INVOLUTIVITY APPLIED TO DIFFERENTIAL EQUATIONS

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1 Introduction

Among the properties of systems of analytical partial differential equations (PDEs) which may be investigated without their explicit integration there are compatibility and formulation of an initial-value problem providing existence and uniqueness of the analytical solution. The classical Cauchy-Kowalevskaya theorem establishes a class of quasilinear PDEs which admit posing such an initial-value problem.

The main obstacle in investigating other classes of PDE systems of some given order q is existence of integrability conditions, that is, such relations for derivatives of order $\leq q$ which are differential but not pure algebraic consequences of equations in the system. An involutive system of PDEs has all the integrability conditions incorporated in it. This means that prolongations of the system do not reveal integrability conditions. Extension of a system by its integrability conditions is called completion. The quasilinear systems of Cauchy-Kowalevskaya type form a particular family of involutive systems.

Extension of a system by its integrability conditions is called completion. In this paper we present some recent achievements in development of constructive methods for completion to involution of polynomial and differential systems. From the completion point of view linear homogeneous PDE systems with constant coefficients can be associated with pure polynomial systems[1, 2, 3, 4], and polynomial involutive systems [6] give a good alternative of the reduced Gröbner bases in commutative algebra [5].

The general involutive approach is based on the concept of involutive monomial division [6] which is defined for a finite monomial set. Every particular division provides for each monomial in the set the self-consistent separation of variables into multiplicative and non-multiplicative. An involutive basis is constructed by combining of non-multiplicative prolongations with multiplicative reductions.

In differential algebra, in addition to their role as canonical bases of differential ideals, linear and quasilinear (orthonomic) involutive systems allow one to pose of an initial value problem providing uniqueness of its solution and to determine the structure of arbitrariness in general analytical solution [4].

Involutive bases of polynomial ideals [6] and linear differential ideals [4] are Gröbner bases of special form. Though involutive bases are generally redundant as Gröbner ones, their use makes more accessible the structural information of polynomial and differential ideals. Janet bases may be cited as typical representatives of involutive bases and have been used in algebraic and Lie symmetry analysis of differential equations. A minimal Janet basis is also a Pommaret basis whenever the latter is finite [7].

In the given paper we briefly describe the concept of involutivity for systems of ordinary differential equations (Sect.2), the notion of involutive polynomial bases (Sect.3) and outline some applications (Sect.4).

2 Involution of Ordinary Differential Equations

The concept of involution was invented by E.Cartan 100 years ago [8] for systems of partial differential equations (PDEs) and led to development of powerful geometric and algebraic methods of their analysis (see historical remarks in [2, 4] and references therein).

Though just for PDEs involuitive methods reveal the full power, their application to ordinary differential equations and finitely-dimensional dynamical systems is also very fruitful. The below definitions taken from the formal theory of PDEs [2] and circumscribed to ODEs.

Consider a system of k ODEs of the order q with m unknown functions $y_1(x), \ldots, y_m(x)$ of the most general form

$$\mathcal{R}_q : \left\{ \Phi_j(x, y_\alpha, y_\alpha^{(1)}, \ldots, y_\alpha^{(q)}) = 0 \quad (1 \le j \le k) \right. \tag{1}$$

with $y_\alpha^{(i)} \equiv d^i y_\alpha / dx^i$. A *prolongation* of system (1) is the system of order $q+1$

$$\mathcal{R}_{q+1} : \left\{ \begin{array}{l} \Phi_j(x, y_\alpha, y_\alpha^{(1)}, \ldots, y_\alpha^{(q)}) = 0\,, \\ \hat{D}\Phi_j(x, y^\alpha, y_\alpha^{(1)}, \ldots, y_\alpha^{(q)}) = 0\,, \end{array} \right. \qquad (1 \le j \le k)$$

where $\hat{D}$ is the total derivation operator.

$$\hat{D} = \frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \sum_{i \ge 0} y_\alpha^{(i+1)} \frac{\partial}{\partial y_\alpha^{(i)}} \qquad (y_\alpha^{(0)} \equiv y_\alpha)\,.$$

Analogously, the r-th prolongation $\mathcal{R}_{q+r}$ of $\mathcal{R}_q$ is defined as enlargement of system (1) with all the total derivatives of its equations up order r inclusive.

