Classification of compatible module orderings

Massimo Caboara [1]

Dip. di Matematica Università di Pisa

Marco Silvestri

Dip. di Matematica Università di Pisa

The aim of this paper is to give a complete classification for the module orderings which are compatible with their ring ordering and that give the same ordering on every module component (Riquier orderings). These orderings have a quite simple classifications, based on the classification of ring orderings. The orderings encountered in practice in Commutative Computer Algebra usually are Riquier orderings.

1991 Math. Subj. Class.: Primary 06F25; secondary 13C99, 13P99

Key words: Polynomial Modules, Ordering Classification

1 Introduction

The orderings on a polynomial ring are strictly related with the computation of Gröbner bases and the efficiency of Buchberger Algorithm; these orderings have been classified by Robbiano [8]. Gröbner bases and Buchberger Algorithm have been generalized to modules on a polynomial ring, see [6]. An unified approach to ring and module computations is desirable, both for theoretical and implementation purposes. One approach is the following : given a free module $M := R^r$ over the polynomial ring $R = k[\mathbf{x}]$ it is possible to embed M in a overring $S := k[\mathbf{x}, \mathbf{y}]$ in such a way that M is a subring of S. Then, module operations may be performed by ring algorithms applied to the image of M with minimal modifications, see [2,5].

Such approach needs a classification of the module orderings of computational interest, preferably one that gives a unified representation of ring and module orderings.

[1] Partially supported by the ESPRIT reactive LTR project 21024 FRISCO and C.N.R., Italy

In this paper, we give that simple classification for the Riquier orderings, i.e. the module orderings which are compatible with their ring ordering and that give the same ordering on every module component (Theorem 17). A brief version of this work, without proofs, has been presented as an ISSAC'96 poster [4]. The classification is obtained by embedding the module in a suitable polynomial ring and then exploiting the ring orderings classification. The orderings encountered in practice in commutative computer algebra usually are Riquier orderings, and they suffices for the classical operations in this field (i.e. ideal/module operations, radical and resolution computations, etc.).

More general module orderings are encountered, especially in problems coming from differential algebra and differential equation solving. In this setting the classification problem has been studied for Riquier orderings by Riquier [9], Thomas [15] and others. Recently, Carrà-Ferro and Sit [3] have studied more general, non-Riquier module orderings, and Reid and Rust [10–12] have classified *all* module orderings, after proving that there exists more module orderings that the ones considered by Carrà-Ferro and Sit. For a complete discussion on the literature, see [10].

2 Term-orderings on rings

Let us start by recalling some notations. We denote by $R := k[x_1, \ldots, x_n]$ the n-indeterminate polynomial ring on a field k, by $T^n := \{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{N}\}$ the set of the power products of R and by *Log* the monoid isomorphism from T^n to $\mathbb{N}^n$ described by $x_1^{a_1} \cdots x_n^{a_n} \longmapsto (a_1, \ldots, a_n)$. This isomorphism will be implicitly used thereafter. To ease the notation, we will often write $\mathbf{a}$ for $(a_1 \cdots a_n)$, $\mathbf{x}$ for $x_1 \cdots x_n$ et cetera throughout this article. Our interest lies in total orderings on R which are compatible with the monoid structure on T^n, and especially in *term-orderings*, the ones for which $t > 1 \forall\, t \in T^n$. Term-orderings are of special interest from the computational point of view since they are the kind of orderings used by the Buchberger algorithm. The set of all the orderings of T^n is $O(T^n)$, and the set of the term-orderings is $TO(T^n)$.
For the proof in this section, we refer to [8,7].

Remark 1

i) An ordering on $\mathbb{N}^n$ may be easily extended to an ordering on $\mathbb{Z}^n$.
ii) An ordering on $\mathbb{Z}^n$ may be easily extended to an ordering on $\mathbb{Q}^n$.

Hence, to give an ordering on $k[\mathbf{x}]$ it is sufficient to give an ordering on T^n (or, by *Log*, on $\mathbb{Z}^n$). The orderings on a polynomial ring has been completely classified; we report here the characterization for term-orderings.

