# Multigraded Hilbert Functions and Buchberger Algorithm

Massimo Caboara *, Gabriel De Dominicis**, Lorenzo Robbiano**
* Dipartimento di Matematica, Università di Pisa
** Dipartimento di Matematica, Università di Genova

Quale uomo fusse, il quale non traesse piacere della villa?
L. B. Alberti (1404–1472)

### Abstract

In this paper we continue the work done in [GMRT]. Namely we describe an algorithm which allows to drive the classical Buchberger Algorithm by means of multigraded Hilbert-Poincaré series. The result is that in the multigraded case one can kill many more useless critical pairs. Besides, the new algorithm, called MGHDriven, is intrinsically parallelizable and it is now implemented in CoCoA (see [CNR]).

## 1 Introduction

The purpose of this paper is to describe an algorithm, which embodies some new ideas about improving the computation of Buchberger Algorithm. Here our point of view is to restrict the attention to the multigraded case. Multigraded ideals arise naturally in Commutative Algebra and Algebraic Geometry, when dealing with many important problems, such as the description of product schemes, Segre varieties, Rees rings of graded ideals and generic determinantal ideals, to name a few of them. Suppose a multigraded ideal is given over a polynomial ring; and assume that for some reasons we know its multigraded Hilbert Poincaré series (this happens under many circumstances, for instance when the ideal is a complete intersection). Then we want to take advantage of such knowledge while computing a Gröbner basis. In the monograded case all the details were explained in the paper [GMRT], but in our case many more complications arise and many more advantages can be achieved, as the actual computations shown at the end of the paper clearly prove. The paper is organized as follows. In the second section some generalities about multigradings are described. Then Sections 3 contains all the relevant theoretical material; in particular the key result is Theorem 3.12, where some properties of the Hilbert-Poincaré series of a multigraded ideal are exploited (for a description of the algorithm, which computes the Hilbert-Poincaré series in the multigraded case see for instance [B]). Section 4 is devoted to the description of the new algorithm, called MGHDriven; instead of using a pseudo code, we prefer to present it in CoCoAL code (see [CNR]). Since the CoCoAL code actually runs, it is very convenient to use it as a specification for a lower level code. Indeed this was the case, and the implementation of MGHDriven in C-code in the kernel of CoCoA allowed us to produce the table of timings of the final Section 5. Moreover the table reports the number of critical pairs killed by MGHDriven, which makes even more clear how intrinsically well it performs.

The authors are very grateful to A. Capani and G. Niesi for their assistance while coding the MGHDriven in the kernel of CoCoA.

The authors are also very grateful to A. Bigatti for her help with some technical aspects of the implementation of Hilbert Functions.

## 2 Multigraded Rings

Let $R := k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ and let $T^n$ be the commutative monoid of the power products in $x_1, \ldots, x_n$. Then $\mathcal{O}(T^n)$ is the set of *monoid orderings*, i.e. orderings on on $T^n$, which are compatible with the monoid structure of $T^n$, and $\mathcal{TO}(T^n)$ is the set of *term orderings* on $T^n$, i.e. monoid orderings, such that all the elements in $T^n \backslash \{1\}$ are greater than 1.

Let $f \in R$ and $\sigma \in \mathcal{O}(T^n)$; then we define

$$\mathrm{Lt}_\sigma(f) := \max_\sigma(Supp(f))$$

and we call it the *leading term* of $f$ with respect to $\sigma$; if $\mathrm{lc}_\sigma(f)$ is the coefficient of $\mathrm{Lt}_\sigma(f)$ in $f$, we call it the *leading coefficient* of $f$ with respect to $\sigma$.

Let $\sigma \in \mathcal{O}(T^n)$ and $I$ an ideal of $R$; we denote by $\mathrm{Lt}_\sigma(I) := (\{\mathrm{Lt}_\sigma(f) \bigm| f \in I\})$ the *leading ideal* of $I$ with respect to $\sigma$.

Let $\sigma \in \mathcal{O}(T^n)$ and $G := \{g_1, \ldots, g_s\} \subset R$; we denote by $\mathrm{Lt}_\sigma(G)$ the ideal of $R$,generated by $\{\mathrm{Lt}_\sigma(g_1), \ldots, \mathrm{Lt}_\sigma(g_s)\}$.

We denote by $Log$ the monoid isomorphism between $T^n$ and $\mathbb{N}^n$ defined by

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \longmapsto (\alpha_1, \ldots, \alpha_n).$$

If $I$ is the ideal of $R$ generated by $G$ we say that $G$ is a Gröbner basis of $I$ with respect to $\sigma$ if $\mathrm{Lt}_\sigma(I) = \mathrm{Lt}_\sigma(G)$.

For an introduction to the theory of Gröbner bases we refer to [R2], [AL], [E].

**Definition 2.1** *Let $w_i \in \mathbb{N}^n$, $1 \le i \le m$ and $\mathcal{W} \in M_{m,n}(\mathbb{N})$ the matrix whose rows are the $w_i$'s. Consider the "$\mathcal{W}$ degree map" induced by $\mathcal{W}$;*

$$\begin{array}{rccc} \partial_{\mathcal{W}}: & T^n & \longrightarrow & \mathbb{N}^m \\ & t & \longmapsto & \mathcal{W} \cdot Log(t) \end{array}$$

*where $\mathcal{W} \cdot Log(t) := (w_1 \cdot Log(t), \dots, w_m \cdot Log(t))$.*
*If $d = (d_1, \dots, d_m) \in \mathbb{N}^m$ we define:*

*i)* $T^n_{d_i} := \{t \in T^n \mid \partial_{\mathcal{W}}(t)_i = d_i\}$.

*ii)* $T^n_d := \bigcap_i T^n_{d_i}$.

