Families of Estimable Terms

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Ancora non resta la villa di dì in dì
mandarti de' frutti suoi più serotini.
(L.B. ALBERTI)

ABSTRACT

In this paper we present a full solution to an open problem in Design of Experiments, a branch of Statistics, which can equally be seen as a problem in Algebraic Geometry. Given a complete set O of estimable terms, we are able to find all the fractions $\mathcal{F}$ of a full factorial design, such that $\overline{O}$ is a basis of $P/I(\mathcal{F})$ as a K-vector space. This fact can be rephrased as a result in the theory of zero-dimensional schemes.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algebraic Algorithms; G.3 [**Experimental Design**]:

Keywords

Gröbner Bases, Border Bases, Estimable Terms, Statistics

1. INTRODUCTION

The concluding sentence of [9] was: *The problem of computing* $\mathrm{Sol}(\mathcal{O})$ *in general is still unsolved.* So, what is $\mathrm{Sol}(\mathcal{O})$? What kind of difficulties are related to the computation of $\mathrm{Sol}(\mathcal{O})$? And what has all this stuff to do with statistics and with computational commutative algebra?

To answer these questions and motivate the work described in the present paper, we must go back to the introduction of [9], where all the necessary background is described. It is explained how the pioneering paper [7] introduced the new methods of computational commutative algebra, in particular Gröbner bases, into the realm of design of experiments. A more complete description of this new branch of mathematics is contained in [6].

Basically, a full factorial design is a finite grid of points, whose defining ideal is the complete intersection of n univariate polynomials in $P = K[x_1, \ldots, x_n]$. In statistical jargon a subset $\mathcal{F}$ of a full factorial design is called a fraction and one of the main problems in Design of Experiments (DoE) is to describe the sets of terms (power products), whose residue classes form a basis of $P/I(\mathcal{F})$ as a K-vector space. These are described as the sets of terms identified by $\mathcal{F}$. But even more important is the inverse problem. In other words, suppose we are given a tuple $\mathcal{O}$ of power products. We would like to be able to find $\mathrm{Sol}(\mathcal{O})$, i.e. the set of all those fractions $\mathcal{F}$ such that the residue classes of $\mathcal{O}$ form a basis of $P/I(\mathcal{F})$ as a K-vector space. The main achievement of [3] was to find a solution of the inverse problem in the particular case that $\mathcal{O}$ is the complement of a leading term ideal, and Gröbner basis theory played the key role. However, it was pointed out that not all the cases fit into this scheme (see Example 2.4).

In this paper we employ a new approach to the problem. It uses the notion of a border basis (see Definition 3.4), which is closely related to a result of Mourrain contained in the paper [5]. In this way we are able to give a full answer to the problem. This paper features five sections.

1) Introduction
2) Estimable Terms and Multiplication Matrices
3) Border Bases and Multiplication Matrices
4) An Inverse Problem in Design of experiments
5) Two Examples

In Section 2 we study the relationship between sets of estimable terms (see Definition 2.3) and multiplication matrices (see Definition 2.6). The main results are Proposition 2.5 and Theorem 2.9, which rephrase and broaden some results of [5] (see also [1]).

Section 3 introduces the notion of a marked polynomial, which is borrowed from the paper [8]. Then, as we said, we give the definition of a border basis (see Definition 3.4) and prove the main Theorem 3.7. Inspired by results in [5], it clarifies the relationship between the fact that suitable matrices are pairwise commuting and the fact that a certain set of polynomials is a border basis.

The theoretical background of the results contained in Sec-

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tions 2 and 3 is more extensively explained in [10].

In Section 4 we develop an algorithm which constructs the solution to the main problem stated at the beginning of the introduction. Our result is contained in the main Theorem 4.6.

Finally, Section 5 presents two examples fully computed with the help of CoCoA (see [2]).

Throughout the entire paper we adopt the conventions and terminology used in the book [4]. In particular P denotes the multivariate polynomial ring $K[x_1, \ldots, x_n]$ over a field K, and $\mathbb{T}^n$ the monoid of terms (power products) in the indeterminates $x_1, \ldots, x_n$.

Another important convention that we are using consistently is the following. Tuples of elements are denoted with calligraphic letters, while their underlying set is denoted with the corresponding roman character. For example, when $\mathcal{G} = (g_1, \ldots g_r)$, then automatically G denotes $\{g_1, \ldots g_r\}$.

The already quoted Example 2.4 is used as a running example to illustrate the various definitions and results.