Theorem 2 *Let τ be an ordering on $\mathbb{Q}^n$. Then there exists a list*
$\mathbf{u} := (u_1, \ldots, u_s)$ *of rationally independent vectors in $\mathbb{R}^n$ such that*

a) The map

$$\begin{array}{rcl}\alpha : (\mathbb{Q}^n, \tau) & \longrightarrow & (\mathbb{R}^s, lex) \\ v & \rightsquigarrow & (v \cdot u_1, \ldots, v \cdot u_s)\end{array}$$

is linear, ordered and injective.

b) If $d(u_i)$ is the dimension of the rational vector space spanned by the coordinates of u_i, then

$$\sum_{i=1}^{s} d(u_i) = n$$

If τ is a term-ordering, then

c) the first non zero entry for every component is positive.

Vice versa, every list $\mathbf{u} := [\, u_1, \ldots, u_s \,]$ of vectors which satisfy the conditions a), *and* b) *defines an ordering on $\mathbb{Q}^n$. If $\mathbf{u}$ satisfy condition* c) *too, then it defines a term-ordering.*

Example 3 *The vector list (with one element) $[(1, \sqrt{37})]$ defines a term-ordering τ on $k[x, y]$ for which $y > x^3$.*
The checking is easy, $y >_\tau x^3 \Leftrightarrow (-3, 1)\dot{(}1, \sqrt{37}) > 0 \Leftrightarrow -3 + \sqrt{37} > 0$.

Definition 4 *If $u_i \in \mathbb{Q}^n$ for each $i : 1 \ldots n$, it is useful to view the vector list $\mathbf{u}$ as a matrix, whose rows are $u_1, \ldots, u_s$. In this case, for $\mathbf{u}$ to represent a total ordering, $Det(\mathbf{u})$ has to be non zero. Such orderings are* rational *ordering. Matrices associated to rational term-orderings have the first non zero entry for every column positive.*

Different matrices may give the same ordering. We refer to Schwartz [14] for a discussion of these problems.

Definition 5 *Let τ be an ordering on T^n. The* set of positive terms *of τ is the set*

$$P_\tau := \{\mathbf{a} \in \mathbb{N}^n \mid x^{\mathbf{a}} >_\tau 1\}$$

Note that giving the set of positive terms of an ordering is a way to define it.

Remark 6 *If $\mathbf{u} := [u_1, \ldots, u_s]$ is a list of vectors defining the term-ordering τ, then the set of positive terms of τ is the subset of the points $p \in \mathbb{Z}^n$ satisfying one of the following systems of equations/inequalities*

$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n > 0 \end{cases}$$

$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n = 0 \\ u_{21}x_1 + \cdots + u_{2n}x_n > 0 \end{cases}$$

$$\vdots$$

$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n & = 0 \\ \vdots & \vdots \ \vdots \\ u_{s-1\,1}x_1 + \cdots + u_{s-1\,n}x_n & = 0 \\ u_{s1}x_1 + \cdots + u_{sn}x_n & > 0 \end{cases}$$

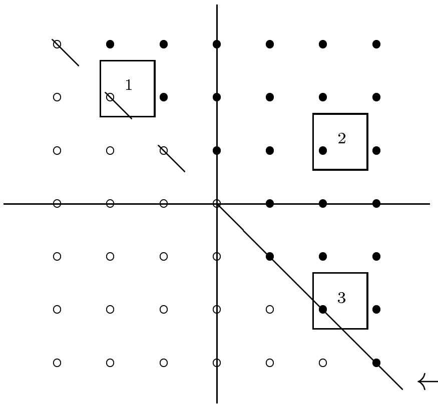
Example 7 *The term-ordering* $\tau := \mathrm{deglex}(x > y)$ *on* $k[x, y]$ *is described by the matrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The systems of equations/inequalities associated to τ *are*

$$\{x + y > 0 \quad \textit{and} \quad \begin{cases} x + y = 0 \\ x > 0 \end{cases}$$

In the picture below, the set $P_\tau \subset \mathbb{Z}^2$ *is described by the full dots.*



1) $y^2 \not>_\tau x^2 \Rightarrow (-2, 2) \notin P_\tau$

2) $x^2y >_\tau 1 \Rightarrow (2, 1) \in P_\tau$

3) $x^2 >_\tau y^2 \Rightarrow (2, -2) \in P_\tau$

$\longleftarrow x + y = 0$

2.1 Semi-flags

It is a natural question to characterize the conditions a subset of $\mathbb{Z}^n$ has to satisfy to be the set of positive terms of some ordering. To answer, we have to examine the geometry of such sets; we need some geometric definition. The topology is the usual euclidean topology.

