differential monomials[3]. Consider monomials

$$M_1 = \prod_{i=1}^{s} (D^{\alpha_i} u_{p_i})^{k_i}, \quad M_2 = \prod_{j=1}^{t} (D^{\beta_j} u_{q_j})^{l_j}, \quad 1 \leq p_i, q_j \leq m, \quad \alpha_i, \beta_j \in \mathbb{N}^n \quad k_i, l_j \in \mathbb{N}$$

with

$$D^{\alpha_1} u_{p_1} \succ D^{\alpha_2} u_{p_2} \succ \ldots \succ D^{\alpha_s} u_{p_s}\,, \quad D^{\beta_1} u_{q_1} \succ D^{\beta_2} u_{q_2} \succ \ldots \succ D^{\beta_t} u_{q_t}\,.$$

Then $M_1 \succ M_2$ if there exist $r \leq s, t$ such that $D^{\alpha_i} u_{p_i} = D^{\beta_i} u_{q_i},\ k_i = l_i$ for $i < r$ and either $D^{\alpha_r} u_{p_r} \succ D^{\beta_r} u_{q_r}$ or $D^{\alpha_r} u_{p_r} = D^{\beta_r} u_{q_r}$ but $k_r > l_r$. If now $\delta_i M$ is the leading monomial of polynomial $\partial M / \partial x_i$ then the above monomial ordering will satisfy the admissibility conditions:

(a) $M \succ 1$ for any $M \neq 1$.
(b) If $M_1 \succ M_2$ then $M_1 M \succ M_2 M$ for any monomial $M \in R$.
(c) $\delta_i M \succ M$ for any $1 \leq i \leq n$ and $M \neq 1$.
(d) If $M_1 \succ M_2$ then $\delta_i M_1 \succ \delta_i M_2$ for any $1 \leq i \leq n$.

The **differential Lex** and **DegRevLex** orderings can be defined much as in the algebraic case (Sect. 2) and they play a similar part for PDEs as their algebraic analogs do for the pure polynomial equation systems.

If $(x_1 \succ x_2 \succ \ldots \succ x_n)$ and $(u_1 \succ u_2 \succ \ldots \succ u_m)$, then we shall use

- **Lex** $(\succ_{Lex})$:
  $D^\alpha u_i \succ_{Lex} D^\beta u_j$ if $i < j$ or if $i = j$ then $\alpha \succ_{Lex} \beta$.

- **DegRevLex** $(\succ_{DegRevLex})$:
  $D^\alpha u_i \succ_{DegRevLex} D^\beta u_j$ if $\alpha \succ_{DegRevLex} \beta$ or $\alpha = \beta$ and $i < j$.

Again, as earlier, the leading term of $f \in R$ is to be presented as a product $lt(f) = lc(f)lm(f)$ of its leading coefficient $lc(f) \in K < x >$ and leading monomial $lm(f)$.

Now the concept of **reduction** as given in Sect.2 can be directly extended to the differential case. A differential polynomial p is said to be **reducible** by another differential polynomial q if it can be rewritten as $p - uD^\alpha q$, where $u \in R$ is a term and $lt(uD^\alpha q)$ coincides with one of the terms in p. Then a **normal form** is defined in a similar manner as above.

The **differential S-polynomial** of $p, q \in R$ is their combination of the form

$$DiffSpoly(p, q) = c_1 M_1 D^\alpha p - c_2 M_2 D^\beta q$$

with the properties:

- $M_1, M_2 \in R$ are minimal (w.r.t. $<$) differential monomials such that
  - $lm(M_1 D^\alpha p) = lm(M_2 D^\beta q)$,
  - there are no other polynomials $N_1 D^\mu p$ and $N_2 D^\nu q$ satisfying the equalities $lm(M_1 D^\alpha p) = lm(D^\gamma(N_1 D^\mu p))$ and $lm(M_2 D^\beta lq) = lm(D^\gamma(N_2 D^\mu lq))$ for some γ.

- $c_1 = lcm(lc(D^\alpha p), lc(D^\beta q))/lc(D^\alpha p)$, $c_2 = lcm(lc(D^\alpha p), lc(D^\beta q))/lc(D^\beta q)$.

A **differential Gröbner basis** for $DiffIdeaf(F)$ can be defined and algorithmically characterized exactly as does the algebraic one. It turns out, however, that, unlike algebraic polynomial ideals, by far not each differential ideal, even if finitely generated, has a finite Gröbner basis[2,3]. If the latter is **infinite**, as it holds, generally, for nonlinear $f_i \in F$, it is called **standard basis**[3] rather than Gröbner basis.

For this reason, although the normal form algorithm of Sect.2 terminates for any finite set F of differential polynomials, the straightforward analog of algorithm **Gröbner Basis** does not terminate in the general nonlinear case[2,3]. To avoid this problem one can use the **pseudo-normal form** algorithm rather than the normal form one. The former is obtained from the latter if one replaces **reductions** by **pseudo-reductions**. In addition to the above reduction procedure of p modulo q, in order to cancel a term of p, a **pseudo-reduction** admits also the transformation

$$p' = u_1 p - u_2 D^\alpha q$$

with multiplication of p by a term u_1, which, for given p, is uniquely determined by the structure of $D^\alpha q$.