*and we call $T^n_d$ the homogeneous component of $T^n$ of $\mathcal{W}$-degree $d$.*

It is clear that $T^n_d$ is the preimage of $d$ via the $\mathcal{W}$ degree map, therefore $T^n_d$ is invariant under any permutation of the $w_i$'s. We also remark that $T^n_d \cap T^n_{d'} = \emptyset$ if $d \neq d'$, hence $Span_k(T^n_d) \cap Span_k(T^n_{d'}) = \{0\}$ if $d \neq d'$.

Let $R_d := Span_k(T^n_d)$;if $t \in R^n_d$ and $u \in R^n_{d'}$ then clearly $tu \in R^n_{d+d'}$. Therefore $R = \bigoplus_{d \in \mathbb{N}^m} R_d$ and it is a $\mathbb{N}^m$-graded $k$-algebra.

**Definition 2.2** *We say that the matrix $\mathcal{W}$ is a system of weights if*

*i) none of the $w_i$'s can be expressed as a linear combination $\sum_{i \neq j} \alpha_j w_j$ with $\alpha_j \in \mathbb{Z}$*

*ii) for each $1 \le j \le n$ there exists $1 \le i \le m$ such that $w_{ij} \neq 0$*

**Remark 2.3** *If a matrix $\mathcal{W}$ is a system of weights, then $R_0 = k$ and $R_d$ is a finite dimensional $R_0$-vector space for every $d \in \mathbb{N}^m$.*

## 3 Multigraded Hilbert Functions, Poincaré series and Gröbner Bases

Given a system of weights $\mathcal{W} = (w_1, \dots, w_m)$ and a $\mathcal{W}$-homogeneous ideal $I$ of $R$ we may consider the Hilbert function of $R/I$ w.r.t $\mathcal{W}$, namely $\mathcal{H}^{\mathcal{W}}_{R/I} : \mathbb{N}^m \longrightarrow \mathbb{N}$, which is defined by $\mathcal{H}^{\mathcal{W}}_{R/I}(d) := dim_k(R_d/I_d)$, and the associated Poincaré series $\mathcal{HP}^{\mathcal{W}}_{R/I}(\lambda) := \sum_{d \in \mathbb{N}^m} \mathcal{H}^{\mathcal{W}}_{R/I}(d)\lambda^d \in \mathbb{Z}[[\lambda_1, \dots, \lambda_m]]$ where $\lambda^d := \lambda_1^{d_1} \cdots \lambda_m^{d_m}$. When there is no danger of ambiguity, we write $\mathcal{H}_{R/I}$ instead of $\mathcal{H}^{\mathcal{W}}_{R/I}$ and $\mathcal{HP}_{R/I}$ instead of $\mathcal{HP}^{\mathcal{W}}_{R/I}$.

The following Lemma collects several basic facts, which are part of the folklore on the subject and whose easy proof is left to the reader.

**Lemma 3.1** *Let $\mathcal{W} := (w_1, \dots, w_m)$ be a system of weights, $I \subset R$ a $\mathcal{W}$-homogeneous ideal and $\sigma \in \mathcal{TO}(T^n)$. Then*

*i)* $\mathcal{HP}_R(\lambda) = \prod_{i=1}^n \frac{1}{\left(1-\lambda^{\partial_{\mathcal{W}}(x_i)}\right)}$

*ii)* $\mathcal{HP}_{R/I} = \mathcal{HP}_{R/\mathrm{Lt}_\sigma(I)}$

*iii) There exists $f \in \mathbb{Z}[\lambda_1, \dots, \lambda_m]$ s.t.*

$$\mathcal{HP}_{R/I}(\lambda_1, \dots, \lambda_m) = \frac{f(\lambda_1, \dots, \lambda_m)}{\prod_{i=1}^n \left(1 - \lambda^{\partial_{\mathcal{W}}(x_i)}\right)}$$

As usual we denote by $\langle I \rangle^{\mathcal{W}}$ or simply by $\langle I \rangle$ the polynomial $f(\lambda_1, \dots, \lambda_m)$ in 3.1 and we call it the $\mathcal{HP}$-numerator of $I$ with respect to $\mathcal{W}$.

**Definition 3.2** *If $S = \sum_{d \in \mathbb{N}^m} \alpha_d \lambda^d$, $T = \sum_{d \in \mathbb{N}^m} \beta_d \lambda^d$ are elements of $\mathbb{Z}[[\lambda_1, \dots, \lambda_n]]$ we say that $S \le T$ if $\alpha_d \le \beta_d$ for each $d \in \mathbb{N}^m$ and $S < T$ if $\alpha_d < \beta_d$ for each $d \in \mathbb{N}^m$.*

**Definition 3.3** *Let $\mathcal{W} = (w_1, \dots, w_m)$ be a system of weights and $1 \le k < m$. We say that $\mathcal{U} := (w_1, \dots, w_k)$ is a subsystem of weights of $\mathcal{W}$ if it is a system of weights. Under these circumstances we can also say that $\mathcal{U} := (w_1, \dots, w_k)$ is a system of weights and $\mathcal{W} = (w_1, \dots, w_m)$ is a refinement of $\mathcal{U}$.*

**Lemma 3.4** *Let $\mathcal{U} = (w_1, \dots, w_k)$ be a system of weights, $\mathcal{W} := (w_1, \dots, w_m)$ a refinement of $\mathcal{U}$ and $r \in \mathbb{N}^k$. Then*

*i) $T^n_r$, the set of terms of $\mathcal{U}$-degree $r$, is the disjoint union of the $T^n_{(r,s)}$'s, the set of terms of $\mathcal{W}$-degree $(r, s)$, where $s \in \mathbb{N}^{m-k}$*

*ii) Let $I$ be a $\mathcal{W}$-homogeneous ideal of $R$. Then $(R/I)_r = \bigoplus_{s \in \mathbb{N}} (R/I)_{r,s}$. Moreover, if $\mathcal{H}^{\mathcal{U}}_{R/I}$ and $\mathcal{H}^{\mathcal{W}}_{R/I}$ are the corresponding Hilbert functions, then*

$$\mathcal{H}^{\mathcal{U}}_{R/I}(r) = \sum_{s \in \mathbb{N}^{m-k}} \mathcal{H}^{\mathcal{W}}_{R/I}(r, s)$$