Definition 8 *A maximal convex subset of* $\mathbb{R}^n$ *not containing the origin is a* semi-flag *of* $\mathbb{R}^n$*.*

Proposition 9 *Let* $B \subset \mathbb{R}^n$ *be a semi-flag. Then* $B = B_n \cup \ldots \cup B_1$*, where*

a) B_n is an open n-dimensional halfspace of $\mathbb{R}^n$ not containing the origin, whose closure contains the origin;
b) for each $n-1 \geq i \geq 1$, B_i is an open i-dimensional halfspace of $\mathbb{R}^n$, not containing the origin, contained in the border of B_{i+1}, and whose closure contains the origin.

PROOF. The set B is obviously a semi-flag. To show that every semi-flag has the same structure of B, let C be a semi-flag and then take the interior of the closure of C. This is an open n-dimensional halfspace of $\mathbb{R}^n$ not containing the origin, but whose closure contains the origin. We have thus found C_n. The set $C - C_n$ is a semi-flag of $\mathbb{R}^{n-1}$ contained in the border of C, and thus iterating this procedure we find $C_{n-1}, \ldots, C_1$. $\square$

Definition 10 *Let B be a semi-flag of $\mathbb{R}^n$; the subset $\mathcal{S}_B := B \cap \mathbb{Z}^n$ is the* skeleton *of B, and the sets B_i are the components of B.*

An ordering defines a semi-flag and conversely.

Lemma 11

a) Let τ be an ordering on T^n. Then there exists a (not necessarily unique) B semi-flag of $\mathbb{R}^n$ whose skeleton is equal to P_τ;
b) if B is a semi-flag of $\mathbb{R}^n$, then there exists an ordering τ on T^n such that $P_\tau = \mathcal{S}_B$.

PROOF.

a) Let B be a semi-flag of $\mathbb{R}^n$. Then it is easy to check that the relation defined by

$$\mathbf{x}^{\boldsymbol{a}} \succ \mathbf{x}^{\boldsymbol{b}} \Leftrightarrow \mathbf{a} - \mathbf{b} \in \mathcal{S}_B$$

is a total ordering on T^n compatible with the monoid operation.
b) Let $\mathbf{u} := [u_1, \ldots, u_s]$ be the list of vectors in $\mathbb{R}^n$ describing τ, and let the first r, $0 \leq r \leq s$ vectors be in $\mathbb{Q}^n$. Then the system of equation/inequalities

$$\left\{ u_{11}x_1 + \cdots + u_{1n}x_n > 0 \right.$$

$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n = 0 \\ u_{21}x_1 + \cdots + u_{2n}x_n > 0 \end{cases}$$

$$\vdots$$

$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n & = 0 \\ \vdots & \vdots \ \vdots \\ u_{r-1\,1}x_1 + \cdots + u_{r-1\,n}x_n & = 0 \\ u_{r1}x_1 + \cdots + u_{rn}x_n & > 0 \end{cases}$$

describe the subsets of $\mathbb{R}^n$ $B_n, B_{n-1}, \ldots, B_{n-r+1}$.