An *integrability condition* of $\mathcal{R}_q$ is the equation of order $\le q$ which is differential but not pure algebraic consequence of $\mathcal{R}_q$. In other words, the integrability conditions of $\mathcal{R}_q$, if they exist, are obtained by algebraic manipulations (by elimination of the derivatives of the order greater than q) over the prolonged systems.

As an example consider the following second order system in two unknowns $x(t), y(t)$

$$\left\{ \begin{array}{l} \dot{x} + y^2 = 0\,, \\ \ddot{x} + x\,y + t\,x = 0\,. \end{array} \right.$$

Here the integrability condition $2\,y\,\dot{y} - x\,y - t\,x = 0$ is obtained from the first prolongation by elimination of $\ddot{x}$.

A system (1) is called *involutive* if it has no integrability conditions. Thus, involutive systems have all integrability conditions incorporated in them. Enlargement of a system with its integrability conditions is *completion*. Any system can be *completed to involution* in finitely many steps by sequential prolongations and elimination [2, 9, 10].

3 Involutive Polynomial Bases

The basic algorithmic ideas go back to M.Janet [1] who invented the constructive approach to study of PDEs in terms of the corresponding monomial sets which is based on the following association between derivatives and monomials:

$$\frac{\partial^{\mu_1+\cdots+\mu_n} u^\alpha}{\partial x_1^{\mu_1}\cdots\partial x_n^{\mu_n}} \Longleftrightarrow [x_1^{\mu_1}\cdots x_n^{\mu_n}]_\alpha \,. \tag{2}$$

The monomials associated with the different dependent variables u^α are to be considered as belonging to different monomial sets.

The association (2) allows one to reduce the involutivity analysis of linear homogeneous systems of PDEs to one of pure algebraic systems [1, 2, 3, 4]. Having in mind this fact consider consider below the involutivity of algebraic systems.

Let $R = K[x_1,\ldots,x_n]$ be a ring of multivariate polynomials over a zero characteristic coefficient field K. Then a finite set $F = \{f_1,\ldots,f_m\} \subset R$ of polynomials in R is *a basis of the ideal*

$$< F >=< f_1,\ldots,f_m >= \{\ \sum_{i=1}^{m} h_i f_i \mid h_j \in R\ \} \,.$$

In the involutive approach to commutative (polynomial) algebra [6], which is a mapping of the involutivity analysis of linear PDEs [4, 2], for every polynomial in a finite set F the set variables $x_1,\ldots,x_n$ are separated into disjoint subsets of *multiplicative* and *nonmultiplicative* variables.

To be self-consistent such a separation must satisfy some axioms [6], and every appropriate separation generates an *involutive monomial division* in the following sense. Fix a linear *admissible monomial order* $\succ$ such that

$$\begin{aligned} &m \neq 1 \Longrightarrow m \succ 1\,,\\ &m_1 \succ m_2 \ \Longleftrightarrow m_1 m\ \succ m_2 m \end{aligned}$$

holds for any monomials (power products of the variables with integer exponents) m, m_1, m_2. Then for every polynomial f in F one can select its *leading monomial* $lm(f)$ (with respect to $\succ$). All leading monomials in F form a finite monomial set U. If $u \in U$ divides a monomial w such that all the variables which occur in w/u are multiplicative for u, then u is called *involutive divisor* of w. We shall denote by L an involutive division, which specifies a set of multiplicative (resp. nonmultiplicative) variables for every monomial u in any given finite monomial set U and write $u|_L w$ if u is ($L-$)involutive divisor of w. In the latter case we shall also write $w = u \times v$ where, by the above definition, monomial $v = w/u$ contains only multiplicative variables.

In papers [6, 11, 12] a number of particular involutive divisions was introduced and studied. As an example we present here one of them called after M.Janet who was one of the founders of the involutive approach to PDEs and who devised the related separation of variables [1].