Although the pseudo-reduction provides termination of the algorithm, its output, being some finite subset of the differential ideal, may not be its Gröbner basis at all. To ascertain this, one has to check certain extra conditions[8]. Unfortunately, that problem is far from being mathematically and, hence, algorithmically solved.

4. Involutive Algebraic Bases

In the involutive approach, each nonzero term is characterized by some self-consistent partitioning of a set of independent variables

$$x = (x_1, \ldots, x_n) = \mathcal{M} \cup \mathcal{NM}$$

into two disjoined subsets $\mathcal{M} \cap \mathcal{NM} = \emptyset$. The variables $x_i \in \mathcal{M}$ are called **multiplicative** and the others $x_j \in \mathcal{NM}$ **non-multiplicative**.

In the present paper we consider the definition of multiplicative and non-multiplicative variables given in[6,12]. Another possible definition is used in[5,11]. The general definition which includes both definitions as particular cases, as well as related analysis of the involutive polynomial bases is given elsewhere[16].

Let $\succ$ be an admissible monomial ordering and $x_1 \succ x_2 \succ \cdots \succ x_n$. Then for the term

$$u = c \cdot x_1^{k_1} \cdots x_i^{k_i}, \quad c \in K \setminus \{0\}, \quad k_i > 0, \quad i \leq n$$

we define

$$\mathcal{M}(u) = \{x_i, \ldots, x_n\}, \quad \mathcal{NM}(u) = \{x_1, \ldots, x_{i-1}\}.$$

If $u = v \cdot w \neq 0$ and either all the variables which present in w with nonzero exponents are multiplicative for v, or $w = 1$ we say that term w is **multiplicative** for v and write $u = v \times w$. In this case v is called **involutive divisor** of u and u is called **involutive multiple** of v, respectively.

For a polynomial p the multiplicative (non-multiplicative) variables are defined as those of its leading term:

$$\mathcal{M}(p) = \mathcal{M}(lt(p))\,, \quad \mathcal{NM}(p) = \mathcal{NM}(lt(p))\,.$$

Now one can introduce the concept of **involutive reduction**. Namely, given a pair of polynomials (p, q) we say that p is **involutively reduced** by q if p has a term t, an involutive multiple of $lt(q)$, that is, $t = lt(q) \times u$. It yields the reduction $p \to p' = p - q \times u$. Respectively, the **involutive normal form** $NF_I(p, q)$ of p modulo q or that $NF_I(p, F)$ modulo a finite polynomial set F can be defined as well as in Sect.2. Furthermore, algorithm **NormalForm** immediately gives a corresponding one for the involutive normal form if one replaces the reduction step by the corresponding involutive one. Obviously, the involutive reductions are the special subset of those of algorithm **NormalForm** and, hence, the corresponding algorithm **InvolutiveNormalForm** terminates.

Similar, a polynomial set F can be **involutively autoreduced**, that is, transformed to F'=$\mathbf{Autoreduce}_I(F)$, such that for any $f \in F'$

$$NF_I(f, F' \setminus \{f\}) = f\,.$$

It is easy to see that any term $u \in R$ has at most one involutive divisor among the leading terms of any involutively autoreduced polynomial set [*].

The involutive normal form $NF_I(p, F)$ has two remarkable properties different from those of $NF(p, F)$[12]

uniqueness: if $h_1 = NF_I(p, F)$ and $h_2 = NF_I(p, F)$, then $h_1 = h_2$,

linearity: $NF_I(c_1p_1 + c_2p_2, F) = c_1NF_I(p_1, F) + c_2NF_I(p_2, F)\,,\ c_1, c_2 \in K.$

In the involutive approach an initial polynomial system is transformed into another so-called **involutive form** by successive **prolongations**, that is, multiplications by the variables[†]. A prolongation is called **non-multiplicative** or **multiplicative** depending on whether the variable is non-multiplicative or multiplicative for the polynomial.

The central definition of the involutive approach is given as follows. A polynomial set $G = (g_1, \ldots, g_m)$ is called **involutive system** if it is involutively autoreduced and for any $g_i \in G$ and any $x_j \in \mathcal{NM}(g_i)$

$$NF_I(g_i \cdot x_j, G) = 0\,.$$

An involutive system G has the properties[12,13]:

- $f \in Ideal(G)$ if and only if $NF_I(f, G) = 0$;
- G is a Gröbner basis, generally, not the reduced one;
- $NF_I(h, G) = NF(h, G)$ for any $h \in K[x]$;
- if two involutive systems G and H generate the same ideal, that is, $Ideal(G) = Ideal(H)$ then they coincide up to multiplication by non-zero elements of K;
- an **involutive basis** exists for any zero-dimensional polynomial ideal, and also for a positive-dimensional one after a generic linear transformation of variables.

It is remarkable that the first three properties hold not only for the above partitioning in multiplicative and non-multiplicative variables, but also for any other one conforming to the general involutivity concepts[16]. In particular, they hold also for the partitioning used in[5,11]. The latter one provides the existence of an involutive basis

[*]In fact, to provide this property and the noted ones for NF_I it is sufficient to autoreduce the leading terms.