Let B_j, $n-r+1 \leq j \leq n-s+1$ be the subsets of $\mathbb{R}^n$ described by

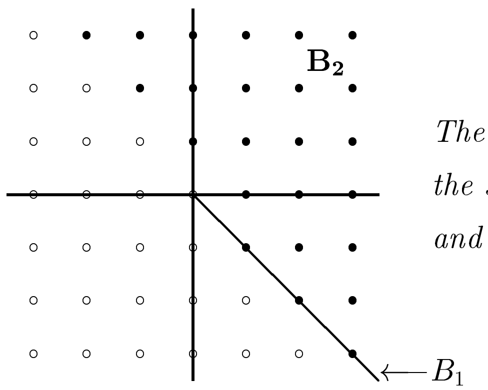
$$\begin{cases} u_{11}x_1 + \cdots + u_{1n}x_n & = 0 \\ \vdots & \vdots \ \vdots \\ u_{j-1\,1}x_1 + \cdots + u_{j-1\,n}x_n & = 0 \\ u_{j1}x_1 + \cdots + u_{jn}x_n & > 0 \end{cases}$$

It is easy to check that $B_n, \ldots, B_{n-s+1}$ are the first s components of a semi-flag. There exists many semi-flags which have first components $B_n, \ldots, B_{n-s+1}$, and every one of these satisfies the thesis. By the definition of $B_n \ldots B_{n-s+1}$ the skeleton of B is equal to P_τ (cfr. remark 6). $\square$

Remark 12 *From the proof of the lemma 11, it follows that for rational orderings the associated semi-flag is unique. For more general orderings, this is not true.*

Example 13 *The term-ordering* $\tau := \mathrm{deglex}(x > y)$ *has been examined in the example 7. The semi-flag* B *associated to it is defined by*

$$B_2 := \{(x,y) \in \mathbb{R}^2 \ \mid \ x+y > 0\} \cup B_1 := \left\{ (x,y) \in \mathbb{R}^2 \ \middle| \ \begin{cases} x+y=0 \\ x>0 \end{cases} \right\}$$



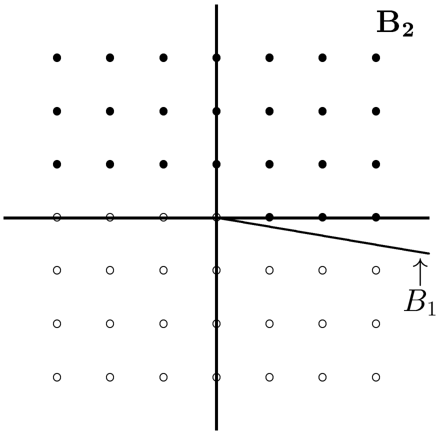
The full dots represent the skeleton of B*,*
the shadowed area B_2
and the halfline from the origin B_1*.*

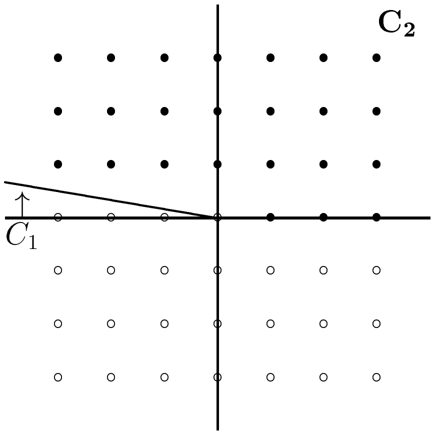
Example 14 *Consider the term-ordering* τ *described by the vector list* $\mathbf{u} := [(1, \sqrt{37})]$ *introduced in the example 3. Two of the associated semi-flags are* B *and*

C, whose components are, respectively,

$$B_2 := \{(x,y) \in \mathbb{R}^2 \;\mid\; \{x + \sqrt{37}y > 0\} \qquad\qquad C_2 := \{(x,y) \in \mathbb{R}^2 \;\mid\; \{x + \sqrt{37}y > 0\}$$

$$B_1 := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{cases} x + \sqrt{37}y = 0 \\ x > 0 \end{cases} \right\} \quad \textit{and} \quad C_1 := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{cases} x + \sqrt{37}y = 0 \\ y > 0 \end{cases} \right\}$$





Note that $\mathcal{S}_B \cap B_1 = \emptyset = \mathcal{S}_C \cap C_1$

3 Module orderings

We are interested in the orderings on a finitely generated, free module over a polynomial ring.