Given a finite set U of monomials in $\{x_1, \ldots, x_n\}$ and a monomial $u = x_1^{d_1} \cdots x_n^{d_n} \in U$, a variable x_i $(i > 1)$ is Janet multiplicative for u if its degree d_i in u is maximal among all the monomials in U having the same degrees in variables $x_1, \ldots, x_{i-1}$. As for x_1, it is *Janet multiplicative* for u if d_1 takes the maximal value among degrees in x_1 of monomials in U. If a variable is not Janet multiplicative for u in U it is considered as *Janet nonmultiplicative.*

Consider, for example, a monomial set

$$U = \{u = x_1x_2, v = x_2x_3^2, w = x_3^3\}\,.$$

This gives the following Janet multiplicative and nonmultiplicative variables for monomials in U:

Monomial	Variables	
	Multiplicative	Nonmultiplicative
x_1x_2	x_1, x_2, x_3	$-$
x_2x_3	x_2, x_3	x_1
x_3^2	x_3	x_1, x_2

Given a finite polynomial set F, a noetherian [6] involutive division L, for instance, Janet division, and an admissible monomial order $\succ$, one can algorithmically construct [6] a *minimal* $L-$*involutive basis* or $L-$basis $G \subset R$ of ideal $< F >=< G >$ such that for any polynomial f in the ideal there is a polynomial g in G satisfying $lm(g)|_L lm(f)$, and every polynomial g in G does not contain monomials having involutive divisors among the leading monomials of other polynomials in G.

If $F = \{f_1, \ldots, f_m\} \subset R$ is a polynomial set, L is an involutive division and $\succ$ is an admissible monomial order, then any polynomial p in R can be rewritten (reduced) modulo ideal $< F >$ as

$$p = h - \sum_{i=1}^{m} \sum_{j} a_{ij} f_i \times u_{ij}$$

where a_{ij} are elements (coefficients) of the base field K, u_{ij} are $L-$multiplicative monomials for $lm(f_i)$ such that $lm(f)\, u_{ij} \preceq lm(p)$ for all i, j, and there are no monomials occurring in h which have $L-$ involutive divisors among $\{lm(f_1), \ldots, lm(f_m)\}$. In this case h is said to be in the $L-$*normal form* modulo F and written as $h = NF_L(p, F)$.

If G is $L-$basis, then $NF_L(p, G)$ is uniquely defined[1] for any polynomial p. In this case $NF_L(p, G) = 0$ if and only if p belongs to the ideal $< G >$ generated by G. Moreover, if the ideal is *radical* for which any its element (polynomial) vanishes at the common roots of all the polynomials in G if and only if this polynomial belongs to the ideal, then it follows that the condition $NF_L(p, G) = 0$ is necessary and sufficient for vanishing p on those common roots.

It is important to emphasize that any involutive basis is a Gröbner basis, generally redundant, and can be used in the same manner as the reduced Gröbner basis [5].

[1] For other properties of involutive bases, proofs and illustrating examples see [6].

4 Some Applications

Completion of differential equations to involution is the most universal algorithmic method for their algebraic analysis [1, 2, 4, 10] and and can be applied in particular for the following purposes.

- Check the compatibility of PDE systems. In the case of the system inconsistency there is an integrability condition of form

$$1 = 0$$

  which is revealed in the course of completion.

- Analysis of arbitrariness in general analytic solution of analytic systems of PDEs.

- Elimination of a subset of dependent variables, that is, obtaining the differential consequences of a given system, if they exist, which do not contain the dependent variables specified.

- Posing of an initial value problem for a system of analytic PDEs providing existence and uniqueness of locally holomorphic solutions.

- Lie symmetry analysis of DEs. Completion to involution of the determining equations for Lie symmetry generators is the most general algorithmic method of their integration.

- Computation of "hidden" constraints for constrained dynamical systems and their numerical indices.

- Pre-analysis for numerical integration. Revealing the integrability conditions and their explicit involvement into design of numerical schemas makes the numerical solution more relevant to the algebraic structure of the initial system.

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