[†]For a differential case prolongations mean derivations with respect to the corresponding independent variables as explained below.

for any polynomial ideal. In doing so, however, the involutive basis is not uniquely defined, unlike the one above.

If the involutive basis exists, and the monomial ordering is **DegRevLex** (Sect.2), for definiteness, then the following algorithm allows one to construct the basis:

Algorithm InvolutiveBasis:

Input: F
Output: G, an involutive basis for $Ideal(F)$
begin
    G:=$\mathbf{Autoreduce}_I(F)$
    $P := \emptyset$
    while $\{(lt(g), x) \mid g \in G, x \in \mathcal{NM}(g)\} \setminus P \neq \emptyset$ **do**
        choose (g, x) **such that**
            $g \in G$, $x \in \mathcal{NM}(g)$, $(lt(g), x) \notin P$
            and $lt(g) \cdot x$ is minimal w.r.t. $\succ_{DegRevLex}$
        $P := P \cup \{(lt(g), x)\}$
        $h := NF_I(g \cdot x, G)$
        if $h \neq 0$ **then**
            G:=$\mathbf{Autoreduce}_I(G \cup \{h\})$
    end
end

This form of the algorithm is a slightly improved version of the one of paper[13] to avoid unnecessary repeated prolongations[14]. Its generalization to linear differential systems is rather straightforward as it is discussed in the next section.

5. Involutive Differential Bases

Unfortunately, for the general nonlinear differential case there is no yet any constructive theoretical platform to design an algorithmic procedure for the transformation of the input system into the involutive output system. In fact, as the analysis of involutivity conditions at both algebraic (Sect.4) and the linear differential level shows, an involutive system is a rather special (generally extended in order to satisfy the involutivity conditions) form of a reduced Gröbner basis. One can expect that such a correspondence is also valid for any system of PDEs. If so, non-existence of a (finite) differential Gröbner basis, as it is noted in Sect.3, implies non-existence of a finite involutive basis for the same input set of nonlinear differential polynomials.

Among nonlinear systems (1) one should separate the special class of quasilinear ones called **orthonomic**. They are such that all their equations are linear just in their highest order derivative term with respect to some admissible ordering $\succ$. Orthonomic differential systems can be considered as some intermediate ones between linear and the most general nonlinear systems. Their involutive algorithmic study, started, in fact, already in[4], seems to be a natural next step for extension of involutive algorithms from linear to nonlinear systems. Construction of involutive bases for linear PDEs is also very important in its own right, for example, in Lie symmetry analysis of nonlinear systems (1). We consider this application in Sect.7.

For those reasons we shall restrict our consideration of differential involutive bases to the case of linear polynomials in (1) which form a set $F = (f_1, \ldots, f_k)$. Presenting the material of this section we shall intensively exploit very useful similarities and connections between the algebraic and differential cases.

First of all, this means the following correspondence between differential terms and monomials

$$\frac{\partial^{\alpha_1+\cdots+\alpha_n} u_j}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Longleftrightarrow x_1^{\alpha_1} \cdots x_n^{\alpha_n}\,, \tag{2}$$

which is reflected, in particular, in the monomial and differential term orderings as presented in Sect.2-3. The monomials corresponding to different undeterminates u_j are to be considered as belonging to different monomial sets.

Respectively, given an admissible differential term ordering, the concept of **multiplicative** and **non-multiplicative** variables for monomials and polynomials as presented in the previous section is nothing else than a simple reflection of that concept for their differential images. Here we talk about the **multiplicative** and **non-multiplicative** prolongations. These are, obviously, nothing else than derivations with respect to the variables of the same type for the underlying monomial or polynomial.

All that immediately results in the intrinsic similarity between polynomial equations and linear homogeneous differential systems in one indeterminate and with constant coefficients[7]. Furthermore, the transformation of both the polynomial and differential system to the involutive form preserves that one-to-one correspondence between differential terms and monomials. In this particular case, therefore, the involutive differential system or differential Gröbner basis can be computed by means of their algebraic analogs. Apparently, it contains a valuable source for optimization of real computations, as we demonstrate in the next section by an explicit example. In addition, the differential image of a zero-dimensional polynomial system has the general solution depending exactly on as many arbitrary constants as there are roots in the system taking into account their multiplicities. The dimension of a polynomial ideal coincides with a number of arbitrary functions in the general solution of the related PDEs.

It is along this line that the differential version of the involutive normal form $NF_I(p, F)$ of a linear differential polynomial p modulo F is achieved by successive reductions of the form

$$p \longrightarrow p - c(x) D^{\alpha_{mult}}(f_i), \quad f_i \in F\,.$$

Here $\alpha_{mult} \in \mathbb{N}^n$ has only those nonzero components $\alpha_i \in \alpha$ ($\alpha_i > 0$) which correspond to the multiplicative variables $x_j \in \mathcal{M}(f_i)$ and $c(x)$ is an element of the differential coefficient field $K < x >$ as defined in Sect.3.

Analogously, an autoreduced system of linear differential polynomials $G \in R$ is called **involutive** if

$$NF_I(\partial g_i/\partial x_j, G) = 0$$

for any $g_i \in G$ and $x_j \in \mathcal{NM}(g_i)$.