*Proof.* The first part is quite straightforward. As to the second one, we observe that a $\mathcal{W}$-homogeneous ideal of $R$ is also $\mathcal{U}$-homogeneous. But $\mathcal{U}$ is a system of weights, hence each $T^n_r$ is finite; this implies that, for a fixed $r \in \mathbb{N}^k$, $T^n_{(r,s)}$ is empty for almost all $s \in \mathbb{N}^{m-k}$. The conclusion follows. $\square$

**Lemma 3.5** *Let $\mathcal{U} = (w_1, \dots, w_k)$ be a system of weights, $\mathcal{W} := (w_1, \dots, w_m)$ a refinement of $\mathcal{U}$, $I$ a $\mathcal{W}$-homogeneous ideal of $R$ and $\langle I \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} a_u \lambda^u$, $\langle I \rangle^{\mathcal{U}} = \sum_{v \in \mathbb{N}^k} a'_v \lambda^v$ the $\mathcal{HP}$-numerator of $I$ with respect to $\mathcal{W}$ and the $\mathcal{HP}$-numerator of $I$ with respect to $\mathcal{U}$. Then*

*i)* $\mathcal{HP}^{\mathcal{U}}_{R/I}(\lambda_1, \dots, \lambda_k) = \mathcal{HP}^{\mathcal{W}}_{R/I}(\lambda_1, \dots, \lambda_k, 1, \dots, 1)$.

*ii)* $\langle I \rangle^{\mathcal{U}} (\lambda_1, \dots, \lambda_k) = \langle I \rangle^{\mathcal{W}} (\lambda_1, \dots, \lambda_k, 1, \dots, 1)$

*iii)* $a'_v = \sum_{s \in \mathbb{N}^{m-k}} a_{(v,s)}$ *for every* $v \in \mathbb{N}^k$.

*Proof.* It is an easy consequence of the previous Lemma. $\square$

The following proposition generalizes a similar proposition in [GMRT], where a *single* weight is considered.

**Proposition 3.6** *Let $\mathcal{W} := (w_1, \ldots, w_m)$ be a system of weights, $I$,$J$ two $\mathcal{W}$-homogeneous ideals of $R$ such that $J \subseteq I$ and $\sigma \in \mathcal{TO}(T^n)$. Let $\langle I \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} a_u \lambda^u$ and $\langle J \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} b_u \lambda^u$; then for each $d \in \mathbb{N}^m$ the following conditions are equivalent:*

*i) $\mathcal{H}_{R/I}(r) = \mathcal{H}_{R/J}(r)$ for each $r <_\sigma d$ and $\mathcal{H}_{R/I}(d) < \mathcal{H}_{R/J}(d)$*

*ii) $a_r = b_r$ for each $r <_\sigma d$ and $a_d < b_d$*

*Moreover, $dim_k(I_d) - dim_k(J_d) = \mathcal{H}_{R/J}(d) - \mathcal{H}_{R/I}(d) = b_d - a_d$.*

*Proof.* We observe that

$$\sum_{r \in \mathbb{N}^m} (b_r - a_r)\lambda^r = (1 + f(\lambda)) \sum_{r \in \mathbb{N}^m} (\mathcal{H}_{R/J}(r) - \mathcal{H}_{R/I}(r))\lambda^r$$

where $1 + f(\lambda) = \prod_{i=1}^n (1 - \lambda^{\partial_{\mathcal{W}}(x_i)}) \in \mathbb{Z}[\lambda]$ and $\partial_{\mathcal{W}}(t) > 0$ for each $t \in Supp(F)$ (see Lemma 3.1). Then the conclusion follows easily $\square$

**Proposition 3.7** *Let $\mathcal{U} = (w_1, \ldots, w_k)$ be a system of weights, $\mathcal{W} := (w_1, \ldots, w_m)$ an extension of $\mathcal{U}$. Let $J \subset I$ be two $\mathcal{W}$-homogeneous ideals of $R$ such that*

$$\langle I \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} a_u \lambda^u \ , \ \langle J \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} b_u \lambda^u$$

*and*

$$\langle I \rangle^{\mathcal{U}} = \sum_{v \in \mathbb{N}^k} a'_v \lambda^v \ , \ \langle J \rangle^{\mathcal{U}} = \sum_{v \in \mathbb{N}^k} b'_v \lambda^v.$$

*If $r \in \mathbb{N}^k$ is s.t. $a'_v = b'_v$ for $v <_{Lex} r$ and $a'_r < b'_r$ then*

*i) $a_{(v,s)} = b_{(v,s)}$ for each $s \in \mathbb{N}^{m-k}$ and $v <_{Lex} r$*

*ii) $b_{(r,s)} - a_{(r,s)} = \mathcal{H}^{\mathcal{W}}_{R/J}(r,s) - \mathcal{H}^{\mathcal{W}}_{R/I}(r,s)$ for each $s \in \mathbb{N}^{m-k}$*

*iii) there exist a finite not empty subset $\mathcal{S}$ of $\mathbb{N}^{m-k}$ such that, for each $s \in \mathcal{S}$, $\mathcal{H}^{\mathcal{W}}_{R/I}(r,s) < \mathcal{H}^{\mathcal{W}}_{R/J}(r,s)$, hence $a_{(r,s)} < b_{(r,s)}$.*