Definition 15 *Let M be a free R module, $\{\mathbf{e}_i\}_{i \in S}$ its unitary vectors and $\tau \in T^n$; then*

(i) the set $MT(M) := \{t\,\mathbf{e}_i \mid t \in T^n\}$ is the set of the module-terms of M;
(ii) let a, $b \in T^n$ and m, $n \in MT(M)$. Then, a Riquier ordering *μ is a total ordering on $MT(M)$such that*

$$a >_\tau b \Rightarrow am >_\mu bm \,, m >_\mu n \Rightarrow am >_\mu an \ \textit{and} \ m\,\mathbf{e}_i > n\,\mathbf{e}_i \Rightarrow m\,\mathbf{e}_j > n\,\mathbf{e}_j \ \forall\ j \in S$$

If $a\,\mathbf{e}_1 > \mathbf{e}_1\ \forall\ a \in TO(T^n)$, then μ is a Riquier term-ordering.

Examples of Riquier orderings are given in examples 21,22,23

Example 16 *Let $R := k[x]^2$ and μ the total module ordering on $MT(R)$ defined by*

$$x^a\,\mathbf{e}_i >_\mu x^b\,\mathbf{e}_j \Leftrightarrow \begin{cases} i = j = 1 \text{ and } a > b \\ i = j = 2 \text{ and } a < b \\ i > j \end{cases}$$

It is easy to see that μ is a total ordering on R^2, but μ is not a Riquier ordering, since $x\,\mathbf{e}_1 >_\mu 1\,\mathbf{e}_1$ but $x\,\mathbf{e}_2 <_\mu 1\,\mathbf{e}_2$.

Theorem 17 (Classification theorem) *Let τ be an ordering on T^n and M be a free, finitely generated R-module of rank r. Let μ be a Riquier module ordering on $MT(M)$. Then there are k, $(k \geq 1)$, variables $y_1, \dots, y_k$, an ordering μ' on $k[\mathbf{x}, \mathbf{y}]$ and $t_1, \dots, t_r$ power products of $k[\mathbf{x}, \mathbf{y}]$ linearly independent on $k[\mathbf{x}]$ such that*

(i) the module morphism

$$\begin{array}{rcl} \phi : (M, \mu) & \rightarrow & (k[\mathbf{x}, \mathbf{y}], \mu') \\ \mathbf{e}_i & \rightsquigarrow & t_i \end{array}$$

is ordered and injective.

(ii) μ is the restriction of μ' to $\phi(M)$.

PROOF. Since ϕ is clearly injective, M is isomorphic to $\phi(M)$. To prove the theorem, it suffices to define a semi-flag B of $\mathbb{R}^{n+k}$ such that ϕ is ordered with respect to μ and the ordering μ' described by the skeleton of B on T^{n+k}.

Let t_i be $x_1^{\alpha_{i,1}}, \dots, x_n^{\alpha_{i,n}}$ for each $1 \leq i \leq r$. We associate to every module-term in M a vector in $\mathbb{R}^{n+k}$ in a natural way

$$\mathbf{x}^{\boldsymbol{a}}\mathbf{e}_i \rightsquigarrow P_{a,i} := (a_1 + \alpha_{i,1}, \dots, a_n + \alpha_{i,n}, 0, \dots, \underbrace{1}_{n+i}, \dots, 0)$$

Let us consider the subset $\triangle$ of $\mathbb{Z}^n \times \mathbb{Z}^r$ defined by

$$\triangle := \{P_{a,i} - P_{b,j} \mid \mathbf{x}^{\boldsymbol{a}}\mathbf{e}_i > \mathbf{x}^{\boldsymbol{b}}\mathbf{e}_j\}$$

note that there are two kinds of the elements in $\triangle$:

$(a_1 - b_1, \dots, a_n - b_n, 0, \dots, 0)$ given by $P_{\boldsymbol{a},i} - P_{\mathbf{b},i}$
$(a_1 - b_1, \dots, a_n - b_n, 0, \dots, \underbrace{1}_{n+i}, \dots, \underbrace{-1}_{n+j}, \dots, 0)$ given by $P_{\boldsymbol{a},i} - P_{\mathbf{b},j}$.