Algorithm **InvolutiveBasis** of Sect.4 can be easily rewritten to that for the computation of an involutive differential basis for a linear polynomial system by replacing $lt(g) \cdot x$ and $g \cdot x$ with $lt(\partial g/\partial x)$ and $\partial g/\partial x$, respectively. Note, that in doing so we do not assume just the only indeterminate. If the algorithm terminates, it provides a computation of the involutive basis for any system of linear differential polynomials. The termination holds [‡]for any differential system with a finite number of arbitrary constants in its general solution. If the latter depends on arbitrary functions then one should, generally, either use another definition of multiplicative and non-multiplicative variables[5,9,11,16] instead of that given in Sect.4 or to perform an appropriate linear change of variables.

An involutive form of linear PDE systems can be called **standard**, not only because it is a differential Gröbner basis, but also due to the fact that involutive systems are especially convenient for selecting the initial conditions providing the existence of a uniquely defined and locally holomorphic solution[4,5]. Moreover, one can compute such lists of initial conditions which are in one-to-one correspondence with arbitrary constants or/and functions in the general solution. For the underlying algorithms and implementation see[9,10].

One should be noted that the algorithms of papers[9,10,11] for computation of an involutive form for a linear system of PDEs are conceptually different from that of Sect.4. Those algorithms are essentially the versions of Buchberger's algorithm. Contrastingly, the use of the involutive normal form NF_I rather that the usual normal form NF, as is done in algorithm **InvolutiveBasis** and its differential analog, leads to a number of significant dissimilarities.

6. Illustrative Example

We consider the following system of linear PDEs

$$\begin{cases} u_{xxx} - u_{yy} + u_z - u = 0, \\ u_{yyy} - u_{zz} + u_x - u = 0, \\ u_{zzz} - u_{xx} + u_y - u = 0, \end{cases} \tag{3}$$

with the single unknown function $u(x, y, z)$ which is a complex-valued one. We would like to end up with the general solution. In principle, for such a type of PDEs a number of different special analytical methods can be tried[20]. We prefer, however, to use the most universal algorithmic approach based on construction of an appropriate Gröbner basis. For explicit integration the pure lexicographical Gröbner basis is

[‡]To prove it one can use the same ideas used in[13] for the algebraic case.

usually the best one. It provides the elimination of the variables as dictated by the chosen ordering.

In our case, we choose, for definiteness, an order $x \succ y \succ z$. One could try to apply any package implementing the differential form of Buchberger's algorithm[2], for example, the one of paper[8§]. However, while computing the lexicographical Gröbner basis for (3), the intermediate differential polynomials are too large to be handled in a reasonable time. To deal with the expression swell problem, the differential Gröbner basis algorithms and the S-polynomial selection strategies should be optimized in an extent comparable with their algebraic versions. That has not been done yet and needs future research, both theoretical and experimental, with treatment of a large number of examples.

A much more practical way is to use the correspondence between algebraic equations and PDEs as outlined in Sect.5. In accordance with (2), the algebraic analog of (3) is given by

$$\left\{ \begin{array}{lcl} f_1 & = & x^3 - y^2 + z - 1 = 0, \\ f_2 & = & y^3 - z^2 + x - 1 = 0, \\ f_3 & = & z^3 - x^2 + y - 1 = 0. \end{array} \right. \tag{4}$$

One can use any appropriate modern computer algebra system with built-in Buchberger's algorithm, for example, Reduce or Maple.[¶] The lexicographical Gröbner basis $G = (g_1, g_2, g_3)$ for $Ideal(F = (f_1, f_2, f_3))$ has the form

$$\begin{aligned} g_1 := {} & 1395367452523974847088496\, x - 3697401204372060660 2747\, z^{26} \\ & -1804238714998140594993 1\, z^{25} + 3303064242798878948329 3\, z^{24} \\ & +2908488764581359132251 12\, z^{23} + 1290763920427100835979 36\, z^{22} \\ & -2199529105434900076514 08\, z^{21} - 7350434974529662992061 59\, z^{20} \\ & -3102727946171409126622 71\, z^{19} + 1606334750015907560768 07\, z^{18} \\ & +9823429161674280592335 58\, z^{17} - 2092043269584422646277 2\, z^{16} \\ & +1237879512449968788217761\, z^{15} - 6493647394921041092941 11\, z^{14} \\ & +1770783798251395989600640\, z^{13} - 3269570785655865115267646\, z^{12} \\ & +1017137392573548569160425\, z^{11} - 5380290668716432422480115\, z^{10} \\ & +2511318414925313271358736\, z^{9} - 5820765926743222198514292\, z^{8} \\ & +3916234105006809425362144\, z^{7} - 1271740986960481649374276\, z^{6} \\ & +4526571866128344763445768\, z^{5} + 4225399407124480064647579\, z^{4} \\ & -1702488139911346623054830\, z^{3} + 2599321588208462787828165\, z^{2} \\ & -6499050198343010356877518\, z + 1138531641035453825071656, \end{aligned} \tag{5}$$

[§]In fact, as far as we know, it is the only currently available universal package for computing differential Gröbner bases.