2. ESTIMABLE TERMS AND MULTIPLICATION MATRICES

We start off with a zero-dimensional proper ideal I in P. We let $\pi : P \longrightarrow P/I$ be the canonical surjective map and $\mu = \dim_K(P/I)$. Then we let $\mathcal{O} = (h_1, \ldots, h_\mu)$ be a tuple of polynomials such that $\overline{\mathcal{O}} = (\overline{h}_1, \ldots, \overline{h}_\mu)$ is a basis of P/I as a K-vector space. We denote by $V(\mathcal{O})$ the K-vector subspace of P generated by the set of elements of $\mathcal{O}$. We consider three linear maps

$$\mathrm{NF}_{\mathcal{O},I} : P \longrightarrow V(\mathcal{O}) \qquad \vartheta : V(\mathcal{O}) \longrightarrow K^\mu$$

$$\mathrm{NFV}_{\mathcal{O},I} : P \longrightarrow K^\mu$$

so that $\mathrm{NF}_{\mathcal{O},I}(f) = \sum_{i=1}^{\mu} a_i h_i$, $\vartheta(\sum_{i=1}^{\mu} a_i h_i) = \sum_{i=1}^{\mu} a_i e_i$, $\mathrm{NFV}_{\mathcal{O},I} = \vartheta \circ \mathrm{NF}_{\mathcal{O},I}$, where $e_1, \ldots, e_\mu$ are the vectors of the canonical basis of K^μ and the elements a_i are defined by $\pi(f) = \sum_{i=1}^{\mu} a_i \overline{h}_i$.

Definition 2.1. *With this notation we give the following definitions*

a) *The polynomial* $\mathrm{NF}_{\mathcal{O},I}(f)$ *is called the* **normal form** *of f with respect to $(\mathcal{O}, I)$ and the linear map* $\mathrm{NF}_{\mathcal{O},I}$ *is called the* **normal form map** *with respect to* $(\mathcal{O}, I)$.

b) *The vector* $\mathrm{NFV}_{\mathcal{O},I}(f)$ *is called the* **normal form vector** *of f with respect to $(\mathcal{O}, I)$ and the linear map* $\mathrm{NFV}_{\mathcal{O},I}$ *is called the* **normal form vector map** *with respect to* $(\mathcal{O}, I)$.

c) *If there exists an algorithm which computes* $\mathrm{NF}_{\mathcal{O},I}(f)$ *and* $\mathrm{NFV}_{\mathcal{O},I}(f)$ *for every $f \in P$, we say that the map* $\mathrm{NF}_{\mathcal{O},I}$ *is an* **explicit normal form map**, *and* $\mathrm{NFV}_{\mathcal{O},I}$ *is an* **explicit normal form vector map**.

Remark 2.2. *It follows from the above definitions that P is the direct sum of the two vector subspaces I and $V(\mathcal{O})$.*

Gröbner basis theory tells us that a typical example of a set of polynomials O such that $\overline{O}$ is a a basis of P/I is given by $O = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$. In that case O has the important property defined below.

Definition 2.3. *Let O be a finite set of power products such that $t \in O$ together with $t'|t$ implies that $t' \in O$. Then O is called an* **order ideal of monomials** *or a* **standard set of power products** *or a* **complete set of estimable terms**.

However, there are complete sets of estimable terms which form a basis modulo an ideal I, but they are not the complement of $\mathrm{LT}_\sigma(I)$, whatever term ordering σ is chosen. To make it clear we recall the following example from [9].

Example 2.4. *Consider the following set*

$$\{(0,0),\ (0,-1),\ (1,0),\ (1,1),\ (-1,1)\}$$

of five points in $\mathbb{A}^2_{\mathbb{Q}}$, denote by $I \subset P = \mathbb{Q}[x,y]$ the vanishing ideal of this set, and let $\mathcal{O} = (1, x, y, x^2, y^2)$. The evaluation matrix of the elements of $\mathcal{O}$ at the points has determinant -4, hence $(\overline{1}, \overline{x}, \overline{y}, \overline{x}^2, \overline{y}^2)$ is a $\mathbb{Q}$-basis of P/I. Consequently, there is a K-linear surjective map $P \longrightarrow K^5$ which sends every polynomial f to the uniquely defined tuple $(a_1, a_2, a_3, a_4, a_5)$ such that the residue class of f in P/I is $a_1 + a_2\overline{x} + a_3\overline{y} + a_4\overline{x}^2 + a_5\overline{y}^2$. However, this map is not induced by a map of the type $\mathrm{NF}_{\sigma,I}$, since $\{1, x, y, x^2, y^2\}$ is not of the form $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma(I)$ for some term ordering σ. To prove this fact, consider $f = x^2 + xy - x - \frac{1}{2}y^2 - \frac{1}{2}y$. It is in I, since it vanishes at the five points. For any term ordering σ, we have $x^2 >_\sigma x$ and $y^2 >_\sigma y$. If $x >_\sigma y$, then $x^2 >_\sigma xy >_\sigma y^2$. If $y >_\sigma x$, then $y^2 >_\sigma xy >_\sigma x^2$. This means that there are only two possibilities for the leading term of f: either it is x^2 or y^2. In both cases we get that $\mathrm{LT}_\sigma(f) \in \mathrm{LT}_\sigma(I)$ is an element of O.