*Proof.* From 3.4 we have $\mathcal{H}^{\mathcal{W}}_{R/I}(v,s) = \mathcal{H}^{\mathcal{W}}_{R/J}(v,s)\ \forall\ v <_{Lex} r$. Following 3.6 we get $a_{(v,s)} = b_{(v,s)}\ \forall\ v <_{Lex} r$. From $a'_r < b'_r$ we get $\mathcal{H}^{\mathcal{U}}_{R/I}(r) < \mathcal{H}^{\mathcal{U}}_{R/J}(r)$, hence

$$\sum_{s \in \mathbb{N}^{m-k}} \mathcal{H}^{\mathcal{W}}_{R/I}(r,s) < \sum_{s \in \mathbb{N}^{m-k}} \mathcal{H}^{\mathcal{W}}_{R/J}(r,s). \quad (1)$$

Now $\mathcal{H}^{\mathcal{W}}_{R/I}(r,s)$ and $\mathcal{H}^{\mathcal{W}}_{R/J}(r,s)$ are zero for almost all $s \in \mathbb{N}^{m-k}$ and inequality (1) implies that there exists at least one $\nu$ such that $\mathcal{H}^{\mathcal{W}}_{R/I}(r,\nu) < \mathcal{H}^{\mathcal{W}}_{R/J}(r,\nu)$.
We define $\mathcal{S} := \{s \in \mathbb{N}^{m-k} \mid \mathcal{H}^{\mathcal{W}}_{R/I}(r,s) \neq \mathcal{H}^{\mathcal{W}}_{R/J}(r,s)\}$ and we have $\mathcal{H}^{\mathcal{W}}_{R/I}(r,s) < \mathcal{H}^{\mathcal{W}}_{R/J}(r,s)$ for every $s \in \mathcal{S}$. Therefore we have shown that i) and ii) imply iii). Let $s \in \mathcal{S}$. As in the proof of 3.6 we can write:

$$\sum_{(r,s) \in \mathbb{N}^m} (b_{(r,s)} - a_{(r,s)})\lambda^{(r,s)} =$$

$$(1 + f(\lambda)) \sum_{(r,s) \in \mathbb{N}^m} (\mathcal{H}^{\mathcal{W}}_{R/J}(r,s) - \mathcal{H}^{\mathcal{W}}_{R/I}(r,s))\lambda^{(r,s)}$$

where $f \in \mathbb{Z}[\lambda]$ and $\partial_{\mathcal{W}}(t) \neq 0$ for each $t \in Supp(f)$, due to the fact that $\mathcal{U}$ is a system of weights. Then, from the above equality it follows

$$b_{(r,s)} - a_{(r,s)} =$$

$$\mathcal{H}^{\mathcal{W}}_{R/J}(r,s) - \mathcal{H}^{\mathcal{W}}_{R/I}(r,s) + f(\lambda) \sum_{(v,t) \in \mathbb{N}^m} (\mathcal{H}^{\mathcal{W}}_{R/J}(v,t) - \mathcal{H}^{\mathcal{W}}_{R/I}(v,t))$$

with $v <_{Lex} r$. By assumption,

$$\mathcal{H}^{\mathcal{W}}_{R/J}(v,t) - \mathcal{H}^{\mathcal{W}}_{R/I}(v,t) = 0 \quad \text{if } v <_{Lex} r.$$

Therefore, for each $s \in \mathbb{N}^{m-k}$ we have:

- $b_{(u,s)} = a_{(u,s)}$ if $u <_{Lex} r$
- $b_{(r,s)} - a_{(r,s)} = \mathcal{H}^{\mathcal{W}}_{R/J}(r,s) - \mathcal{H}^{\mathcal{W}}_{R/I}(r,s)$

Hence i) and ii) are proved. $\square$

**Example 3.8** *Let $R := k[x,y,z,t]$, $\mathcal{W} := \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix}$ a system of weights, $\mathcal{U} := \begin{pmatrix} 1 & 1 & 2 & 1 \end{pmatrix}$ a subsystem of weights of $\mathcal{W}$.*

*The ideals $I := (yt - t^2, xy - xt)$ and $J := (0) \subset I$ are $\mathcal{W}$-homogeneous ideals of $R$.*
*The Poincaré series of $I$ and $J$ w.r.t. $\mathcal{W}$ and $\mathcal{U}$ are*

$$\mathcal{HP}^{\mathcal{W}}_{R/I}(\lambda_1,\lambda_2) = \frac{1 - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_2^4 + \lambda_1^3\lambda_2^5}{(1-\lambda_1\lambda_2^3)(1-\lambda_1\lambda_2)^2(1-\lambda_1^2\lambda_2)}$$

$$\mathcal{HP}^{\mathcal{W}}_{R/J}(\lambda_1,\lambda_2) = \frac{1}{(1-\lambda_1\lambda_2^3)(1-\lambda_1\lambda_2)^2(1-\lambda_1^2\lambda_2)}$$

$$\mathcal{HP}^{\mathcal{U}}_{R/I}(\lambda) = \frac{1 - 2\lambda^2 + \lambda^3}{(1-\lambda)^3(1-\lambda^2))}$$

$$\mathcal{HP}^{\mathcal{U}}_{R/J}(\lambda) = \frac{1}{(1-\lambda)^3(1-\lambda^2))}$$

The first differences between $I$ and $J$ are generated by the power products $yt$ and $xy$, of $\mathcal{W}$-degree $(2,2)$ and $(2,4)$ respectively, so we expect difference 2 in $\mathcal{U}$-degree 2 and differences 1 in $\mathcal{W}$-degrees $(2,2)$ and $(2,4)$.
By subtracting the numerators we compute the first differences and verify that this example fits with 3.6.

$$\langle J \rangle^{\mathcal{U}} - \langle I \rangle^{\mathcal{U}} = 2\lambda^2 + 7\lambda^3 + 17\lambda^4 + \cdots$$

$$\langle J \rangle^{\mathcal{W}} - \langle I \rangle^{\mathcal{W}} =$$

$$\underbrace{\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_2^4}_{\mathcal{U}-degree=2} + \lambda_1^3\lambda_2^3 + \lambda_1^3\lambda_2^5 + \lambda_1^3\lambda_2^7 + \lambda_1^4\lambda_2^3 + \cdots$$

Moreover, the reader can easily check how this example fits with 3.5, and 3.7.