[¶]The timings on an Sparc station IPX are about 10 seconds and 1,5 minutes for Reduce 3.5 and Maple 5.3, respectively.

$$
\begin{aligned}
g_2 = {} & 1395367452523974847088496\, y - 242632459056334226660089\, z^{26} \\
& -256410179757740270610 5\, z^{25} - 761271693439804295251 3\, z^{24} \\
& +1837084738435086696768 88\, z^{23} + 2738259049306470656172 8\, z^{22} \\
& +933828836590525029178 72\, z^{21} - 445019735963407736728965\, z^{20} \\
& -1245803256938270529676 53\, z^{19} - 496913514381305731286115\, z^{18} \\
& +6860285684079607582747 70\, z^{17} - 293264534395468068345180\, z^{16} \\
& +175491631773986965591308 3\, z^{15} - 1494161044420719326758149\, z^{14} \\
& +3166864848665922491507168\, z^{13} - 4324148866718107366719898\, z^{12} \\
& +5118193353820594354340467\, z^{11} - 9669988041948198759925505\, z^{10} \\
& +8234964392530939187961184\, z^{9} - 14800290986146424015670204\, z^{8} \\
& +14407709710372641542274656\, z^{7} - 17110238558169309305535500\, z^{6} \\
& +19289855195567445653291224\, z^{5} - 14071675248685297091156215\, z^{4} \\
& +17208495316564910541540422\, z^{3} - 9780261032315595871168233\, z^{2} \\
& +5684543607089979672443254\, z - 4606430757804595389204888,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
g_3 = {} & z^{27} - 9\, z^{24} + 29\, z^{21} + 6\, z^{19} - 53\, z^{18} + 22\, z^{17} - 63\, z^{16} + 96\, z^{15} \\
& -149\, z^{14} + 242\, z^{13} - 261\, z^{12} + 484\, z^{11} - 545\, z^{10} + 740\, z^{9} - 908\, z^{8} \\
& +972\, z^{7} - 1220\, z^{6} + 1047\, z^{5} - 1045\, z^{4} + 943\, z^{3} - 535\, z^{2} + 422\, z - 216.
\end{aligned} \tag{7}
$$

Formulae (5-7) provide the desirable **elimination**. The polynomial g_3 is univariate in the smallest, with respect to $\succ$, variable z. Two other polynomials g_1 and g_2 are linear in x in y. Hence, the multivariate problem (4) has been reduced to the univariate one. In particular, the number of roots is just that of g_3. One can easily check, using a computer algebra system,[‖] that there are no multiple roots. Using relation (2) we obtain the following form of the general solution for the differential analog of $g_3 = 0$:

$$
u = \sum_{i=1}^{27} C_i(x, y) \exp\{\lambda_i z\}\,, \tag{8}
$$

where λ_i are the roots of equation $g_3 = 0$. Functions $C(x, y)$ are defined by the remaining two equations. Except for three roots $(1, \pm i)$ all the others can only be computed approximately by numerical methods.

To determine functions $C_i(x, y)$ we substitute (8) in the differential analog of equation $g_2 = 0$ written as $y - \sum_{j=1}^{26} b_j z^j = 0$ with the rational number coefficients $b_j \in \mathbb{Q}$. It yields equation $\partial c_i/\partial y - \sum_{j=1}^{26} b_j \lambda^j = 0$ with solution

$$
C_i(x, y) = \tilde{C}_i(x)\, \exp\left\{ y \sum_{j=1}^{26} b_j \lambda_i^j \right\}.
$$

[‖]For example, one can compute the greatest common divisor of g_3 and dg_3/dz.

Finally, the differential version of equation $g_1 = 0$ written as $x - \sum_{k=1}^{26} a_j \lambda^j = 0$ $(a_j \in Q)$ leads to the general solution of (1)

$$u(x,y,z) = \sum_{i=1}^{27} c_i \, \exp\left\{ x \sum_{j=1}^{26} a_j \lambda_i^j + y \sum_{j=1}^{26} b_j \lambda_i^j + \lambda_i z \right\},$$

where $c_i \in \mathbb{C}$ are arbitrary complex constants.

We established the arbitrariness in the general solution, namely, its dependence on twenty seven arbitrary constants, by explicit integration. However, as it was said above, it can be easily obtained directly from the Gröbner basis structure. Equations (5-7) reveal that number immediately as the number of roots of (7). In the general case, given any Gröbner basis, the number of roots with their multiplicities is the number of different monomials which have no divisors among the leading terms of the Gröbner basis[1]. If such a monomial number is infinite it means, in fact, the presence of a parametric arbitrariness in the solution space. The maximal number of arbitrary parameters defines the dimension of the polynomial ideal.