Note also that $0 \notin \triangle$.

Claim. There exists a convex subset $C \subset \mathbb{R}^{n+k}$ which contains $\triangle$ but not the origin.

Since a semi-flag of $\mathbb{R}^{n+k}$ is a maximal convex subset of $\mathbb{R}^{n+k}$ which does not contain the origin, there exists a semi-flag B which contains C, and hence $\triangle$. The skeleton of B defines a term-ordering μ'. Since the skeleton of B contains $\triangle$, we have that

$$\mathbf{x}^{\boldsymbol{a}}\mathbf{e}_i >_\mu \mathbf{x}^{\boldsymbol{b}}\mathbf{e}_j \Leftrightarrow (a_1 - b_1, \ldots, a_n - b_n, 0, \ldots, \underbrace{1}_{n+i}, \ldots, \underbrace{-1}_{n+j}, \ldots, 0) \in \triangle$$
$$\Leftrightarrow \phi(\mathbf{x}^{\boldsymbol{a}}\mathbf{e}_i) = \mathbf{x}^{\boldsymbol{a}} y_i >_{\mu'} \mathbf{x}^{\boldsymbol{b}} y_j = \phi(\mathbf{x}^{\boldsymbol{b}}\mathbf{e}_j)$$

Hence ϕ is ordered with respect to μ and μ'.

Proof ***of the claim.***

Let $C := \{\sum_i c_i P_i \mid P_i \in \triangle \text{ and } \sum_i c_i = 1\}$ (the convex hull of $\triangle$) . This is a convex subset of $\mathbb{R}^{n+k}$ which contains $\triangle$. We have that $\triangle \subset C \subset \mathbb{R}^{n+k}$. Then we have only to prove that it does not contain the origin. Looking for a contradiction, we suppose that there exist $\delta_1, \ldots \delta_v \in \triangle$ such that

$$\sum_{i=1}^{v} c_i \delta_i = 0 \tag{1}$$

where $\sum_i c_i \geq 0$. If the set of c_i's which satisfy (1) has real points, then it has rational (and hence integer) points. Thus disproving (1) for integer c_i's is enough.

We then consider the list $L := [\underbrace{\delta_1, \ldots, \delta_1}_{c_1}, \ldots, \underbrace{\delta_v, \ldots, \delta_v}_{c_v}]$ and rewrite equation (1) as

$$\sum_{\delta \in L} \delta = 0 \tag{2}$$

Either all the elements of L have the last r components null, or they don't have them. If they don't have them, there exists an element $\delta \in L$ whose i^{th}, $n+1 \leq i \leq n+r$ component is 1. By equation (2), then there exists δ' in L whose i^{th} component is -1. Then let $L' := L - \{\delta, \delta'\} \cup \{\delta + \delta'\}$. The vector $\delta + \delta'$ belongs to $\triangle$ since μ is compatible with the monoid operation, and $\sum_{\delta \in L'} \delta = 0$. Since every application of the above process to L gives a list with a lesser number of elements, after a finite number of iterations all the elements of L are in $\triangle$ and have the last r components null. Thus, every element $\delta_i = (c_1^i, , c_n^i, 0, \ldots, 0)$ of L comes from an inequality $\mathbf{x}^{\boldsymbol{a}_i}\mathbf{e}_{l_i} > \mathbf{x}^{\mathbf{b}_i}\mathbf{e}_{l_i}$, where $\mathbf{a}_i$ is the vector of the positive c_i^j and $\mathbf{b}_i$ is the vector of the negative c_i^j. Since μ is a Riquier ordering, we may suppose that $l_i = 1\ \forall i$. That implies

$$\mathbf{x}^{\sum_i \mathbf{a}_i} > \mathbf{x}^{\sum_i \mathbf{b}_i} \Rightarrow \sum_i (\mathbf{a}_i - \mathbf{b}_i) = \sum_i \delta_i \neq 0$$

a contradiction. $\square$

Remark 18 *From theorem 17, the classification of Riquier orderings derives easily from the classification of ring orderings.*

4 Some interesting Riquier module term-orderings classes

From the point of view of computational commutative algebra, there are two classes of Riquier term-orderings which are especially significant, the Riquier term-orderings based on a ring term-ordering plus position considerations (cfr. example 21) and the "weighted" Riquier term-orderings associated to syzygies and resolutions computations [2,6] (cfr. example 22-23).