**Definition 3.9** *Let $\mathcal{U} = (w_1, \ldots, w_k)$ be a system of weights, $\mathcal{W} := (w_1, \ldots, w_m)$ a refinement of $\mathcal{U}$. Let $F$ be a $\mathcal{W}$-homogeneous system of generators of $I$ and*

$$\langle I \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} a_u \lambda^u \ , \ \langle \mathrm{Lt}_\sigma(F) \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} b_u \lambda^u.$$

*We define $Min(F,\mathcal{W}) := \min\{r \in \mathbb{N}^m \mid a_r \neq b_r\}$ and we call it the first difference of $F$ with respect to $\mathcal{W}$. In the same way we define $Min(F,\mathcal{U})$. If $r := Min(F,\mathcal{U})$, we denote $D(F,\mathcal{W},\mathcal{U}) := \{(r,s) \mid s \in \mathbb{N}^{m-k}$ and $a_{(r,s)} \neq b_{(r,s)}\}$ and we call it the first difference set of $F$ with respect to $(\mathcal{W},\mathcal{U})$.*

Let now $\sigma$ be a $\mathcal{W}$-compatible term-ordering. We have the concept of truncated Gröbner basis in this setting too.

**Definition 3.10** *Let $I$ be a $\mathcal{W}$-homogeneous ideal, $F$ a $\mathcal{W}$-homogeneous system of generators of $I$ and $d \in \mathbb{N}^m$. Then $F$ is called a $d$-truncated Gröbner basis of $I$ if $\{\mathrm{Lt}_\sigma(f) \mid f \in F_{\leq_{Lex} d}\}$ generates $\mathrm{Lt}_\sigma(I)_{\leq_{Lex} d}$.*

**Proposition 3.11** *Let $I$ be a $\mathcal{W}$-homogeneous ideal, $F$ a $\mathcal{W}$-homogeneous system of generators of $I$ and $\sigma$ a $\mathcal{W}$-compatible term-ordering; then:*

*i) $\boldsymbol{\mathcal{H}}_{R/I}(d) \leq \boldsymbol{\mathcal{H}}_{R/\mathrm{Lt}_\sigma(F)}(d)$ for every $d \in \mathbb{N}^m$*

*ii) $\boldsymbol{\mathcal{H}}_{R/I} = \boldsymbol{\mathcal{H}}_{R/\mathrm{Lt}_\sigma(F)}$ $iff$ $F$ is a Gröbner basis of $I$*

*iii) $F$ is an $r$-truncated Gröbner basis of $I$ for every $r <_{Lex} Min(F, \mathcal{U})$*

*Proof.* The proof is straightforward □

**Theorem 3.12** *Let $\mathcal{W} = (w_1, \ldots, w_m)$ be a system of weights, $I$ a $\mathcal{W}$-homogeneous ideal, $\sigma$ a $\mathcal{W}$-compatible term-ordering; let $G$ be the reduced Gröbner basis of $I$ with respect to $\sigma$ and $F$ an interreduced $\mathcal{W}$-homogeneous system of generators of $I$. We assume*

$$\langle I \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} a_u \lambda^u \ , \ \langle \mathrm{Lt}_\sigma(F) \rangle^{\mathcal{W}} = \sum_{u \in \mathbb{N}^m} b_u \lambda^u$$

*Finally we let $\mathcal{U}$ be a subsystem of weights of $\mathcal{W}$, $r := Min(F, \mathcal{U})$ and $D(F, \mathcal{W}, \mathcal{U})) = \{(r, u_1), \ldots, (r, u_s)\}$; then*

*i) $\#G_{(r,u_i)} = \#F_{(r,u_i)} + b_{(r,u_i)} - a_{(r,u_i)}$ for $i = 1, \ldots, s$.*

*ii) For each $h \in R_{(r,u_j)}$ and $i \neq j$ it holds $\boldsymbol{\mathcal{H}}_{R/\mathrm{Lt}_\sigma(F,h)}(r, u_i) = \boldsymbol{\mathcal{H}}_{R/\mathrm{Lt}_\sigma(F)}(r, u_i)$*

*Proof.* We know that $r$ is the first difference of $F$ with respect to $\mathcal{U}$. Moreover $F$ is assumed to be interreduced. This implies that $F$ is an $r'$-truncated reduced Gröbner basis of $I$ for every $r' <_{Lex} r$. Therefore $F_{<_{Lex} r} = G_{<_{Lex} r}$. By Proposition 3.7

$$\dim_k(Lt_\sigma(I)_{(r,u_i)}) - \dim_k(\mathrm{Lt}_\sigma(F)_{(r,u_i)}) = b_{(r,u_i)} - a_{(r,u_i)}$$

for each $1 \leq i \leq s$. From $F_{<_{Lex} r} = G_{<_{Lex} r}$ it follows that

$$\#G_{(r,u_i)} - \#F_{(r,u_i)} = \dim_k(Lt_\sigma(I)_{(r,u_i)}) - dim_k(Lt_\sigma(F)_{(r,u_i)})$$

for each $1 \leq i \leq s$, and so $i)$ is proved. To prove ii) we observe that if $p$,$q$ are power products and $p \in R_{(r,u_i)}$, $q \in R_{(r,u_j)}$ then no divisibility relation can occur between them. □

## 4 The Algorithm

Let $\mathcal{U} = (w_1, \ldots, w_k)$ be a system of weights and $\mathcal{W} := (w_1, \ldots, w_m)$ a refinement of $\mathcal{U}$. Let $I$ be a $\mathcal{W}$-homogeneous ideal of $R$, $F := \{f_1, \ldots, f_r\}$ a set of $\mathcal{W}$-homogeneous generators of $I$ and $\sigma$ a $\mathcal{W}$-compatible term-ordering. We also assume that we know the $\mathcal{W}$-Poincaré series of $R/I$; our goal is to use the know-how achieved from Theorem 3.12 and device an algorithm for the computation of a Gröbner basis of $I$ with respect to $\sigma$.