For an involutive system those relations still hold because it is a redundant Gröbner basis. Moreover, all that is also valid for the **involutive monomial division**. To see the latter explicitly at our example we consider the involutive polynomial basis for the ideal generated by polynomials (4). We choose the **DegRevLex** ordering (Sect.2) with $x \succ y \succ z$. Then the involutive basis can be constructed by means of algorithm **InvolutiveBasis** of Sect.4. Its implementation in Reduce in the form of a package *INVSYS*[12,13] in fractions of a second returns the output

$$\left\{ \begin{array}{l} x^2y^2z^3 - x^2y^2 + xy^2z + x^2z^2 - xyz^2 + x^2y - xy^2 + x^2 - xy - y^2 + z - 1\,, \\ x^2yz^3 + x^2y^2 - x^2y + xyz - xz^2 + x^2 - xy - x\,, \\ xy^2z^3 - xy^2 + y^2z + xz^2 - yz^2 - x^2 + xy - y^2 + x - y\,, \\ x^2y^3 - x^2z^2 - x^2 + y^2 - z + 1\,, \\ x^2z^3 + x^2y - xy^2 - x^2 + xz - x\,, \\ xyz^3 + xy^2 - xy + yz - z^2 + x - y - 1\,, \\ y^2z^3 - x^2y^2 - y^2 + z^2 - x + 1\,, \\ xy^3 - xz^2 + x^2 - x\,, \\ xz^3 + xy - y^2 - x + z - 1\,, \\ yz^3 - x^2y + y^2 - y\,, \\ x^3 - y^2 + z - 1\,, \\ y^3 - z^2 + x - 1\,, \\ z^3 - x^2 + y - 1\,. \end{array} \right. \tag{9}$$

The structure of system (9) reveals the following features:

- Being an extension of (4) which is already the reduced Gröbner basis for the DegRevLex ordering, system (9) is, therefore, a redundant Gröbner basis. The additional polynomials are reducible to zero modulo (4), but are not involutively reducible.

- The set of monomials indivisible by any of the leading terms of (9) is the same as for those of (4). There are exactly twenty seven such monomials $x^i y^j z^k$ $(0 \leq i, j, k < 3)$, as well as different complex roots of the system.
- Those monomials and only they are also involutively indivisible by the leading terms of (9).
- The sum of degrees in z over the leading terms in (9) is again equal to twenty seven. In such a way the total number of roots with their multiplicities can be computed for any involutive system[14].

One can ask whether there is any practical profit from ten extra equations of (9) with respect to (4). As it was noticed already in[12] and carefully investigated in[14], given an involutive system in ordering **DegRevLex**, it is straightforward to compute the corresponding lexicographical Gröbner basis.

To illustrate that by our example, let us consider (9) as a system of linear algebraic equations over the coefficient ring $\mathbb{Q}[z]$ in all the different monomials in x, y which occur in the system. It means that the smallest, with respect to the given ordering, variable z is treated as a free parameter. Then (9), as a linear system in twelve 'unknowns' x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^2y^2, xy^3, x^3y^3 contains thirteen equations. The **compatibility condition** which can be easily computed, for instance, by Gaussian elimination of those monomial 'unknowns', is nothing else than polynomial equation $g_3 = 0$ with g_3 given in (7). Analogously, solving the linear system with respect to x and y leads to other polynomials (5,6).

By doing so, computation of (5-7) from (4) needs less than two seconds on a Sparc station IPX with the *INVSYS* package[12] extended with a separate module *Invlex*[14] to perform the above linear algebra computations.

Similarly, an efficient algorithm[21] for computation the lexicographical Gröbner basis for (4) can also be applied. That algorithm uses the structure of a Gröbner basis in one ordering to convert it into another ordering, and it is likewise based essentially on linear algebra computations.

7. Symmetry Analysis of Differential Equations

Among numerous methods of analysis and integration of differential equations the most general and universal ones are based on their symmetry properties. Sophus Lie laid a foundation for theory of continuous groups and introduced the concept of infinitesimal group transformation just to find solutions of ordinary differential equations in a closed form. Later on it became clear that the Lie symmetry method is a universal powerful tool for qualitative analysis of nonlinear PDEs and even for constructing some classes of their explicit solutions (see comprehensive bibliography in[15]). Knowledge of those group-invariant solutions not only forms an important step in investigating the general solutions, but also provides a platform for analysis of the underlying physical models. The symmetry technique is now applied to diverse fields of mathematics, mechanics, theoretical and mathematical physics including differential geometry, analytical and quantum mechanics, nonlinear optics, fluid dynamics, general relativity, quantum field theory, etc.

In investigating the Lie symmetry properties of PDEs (1) one considers the one-parameter infinitesimal transformations in the space of dependent and independent variables $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$

$$\begin{cases} \tilde{x}_i(a) = x_i + \xi_i(x, u)a + O(a^2), \\ \tilde{u}_j(a) = u_j + \eta_j(x, u)a + O(a^2). \end{cases} \tag{10}$$

Then a function $\phi(x, u)$ is transformed under (10) as

$$\phi(\tilde{x}, \tilde{u}) = \phi(x, u) + a\,\hat{X}\phi(x, u) + O(a^2)\,,$$

where the differential operator $\hat{X}$ is given by formula

$$\hat{X} = \xi_i\,\frac{\partial}{\partial x_i} + \eta_j\,\frac{\partial}{\partial u_j}\,. \tag{11}$$