The following proposition provides a description of zero-dimensional proper ideals in terms of multiplication matrices.

Proposition 2.5. *Let $\mu \geq 1$, let $\varphi : P \longrightarrow K^\mu$ be a surjective K-linear map such that $I = \mathrm{Ker}(\varphi)$ is a proper ideal in P, and let $w = \varphi(1)$, viewed as a column vector.*

a) *The ideal I is zero-dimensional. Moreover, if we pick $\mathcal{O} = (h_1, \ldots, h_\mu)$ so that $\varphi(h_i) = e_i$ for $i = 1, \ldots, \mu$, then $\varphi = \mathrm{NFV}_{\mathcal{O},I}$.*

b) *There exists a unique n-tuple of pairwise commuting matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n$ in the ring $\mathrm{Mat}_\mu(K)$ such that $\varphi(x_i f) = \mathcal{M}_i \cdot \varphi(f)$ for all $f \in P$ and all $i = 1, \ldots, n$*

c) *We have $\varphi(f) = f(\mathcal{M}_1, \ldots, \mathcal{M}_n) \cdot w$ for all $f \in P$.*

Proof. To give a proof of a) it is sufficient to observe that $\dim_K(P/I) < \infty$, and then use the definition of $\mathrm{NFV}_{\mathcal{O},I}$. To prove b) let $h_1, \ldots, h_\mu$ be polynomials in P such that $\varphi(h_k) = e_k$ for $k = 1, \ldots, \mu$. We define $\mathcal{M}_i$ to be the matrix whose columns are the vectors $\varphi(x_i h_1), \ldots, \varphi(x_i h_\mu)$. To show that $\mathcal{M}_i$ is well-defined and unique, we pick other polynomials $h'_1, \ldots, h'_\mu$ such that $\varphi(h'_k) = e_k$ for $k = 1, \ldots, \mu$. Then $h_k - h'_k \in I$ yields $x_i(h_k - h'_k) \in I$. Hence we have $\varphi(x_i h_k) = \varphi(x_i h'_k)$ for all i, k. For $f \in P$, we have $\varphi(f) = \sum_{k=1}^{\mu} a_k e_k$. Therefore $f - \sum_{k=1}^{\mu} a_k h_k \in I$, and hence $x_i f - \sum_{k=1}^{\mu} a_k x_i h_k \in I$. It follows that

$$\varphi(x_i f) = \sum_{k=1}^{\mu} a_k \varphi(x_i h_k) = \sum_{k=1}^{\mu} a_k \mathcal{M}_i e_k = \mathcal{M}_i \varphi(f)$$

Now we prove that the matrices $\mathcal{M}_i$ are pairwise commuting. By what we have seen before

$$\mathcal{M}_i \cdot \mathcal{M}_j e_k = \varphi(x_i x_j h_k) = \varphi(x_j x_i h_k) = \mathcal{M}_j \cdot \mathcal{M}_i e_k$$

Consequently, $\mathcal{M}_i \cdot \mathcal{M}_j = \mathcal{M}_j \cdot \mathcal{M}_i$ for all i, j.

Finally we prove c). By the linearity of φ it suffices to prove the formula in the case of a power product. The formula is correct for a power product of degree 0, and we proceed by induction. Let $f = x_1^{a_1} \cdots x_n^{a_n}$ and without loss of generality we may assume that $a_1 > 0$. Then let $h = x_1^{a_1-1} \cdots x_n^{a_n}$. The inductive assumption and b) imply $\varphi(f) = \varphi(x_1 h) = \mathcal{M}_1 \varphi(h) = \mathcal{M}_1\, h(\mathcal{M}_1, \ldots, \mathcal{M}_n) \cdot w = \mathcal{M}_1 \mathcal{M}_1^{a_1-1} \cdots \mathcal{M}_n^{a_n} \cdot w = \mathcal{M}_1^{a_1} \cdots \mathcal{M}_n^{a_n} \cdot w = f(\mathcal{M}_1^{a_1} \cdots \mathcal{M}_n^{a_n}) \cdot w$. Now the proof is complete. □

Definition 2.6. *The matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n$ described before are called* **the multiplication matrices** *of φ.*