The algorithm displayed below is written in CoCoAL, the Programming Language, which is part of CoCoA (see [CNR]).

**Algorithm 4.1 MGHDriven**

**Input** *:*
*$F := \{f_1, \ldots, f_l\} \subset k[x_1, \ldots, x_n]$,*
*$\mathcal{W}$ a system of weights,*
*$\mathcal{U}$ a subsystem of weightsof $w$,*
*$HPIw := \boldsymbol{\mathcal{HP}}^{\mathcal{W}}_{R/(F)}$.*

**Output** *:*
*$G$ a Gröbner basis w.r.t. the current ordering.*

```
Define MGHDriven(F,W,U,HPIw)
  G := F;
  B := CreateInitialCouples(G);
  HPIu := Poincare(R/I,U);
  DiffList := ComputeDiff(G,W,U,HPIw,HPIu);
  While DiffList <> [ ] Do
    For J := 1 To Len(DiffList) Do
      UDeg := FirstUDeg(DiffList);
      WDeg := FirstWDeg(DiffList);
      CurDiff := FirstDiff(DiffList);
      DiffList := Tail(DiffList);
      Found := 0;
      While Found <= CurDiff Do
        CurPair := FirstPairInUWDeg(B,UDeg,WDeg);
        B := Tail(B);
        If PairIsNotCoprime(CurPair) Then
          S := NF(Spol(CurPair),G);
          If S <> 0 Then
            G := Concat(G,[S]);
            Found := Found+1;
            B := UpgradePairs(B,G,S);
          End;
          B := KillPairsInUWDeg(B,[WDeg,UDeg]);
        End;
        B := KillPairsInUDeg(B,Udeg);
        DiffList := ComputeDiff(G,W,U,HPIw,HPIu);
      End;
      B := KillAllPairs(B);
    End
  End;
  Return G
End;
```

The key words of CoCoAL are written in **bold**. The procedures `Poincare`, `Tail`, `Len`, `Concat`, `NF` (= NormalForm) are part of the standard library of CoCoA. The procedures `CreateInitialCouples`, `ComputeDiff`, `KillPairsInUDeg`, `FirstUDeg`, `FirstWDeg`, `FirstDiff`, `PairIsNotCoprime`, `UpgradePairs`, `KillPairsInUWDeg`, `FirstPairInUWDeg`, `KillAllPairs` were implemented in CoCoAL and then written in C code in the kernel of the last version of CoCoA.

**Remark 4.2** *The first important remark is that in practice it turns out to be very convenient to use a single weight as a subsystem of weights. However it may happen that, to reduce ourselves to a situation of that type, we need to redefine the weights. To explain that let us show an example. Let us consider the following system of weights $\mathcal{W} := \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$*

*Of course no subset is a subsystem of weights, but if an ideal is $\mathcal{W}$-homogeneous, it is also homogeneous with respect to $\mathcal{W}' := \begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$*

*So it is possible to consider the subsystem of weights given by $\mathcal{U} := \begin{pmatrix} 1 & 1 & 3 & 2 \end{pmatrix}$.*

**Remark 4.3** *Following the previous remark, MGHDriven requires the computation of some univariate Hilbert-Poincaré series in order to spot the first difference, say in* $\mathfrak{u}$*-degree* $r$*. Once such a difference is found, MGHDriven requires to compute a multigraded Hilbert-Poincaré series in order to detect the differences, say* $((r,u_1),\dots,(r,u_s))$*, in the more refined graduation. This is the key to discard many more useless critical pairs.*

**Remark 4.4** *Moreover, once a first difference is found, say in* $\mathfrak{u}$*-degree* $r$*, we do not need to compute the full multigraded Hilbert-Poincaré series, but it is sufficient to truncate it to the* $\mathfrak{u}$*-degree* $r$*. This is a substantial optimization of MGH-Driven.*

**Remark 4.5** *Moreover, the computation of the SP-polynomials in the degrees* $((r,u_1),\dots,(r,u_s))$ *is intrinsically parallelizable. Namely,* $R_{(r,u_1)}+\dots+R_{(r,u_s)}$ *is a direct sum, hence no interaction occurs among the elements of different subdegrees. This allows to conclude that the computations in the degrees* $(r,u_1),\dots,(r,u_s)$ *are independent each other.*

*This is a remarkable feature of MGHDriven.*

**Remark 4.6** *The Algorithm was implemented in* CoCoAL *and the* CoCoAL *code was used as a specification for the definitive C-coding in the kernel of* CoCoA*. This part of the work was done by the first two authors.*