Hereafter the summation over the repeated indices is assumed. To be applied to the left hand sides of (1) operator (11) is to be **prolonged** up to the highest derivative order α, in order to take into account the transformation of partial derivatives of u_j under (10). Denote partial derivatives of u_j with respect to x_i by $u_{j;i}$, higher derivatives $D^{i_1 i_2}u_j$ by $u_{j;i_1,i_2}$ and more generally $D^\alpha u_j$ by $u_{j;\alpha}$. Then the necessary prolongation of (11) for equations (1) is given by

$$\hat{X}^{(\alpha)} = \hat{X} + \zeta_{j;i}\frac{\partial}{\partial u_{j;i}} + \zeta_{j;i_1 i_2}\frac{\partial}{\partial u_{j;i_1 i_2}} + \cdots + \zeta_{j;\alpha}\frac{\partial}{\partial u_{j;\alpha}}\,. \tag{12}$$

The functions $\zeta_{j;\ldots}$ are uniquely computed in terms of ξ_i, η_j and their derivatives by the recurrence relations

$$\begin{aligned} &\zeta_{j;i} = \mathbb{D}_i(\eta_j) - u_{j;q}\mathbb{D}_i(\xi_q), \\ &\zeta_{j;i_1\ldots i_p} = \mathbb{D}_p(\zeta_{j;i_1\ldots i_{p-1}}) - u_{j;i_1\ldots i_{p-1}q}\mathbb{D}_{i_p}(\eta_q)\,, \end{aligned} \tag{13}$$

where $\mathbb{D}_i$ is the total derivative operator with respect to x_i

$$\mathbb{D}_i = \frac{\partial}{\partial x_i} + u_{j;i}\frac{\partial}{\partial u_j} + u_{j;ik}\frac{\partial}{\partial u_{j;k}} + \cdots\,. \tag{14}$$

The invariance (1) under the transformation (10) implies

$$\hat{X}^\alpha f_i(x, u, u_i, \ldots, u_\alpha)|_{f_i(x,u,u_i,\ldots,u_\alpha u)=0} = 0\,, \tag{15}$$

where all the **differential variables** $u_j, u_{j;i}, \ldots$ are to be considered as independent ones subject to the constrains (1). It leads to the system of linear homogeneous PDEs in $\xi_i(x, u), \eta_j(x, u)$ which is called **determining system**.

Nowadays, there are quite a number of computer algebra packages for the automatic generation of the determining systems on the basis of formulae (11-15) and for their solving[15]. Sometimes different heuristic methods built into those packages can be successfully applied to a great variety of differential equations to find the explicit form of some particular infinitesimal operators (11) or even of all of them. In the latter case it is not difficult, usually, to compute their commutators and obtain the **symmetry Lie algebra**.

There are, however, no universal algorithms to solve any determining system. The most general approach is to transform the latter into the standard form. As such a form one can use an appropriate differential Gröbner basis[8], but the involutive one is of particular assistance. Possessing all the properties of the former, the involutive basis allows an easy computation of the dimension of the symmetry group[11] as well as to find the structure constants of the symmetry Lie algebra without explicit integration of the determining equations[9].

As an illustration we take the following nonlinear evolution PDE of diffusion type

$$u_t + u\,u_x - t\,u_{xx} = 0 \tag{16}$$

with two independent variables (t,x) and one unknown function u. In this particular case we shall use, for clarity sake, a slightly different indexing of functions ξ_i in the infinitesimal operator (11)

$$\hat{X} = \xi^t\,\frac{\partial}{\partial t} + \xi^x\,\frac{\partial}{\partial x} + \eta\,\frac{\partial}{\partial u}\,. \tag{17}$$

The determining system for functions ξ^t, ξ^x, η has the form

$$\begin{aligned}
&\xi^t_{uu} = 0\,, \quad \xi^x_{uu} = 0\,, \quad t\,\eta_{uu} - 2\,t\,\xi^x_{xu} - 2\,u\,\xi^x_u = 0\,, \quad 2\,t\,\xi^t_{xu} + 2\,\xi_u = 0\,,\\
&2\,t\,\eta_{xu} - t\,\xi^x_{xx} - u\,\xi^x_t + \xi^x_t + \frac{\xi^t u}{t} - \eta = 0\,, \quad t\,\eta_{xx} - u\,\eta_x - \eta_t = 0\,,\\
&t\,\xi^t_{xx} - u\,\xi^t_x + 2\,\xi^x_x - \xi^t_t - \frac{\xi^t}{t} = 0\,, \quad \xi^t_u = 0\,, \quad \xi^x_x = 0\,.
\end{aligned} \tag{18}$$

Choosing the **DegRevLex** ordering defined in Sect.3 with $u \succ x \succ t$, $\xi^t \succ \xi^x \succ u$ and applying the differential form (Sect.5) of algorithm **InvolutiveBasis** (Sect.4) we obtain the **standard form** for (18)

$$\begin{aligned}
&\xi^t_u = 0\,, \quad \xi^x_u = 0\,, \quad \eta_u = 0\,, \quad \xi^t_x = 0\,, \quad \xi^x_x - \xi^t/t = 0\,,\\
&\eta_x = 0\,, \quad \xi^t_t - \xi^t/t = 0\,, \quad \xi^x_t - \eta = 0\,, \quad \eta_t = 0\,.
\end{aligned} \tag{19}$$