Definition 19 *Let M be a free R-module of rank r, $\tau \in TO(T^n)$ and $t_1, \ldots, t_r$ be power products in T^n. A compatible module term-ordering μ on M is a* To+Pos$^+$ *term-ordering if*

$$t\,\mathbf{e}_i >_\mu t'\,\mathbf{e}_j \Longleftrightarrow t >_\tau t' \text{ or } t = t' \text{ and } i > j$$

and a weighted term-ordering *w.r.t. the power products $m_1, \ldots, m_r$ in T^n if*

$$t\,\mathbf{e}_i >_\mu t'\,\mathbf{e}_j \Longleftrightarrow tm_i >_\tau t'm_j \text{ or } tm_i = t'm_j \text{ and } i > j$$

The matrices associated to the ring term-ordering τ and $lex(x_1 > x_2 > \ldots > x_r)$ will be denoted by Γ_τ and Γ_{lex_r} respectively.

Remark 20 *Using the notations of the classification theorem, it is easy to see that given a free, finitely generated R-module M, there is a considerable degree of freedom in the choice of the overring and of the embedding associated to a given module term-ordering. In the following we exhibit two different choices of overring (with term-ordering and associated matrix) and embedding for several examples. Clearly, the smaller the number of new variables introduced, the more efficient is the handling of the ordering.*

Example 21 *With the notation introduced above, A possible choice of overring and embedding for the To+Pos$^+$ module term-ordering is*

$$\begin{array}{rcl} \phi : M & \longrightarrow & k[\mathbf{x}, \mathbf{y}] \\ \mathbf{e}_i & \rightsquigarrow & y_i \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} [\Gamma_\tau] & \mathbf{0} \\ \mathbf{0} & [\Gamma_{lex_r}] \end{bmatrix}$$

Another choice is

$$\begin{array}{rcl} \phi : M & \longrightarrow & k[\mathbf{x}, y] \\ \mathbf{e}_i & \rightsquigarrow & y^i \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} \begin{bmatrix} & & \\ & \Gamma_\tau & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & 1 \end{bmatrix}$$

Example 22 *With the notation introduced above, A possible choice of overring and embedding for the weighted (w.r.t. the power products $m_1, \dots, m_r$ in T^n) module term-ordering is*

$$\begin{array}{rcl} \phi : M & \longrightarrow & k[\mathbf{x}, \mathbf{y}] \\ \mathbf{e}_i & \rightsquigarrow & y_i \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} \begin{bmatrix} & & \\ & \Gamma_\tau & \\ & & \end{bmatrix} & \begin{bmatrix} u_1 \cdot Log(m_1) & \cdots & u_1 \cdot Log(m_r) \\ \vdots & \vdots & \vdots \\ u_n \cdot Log(m_1) & \cdots & u_n \cdot Log(m_r) \end{bmatrix} \\ & \\ \begin{bmatrix} & & \\ & \mathbf{0} & \\ & & \end{bmatrix} & \begin{bmatrix} & & \\ & \Gamma_{lex_r} & \\ & & \end{bmatrix} \end{bmatrix}$$

where $\Gamma_\tau := (u_1, \dots, u_n)$.

If the number of variables $\mathbf{x}$ is n, another choice is

$$\begin{array}{rcl} \phi : M & \longrightarrow & k[\mathbf{x}, \mathbf{y}] \\ \mathbf{e}_i & \rightsquigarrow & m_i y^{n-i+1} \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} \begin{bmatrix} & & \\ & \Gamma_\tau & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & -1 \end{bmatrix}$$

Note that μ' is not a term-ordering on $k[\mathbf{x}, \mathbf{y}]$, but its restriction to $\phi(M)$ is.

Let us examine in detail a weighted term-ordering.