## 5 Examples and Timing

There is a list of test ideals. In every case, the Poincaré series is known because the ideals are complete intersections.

- ```
  -- Example Segre

  Use R::=Z/(32003)[x[1..3]y[1..3]];

  I:=Ideal(x[1]y[1]^2-x[2]y[2]^2+x[3]y[3]^2,
           x[1]y[2]^3-x[2]y[3]^3+x[3]y[1]^3,
           x[1]y[3]^4-x[2]y[1]^4+x[3]y[2]^4);
  Weights:= [ [1,1,1,1,1,1],[0,0,0,1,1,1]];

      The task is to compute   GBasis(I)
  ------------------------------------------
  ```

- ```
  -- Example Det

  Use R::=Z/(32003)[t a b c d e f g h i j k l
  m n o p q r s u ],Lex;

  I:=Ideal(-tc^2e+2tbcf-taf^2-tb^2h+taeh+k,
  -tcde+tbdf+tbcg-tafg-tb^2i+taei+l,
  -td^2e+2tbdg-tag^2-tb^2j+taej+m,
  -tcdf+tc^2g+tbdh-tagh-tbci+tafi+n,
  -td^2f+tcdg+tbdi-tagi-tbcj+tafj+o,
  -tdf^2+tcfg+tdeh-tbgh-tcei+tbfi+p,
  -tdfg+tcg^2+tdei-tbgi-tcej+tbfj+q,
  -td^2h+2tcdi-tai^2-tc^2j+tahj+r,
  -tdgh+tdfi+tcgi-tbi^2-tcfj+tbhj+s,
  -tg^2h+2tfgi-tei^2-tf^2j+tehj+u);

  W:=[ [1,1,1,1,1,1,1,1,1,1,1,4,4,4,4,4,4,4,
        4,4,4] ,
       [1,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,
        1,1,1] ];

      The task is to compute   GBasis(I)
  ------------------------------------------
  ```

- ```
  -- Example Aldo -- This example comes from the
     study of the blow-up algebras of certain
     rational normal scrolls

  Use R ::=  Z/(32003)[y[36],x[0..9]];

  I:=Ideal( x[1]^2   - x[0]x[2] + y[1],
            x[1]x[2] - x[0]x[3] + y[2],
            x[1]x[3] - x[0]x[4] + y[3],
            x[1]x[4] - x[0]x[5] + y[4],
            x[1]x[5] - x[0]x[6] + y[5],
            x[1]x[6] - x[0]x[7] + y[6],
            x[1]x[7] - x[0]x[8] + y[7],
            x[1]x[8] - x[0]x[9] + y[8],
            x[2]^2   - x[1]x[3] + y[9],
            x[2]x[3] - x[1]x[4] + y[10],
            x[2]x[4] - x[1]x[5] + y[11],
            x[2]x[5] - x[1]x[6] + y[12],
            x[2]x[6] - x[1]x[7] + y[13],
            x[2]x[7] - x[1]x[8] + y[14],
            x[2]x[8] - x[1]x[9] + y[15],
            x[3]^2   - x[2]x[4] + y[16],
            x[3]x[4] - x[2]x[5] + y[17],
            x[3]x[5] - x[2]x[6] + y[18],
            x[3]x[6] - x[2]x[7] + y[19],
            x[3]x[7] - x[2]x[8] + y[20],
            x[3]x[8] - x[2]x[9] + y[21],
            x[4]^2   - x[3]x[5] + y[22],
            x[4]x[5] - x[3]x[6] + y[23],
            x[4]x[6] - x[3]x[7] + y[24],
            x[4]x[7] - x[3]x[8] + y[25],
            x[4]x[8] - x[3]x[9] + y[26],
            x[5]^2   - x[4]x[6] + y[27],
            x[5]x[6] - x[4]x[7] + y[28],
            x[5]x[7] - x[4]x[8] + y[29],
            x[5]x[8] - x[4]x[9] + y[30],
            x[6]^2   - x[5]x[7] + y[31],
            x[6]x[7] - x[5]x[8] + y[32],
            x[6]x[8] - x[5]x[9] + y[33],
            x[7]^2   - x[6]x[8] + y[34],
            x[7]x[8] - x[6]x[9] + y[35],
            x[8]^2   - x[7]x[9] + y[36] )

  Weights:=[[2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2,  2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
  [2, 3, 4, 5, 6, 7, 8, 9, 4, 5, 6, 7, 8, 9,
  10, 6, 7, 8, 9, 10, 11, 8, 9, 10, 11, 12,
  10, 11, 12, 13, 12, 13, 14, 14, 15, 16,
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9] ]

      The task is to compute  Elim(x,I)
  ------------------------------------------
  ```

The following examples are Rees algebras

- ```
  -- Example Rees-a
  Use R::=Z/(32003)[y[1..16],x[1..4,1..4],t]
  M:=Mat([ [x[I,J] | J In 1..4] | I In 1..4])
  M[1]:=[x[1,1]+x[2,1]+x[2,4],
  x[1,2]+x[2,2]+x[1,4],x[1,3]+x[2,3]+x[2,4],
  x[1,4]+x[2,4]+x[1,1]]
  I:=Ideal(tMinors(3,M)-[y[K] | K In 1..16])
  ```

```
Weights:=[[4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
4, 4, 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1,1,1,1,1,1],
[9, 9, 9, 9, 10, 10, 10, 10, 12, 12, 12, 12,
12, 12, 12, 12, 2, 2, 2, 2, 2, 2, 2, 2, 4,
4, 4, 4, 5, 5, 5, 5, 1]]

    The task is to compute Elim(t,I)
-----------------------------------------
```

- -- Example Rees-b

```
Use R::=Z/(32003)[y[1..16],x[1..4,1..4],t];

M:=Mat([ [x[I,J] | J In 1..4] | I In 1..4])
M[1]:=[x[1,1]+x[2,1]+x[2,4],
x[1,2]+x[2,2]+x[1,4],x[1,3]+x[2,3]+x[1,4],
x[1,1]+x[2,4]+x[2,2]]
I:=Ideal(tMinors(3,M)-[y[K] | K In 1..16])
Weights:=[[4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
4, 4, 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1,1,1,1,1, 1],
[9, 9, 9, 9, 10, 10, 10, 10, 12, 12, 12,
12, 12, 12, 12, 12, 2, 2, 2, 2, 2, 2, 2, 2,
4, 4, 4, 4, 5, 5, 5, 5, 1]];

    The task is to compute Elim(t,I)
-----------------------------------------
```

- -- Example Rees-c

```
Use R::=Z/(32003)[y[1..16],x[1..4,1..4],t]
M:=Mat([ [x[I,J] | J In 1..4] | I In 1..4])
M[1]:=[x[1,1]+x[1,2]+x[2,1],
x[1,2],x[1,3],x[1,4]]
M[2]:=[x[2,1],x[2,2]+x[2,4]+x[1,3],
x[2,3],x[2,4]]
I:=Ideal(tMinors(3,M)-[y[K] | K In 1..16])
Weights:=[[4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,
4, 4, 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1],
[9, 9, 9, 9, 10, 10, 10, 10, 12, 12, 12, 12,
12, 12, 12, 12, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4,
4, 4, 5, 5, 5, 5, 1]]

    The task is to compute Elim(t,I)
-----------------------------------------
```