Equations (19) are easily integrated giving the general solution

$$\xi^t = c_1\,t, \quad \xi^x = c_1\,x + c_2\,t + c_3\,, \quad \eta = c_2\,,$$

depending on three arbitrary constants c_i. Respectively, the Lie symmetry group is three-dimensional. The infinitesimal symmetry generators are given by

$$\hat{X}_1 = t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x}\,, \quad \hat{X}_2 = t\,\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\,, \quad \hat{X}_3 = \frac{\partial}{\partial x}\,.$$

with their commutator table

$$[\hat{X}_1, \hat{X}_2] = 0\,, \quad [\hat{X}_2, \hat{X}_3] = 0\,, \quad [\hat{X}_1, \hat{X}_3] = -\hat{X}_3\,. \tag{20}$$

As we noted above, the size of the symmetry group, that is the number of arbitrary constants in the general solution, as well as the commutator table for the symmetry operators can be obtained directly from the structure of the standard (involutive) form of the determining system without its explicit integration. It was established still in[4,5] that the number of arbitrary constants coincides with the number of so-called **parametric derivatives**. Those derivatives which are neither present among the leading terms of the involutive system, nor can be obtained by differentiation of these terms, are called **principal derivatives**.

For system (19) the parametric derivatives are just three unknown functions themselves. Using the general algorithm of paper[9], the Lie symmetry algebra (20) can be readily computed directly from (19).

One more important application of computer algebra to mathematical physics likewise based on the symmetry ideas is integrability analysis of nonlinear evolution PDEs by the inverse spectral transform[17]. It reveals another deep internal interconnection between differential and algebraic equations.

Let us consider the multi-parametric family of quasi-linear evolution equations with one-spatial and one-temporal independent variables

$$u_t = \Lambda\, u_n + \sum_{(i,d)} \beta_{(i,d)} \prod_{j=k\geq 0}^{k\leq n-1} (u_j^{i_j})^{d_j}\,,\ u = u(x,t) = (u^1,\ldots,u^m),\ u_i^j = \frac{d^i u^j}{dx^i} \tag{21}$$

with

$$\Lambda = diag(\lambda_1,\ldots,\lambda_m),\ \lambda_i, \beta_{(i,d)} \in \mathbb{C}\,,\ \lambda_i \neq 0,\ (i,d) = (i_1,\ldots,i_k; d_1\ldots,d_k).$$

Here polynomial equations in $\beta_{(i,d)}$ arise as an algebraic equivalent of the integrability conditions for (21)[19]. These conditions in their original form are some differential equalities which imply the existence of a differential polynomial $H(u,\ldots,u_N)$ with $N > n$ which provide the invariance of (21) under infinitesimal transformations

$$t' = t\,, \quad x' = x\,, \quad u' = u + a\, H(u,\ldots,u_N) + O(a^2)\,. \tag{22}$$

Note, that unlike (10), the transformation (22) depends on derivatives of u. That is why the underlying symmetry is called a **higher**, or **generalized**, or **Lie-Bäcklund** one, while that one corresponding to (10) is usually called **classical Lie symmetry**.

However, to solve those algebraic equivalents of the integrability conditions by the Gröbner basis or involutive methods one should try to split them into a finite set of subsystems. Generally, only afterwards each of the subsystems can be transformed to the standard form and solved. Otherwise, the standard form of the initial algebraic system, if it can be computed, is typically a set of polynomials too large to be analyzed and solved (see, for instance, example II in[19]).

8. Conclusion

In spite of notable progress over the last years in extending the Gröbner basis method to differential ideals, and recent steps forward in the design of involutive algorithms, these tools are still of very limited usefulness for nonlinear differential equations. In the symmetry analysis the initial nonlinear problem is reduced, in fact, to a linear one: integration of the determining equations. Generally, however, it is impossible to reduce a nonlinear problem not only to linear but even to quasilinear one.

Nevertheless, very recent results presented in[22] give a certain hope that, at least from the principal algorithmic point of view, nonlinear systems of differential polynomials can be also treated by computer algebra methods. Namely, any system of differential polynomials is algorithmically split into a finite number subsystems which can be considered much like the standard systems if one uses pseudo-reductions instead of reductions.

Another important direction of algorithmic research is the extension of the Gröbner basis technique to non-commutative algebras. The first steps in that direction has been done. For example, there is a class of **solvable** algebras with a finite Gröbner basis[23]. It is notable that in[23] the prolongation is also used for constructing Gröbner bases for two-sided ideals. Although the algorithms for the computation of Gröbner bases for those algebras have been implemented on several platforms, they should be improved to be successfully applied to problems of practical interest. One such problem arises in the computation of two-loop Feynman integrals in quantum field theory by the recurrence relation method[24]. It can be reduced to the elimination problem for a non-commutative polynomial system of a solvable type. For that particular problem it makes sense also to use another computer algebra technique[25].

9. Acknowledgements

The author dedicates this paper to the memory of Alyosha Zharkov who tragically died in March 1994 and with whom we had long-term discussions on all the related topics. I am grateful to Yu. Blinkov and F. Boulier for their fruitful remarks. This work was supported in part by the INTAS project No. 93-1180.

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