Example 23 *Let M be the module $(k[x_1, x_2])^3$. The term-ordering τ on $k[\mathbf{x}]$ is $deglex(x_1 > x_2)$ and the compatible module term-ordering μ on M is the multiweigthed module term-ordering w.r.t. the power products x_1^5, $x_1^4 x_2$, $x_1^3 x_2^2$. A possible choice of over-*

ring and embedding is

$$\begin{array}{rcl} \phi : M & \longrightarrow & k[x_1, x_2, y_1, y_2, y_3] \\ \mathbf{e}_i & \rightsquigarrow & y_i \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} 1 & 1 & 5 & 5 & 5 \\ 1 & 0 & 5 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Another choice of overring and embedding is

$$\begin{array}{rcl} \psi : M & \longrightarrow & k[x_1, x_2, y] \\ \mathbf{e}_1 & \rightsquigarrow & x_1^3 x_2^2 y \\ \mathbf{e}_2 & \rightsquigarrow & x_1^4 x_2 y^2 \\ \mathbf{e}_3 & \rightsquigarrow & x_1^5 y^3 \end{array} \qquad \Gamma_{\mu'} := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are orderings on the overring that are not the ones induced by a Riquier ordering of M via the embedding.

Example 24 *Let M, τ and μ be as in example 23. Then the term-ordering μ' on M given by the matrix*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is not compatible with μ through ψ, since $x_2 \mathbf{e}_3 >_\mu x_1 \mathbf{e}_1$ but $\psi(x_2 \mathbf{e}_3) = x_1^5 x_2 y^3 <_{\mu'} x_1^4 x_2^2 y = \psi(x_1\, \mathbf{e}_1)$.

Acknowledgement

We are pleased to thank Prof. Carlo Traverso for his hints and support.

References

[1] D. Bayer and D. Mumford, (1992). What can be computed in Algebraic Geometry?, Cambridge University Press, Symposia Math. XXXIV, pp. 1-48 .

[2] M. Caboara. A Modified Buchberger Algorithm for Resolution Computations, Preprint (1997).

[3] G. Carra' Ferro and W.Y. Sit, (1994). On term orderings and rankings, in Computational Algebra, Lecture Notes in Pure and Applied Math. 151, pp. 31-77, Dekker, NewYork.

[4] M. Caboara and M. Silvestri. (1996) Compatible module orderings, ISSAC'96 Poster Session, pp. 17-22.

[5] M. Caboara and C. Traverso. (1998) Efficient algorithms for ideal operations. Proceedings of ISSAC98 Conference (extended abstract)s.

[6] H.M. Möller and F. Mora (1986). New constructive methods in classical ideal theory. J. Algebra **100**, pp. 138-178.

[7] F. Mora and L. Robbiano, (1988).The Gröbner fan of an Ideal. Journal of Symbolic Computation **6**, pp. 183-208

[8] L. Robbiano, (1985). Term orderings on the polynomial rings. EUROCAL '85, Springer LNCS **204**, pp. 513-517.

[9] C. Riquier, (1910). Les systèmes d'équation aux d́erivées partielles. Gauthier-Villars, Paris.

[10] C.J. Rust, (in prep) The classification of Rankings of partials derivatives. PhD thesis.

[11] C.J. Rust, On the classification of Rankings of partials derivatives. (1995) May be downloaded from `http://www.math.uchicago.edu/~rust/orderings.ps`.

[12] C.J. Rust, C.J. and G.J. Reid, (1997) Rankings of partials derivatives. Proceedings of ISSAC'97, pp. 9-16.

[13] M. Silvestri (1993). Thesis. Università degli studi di Pisa.

[14] F. Schwartz, (1991). Monomial orderings and Gröbner Bases. ACM SIGSAM Bullettin 25, No 1, pp. 10-23.

[15] J.M. Thomas, (1931). Matrices of Integers Ordering Derivatives. Trans. Amer. Math Soc. **33**, pp. 389-410.

[16] V. Weispfenning, (1987). Differential term orderings. ACM SIGSAM Bullettin **21**/2, pp. 16-18.