- -- Example SecPl (Secant Planes)

```
Use R::=
Z/(32003)[x[0..9]abcy[1..3]z[1..3]w[1..3]];

I:=Ideal(-by[1]^3-az[1]^3-cw[1]^3+x[0],
-by[1]^2y[2]-az[1]^2z[2]-cw[1]^2w[2]+x[1],
-by[1]^2y[3]-az[1]^2z[3]-cw[1]^2w[3]+x[2],
-by[1]y[2]^2-az[1]z[2]^2-cw[1]w[2]^2+x[3],
-by[1]y[2]y[3]-az[1]z[2]z[3]-cw[1]w[2]w[3]
  +x[4],
-by[1]y[3]^2-az[1]z[3]^2-cw[1]w[3]^2+x[5],
-by[2]^3-az[2]^3-cw[2]^3+x[6],
-by[2]^2y[3]-az[2]^2z[3]-cw[2]^2w[3]+x[7],
-by[2]y[3]^2-az[2]z[3]^2-cw[2]w[3]^2+x[8],
-by[3]^3-az[3]^3-cw[3]^3+x[9]);

Weights:=[ [4,4,4, 4,4,4, 4,4,4, 4, 1,1,1,
           1,1,1, 1,1,1, 1,1,1],
           [1,1,1, 1,1,1, 1,1,1, 1, 1,1,1,
            0,0,0, 0,0,0, 0,0,0]];

    The task is to compute Elim(a..w[3],I)
-----------------------------------------
```

We show two tables. In the first one we report the timing in seconds obtained with the above described examples. They have been computed with a **Pentium 90 Mhz** with **16Mb RAM** of memory, running under linux. To compute them we used the latest version of CoCoA (3011b). In the second table we show some statistics related to the performance of our algorithm. We remark that all the examples are in *char* 32003. In *char* 0 the performance of the algorithm improves, since the computational cost of every reduction increases.

And now let us give some explanation of the headings:

**Usual** The usual Buchberger algorithm.

**HDr** The univariate Hilbert-Driven Buchberger algorithm as it is described in [GMRT].

**Ov1** The overhead of the computation of the univariate Poincarè series.

**MGHD** The MGHDriven algorithm.

**Ov2** The overhead of the computation of the univariate and the multigraded Poincarè series.

**Useful** The number of the computed useful pairs after the application of the Gebauer-Möller criteria.

**Total** The total number of the computed critical pairs.

**H-K** The number of pairs recognized as useless (hence killed) by the **HDriven** algorithm.

**MGH-Kk** The number of pairs recognized as useless (hence killed) by the **MGHDriven** algorithm.

**No PS** The number of univariate and multigraded Poincarè series computed.

| Name | Usual | HDr | Ov1 | MgHD | Ov2 |
|---|---|---|---|---|---|
| Segre | 359 s | 289 s | 0.5 s | 34 s | 8.5 s |
| Det | 96 s | 52 s | 2.48 s | 48 s | 13.5 s |
| Aldo | $>$ 24 h | 16908 s | 8.46 s | 4202 s | 61.14 s |
| Rees a | 1129 s | 424 s | 11.39 s | 170 s | 68.8 s |
| Rees b | 3327 s | 876 s | 8.5 s | 241 s | 97.89 s |
| Rees c | $>$ 15896 s | 2510 s | 17.13 s | 1465 s | 350 s |
| SecPl | 210 s | 209 s | 3.76 s | 189 s | 33.32 s |

| Name | Useful | Total | H-K | MGH-K | No PS |
|---|---|---|---|---|---|
| Segre | 272 | 1073 | 89 | 711 | 28+28 |
| Det | 411 | 4123 | 1524 | 1708 | 25+25 |
| Aldo | 1659 | 53697 | 35728 | 46700 | 8+8 |
| Rees a | 623 | 8702 | 1876 | 5815 | 21+21 |
| Rees b | 537 | 7388 | 2355 | 5123 | 20+20 |
| Rees c | 820 | 12097 | 2587 | 7398 | 24+24 |
| SecPl | 1198 | 8875 | 1309 | 1972 | 12+12 |

## References


[AL] ADAMS, W. W., AND LOUSTANOU, P. *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics, AMS (1994)

[BS] BAYER, D., AND STILLMAN M. *Computation of Hilbert functions*, J. of Symbolic Comp. vol. 14 (1992)

[B] BIGATTI, A. M. *Computations of Hilbert-Poincaré Series*, Preprint (1995)

[BCR] BIGATTI, A., CABOARA, M., AND ROBBIANO L. *On the Computation of Hilbert-Poincarè Series*, AAECC 2, 21-33 (1991)

[CNR] CAPANI, A., NIESI, G., AND ROBBIANO, L. CoCoA*, a system for doing Computations in Commutative Algebra*, (1995). Available via anonymous ftp from `lancelot.dima.unige.it`

[E] EISENBUD, D. *Commutative Algebra with a View Toward Algebraic Geometry*, Springer Graduate Texts in Mathematics 150, (1994)

[GMRT] GIANNI, P., MORA, T., ROBBIANO, L., AND TRAVERSO, C. *Hilbert functions and Buchberger algorithm*, Preprint

[MM] MÖLLER, H. M., AND MORA, T. *New Constructive Methods in Classical Ideal Theory*, Journal of Algebra 100, 138-178 (1986)

[R1] ROBBIANO, L. *Term orderings on the polynomial rings*, EUROCAL '85, Springer LNCS 204, 513-517 (1985)

[R2] ROBBIANO, L. *Introduction to the theory of Gröbner bases*, Queen's Papers in Pure and Applied Maths (1988)