Corollary 2.7. *Let I be a zero-dimensional proper ideal in P, let $\mathcal{O} = (t_1, \ldots, t_\mu)$ be a tuple of polynomials such that $\overline{\mathcal{O}} = (\bar{t}_1, \ldots, \bar{t}_\mu)$ is a basis of P/I as a K-vector space, let $w = \mathrm{NFV}_{\mathcal{O},I}(1)$ viewed as a column vector, and let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be the multiplication matrices of $\mathrm{NFV}_{\mathcal{O},I}$.*

a) For $f \in P$ we get $\mathrm{NF}_{\mathcal{O},I}(f) = \mathcal{O} \cdot f(\mathcal{M}_1, \ldots, \mathcal{M}_n) \cdot w$, hence $\mathrm{NF}_{\mathcal{O},I}$ is an explicit normal form map with respect to $(\mathcal{O}, I)$.

b) For $f \in P$ we have $\mathrm{NFV}_{\mathcal{O},I}(f) = f(\mathcal{M}_1, \ldots, \mathcal{M}_n) \cdot w$, hence $\mathrm{NFV}_{\mathcal{O},I}$ is an explicit normal form vector map with respect to $(\mathcal{O}, I)$.

Proof. It suffices to prove b), which follows directly from Proposition 2.5.c applied to the map $\mathrm{NFV}_{\mathcal{O},I}$. We observe that for every given input f there is more than an algorithm which computes $\mathrm{NFV}_{\mathcal{O},I}(f)$, actually there is an explicit formula. □

Example 2.8. *We consider Example 2.4 again, and compute $\mathcal{M}_1$ and $\mathcal{M}_2$, which we conveniently call $\mathcal{M}_x$ and $\mathcal{M}_y$. It turns out that*

$$\mathcal{M}_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \mathcal{M}_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \end{pmatrix}$$

while $w = (1,0,0,0,0)^{\mathrm{tr}}$. It is immediate to check that $\mathcal{M}_x \mathcal{M}_y = \mathcal{M}_y \mathcal{M}_x$. We want to compute $\mathrm{NFV}_{\mathcal{O},I}(x^6 y^2)$. We may use the formula of Corollary 2.7 above and get

$$\mathrm{NFV}_{\mathcal{O},I}(x^6 y^2) = \mathcal{M}_x^6 \mathcal{M}_y^2 \cdot w = (0, 0, \frac{1}{2}, 0, \frac{1}{2})^{\mathrm{tr}}$$

Therefore $\mathrm{NF}_{\mathcal{O},I}(x^6 y^2) = \frac{1}{2} y + \frac{1}{2} y^2$.

The following theorem states a sort of converse to Proposition 2.5. It simply says that n pairwise commuting matrices determine a unique zero-dimensional ideal.

Theorem 2.9. *Let $\mu \geq 1$, let $w \in K^\mu$ be a non-zero vector, and let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be pairwise commuting matrices in $\mathrm{Mat}_\mu(K)$.*

a) There exists a unique map $\varphi : P \longrightarrow K^\mu$ with the following properties:
   - *a_1) $\varphi(1) = w$,*
   - *a_2) φ is K-linear,*
   - *a_3) $\varphi(x_i f) = \mathcal{M}_i \varphi(f)$ for all $f \in P$, $i = 1, \ldots, n$.*

b) The kernel of φ is a proper zero-dimensional ideal.

c) Moreover, if φ is surjective and we denote $\mathrm{Ker}(\varphi)$ by I, then
   - *c_1) For every $\mathcal{O} = (h_1, \ldots, h_\mu)$ such that $\varphi(h_i) = e_i$ for $i = 1, \ldots, \mu$, we have $\varphi = \mathrm{NFV}_{\mathcal{O},I}$.*
   - *c_2) The matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are the multiplication matrices of $\mathrm{NFV}_{\mathcal{O},I}$.*

Proof. Using property a_2), it suffices to show how φ is defined on terms. By a_1), it is defined on 1. By induction on the degree we may assume that it is uniquely defined on terms of degree at most $d-1$. Let t be a term of degree d. Either t is a term in a single indeterminate, say x_i, or at least two indeterminates, say x_i, x_j, have positive exponents. In the first case there a unique way of writing $t = x_i t'$ with $\deg(t') = d-1$. Property a_3) implies that $\varphi(t) = \mathcal{M}_i \varphi(t')$, and we are done. Let us examine the second case. We can write $t = x_i t' = x_j t''$ with terms t', t'' of degree $d-1$. Then there exists a term τ such that $t' = x_j \tau$ and $t'' = x_i \tau$. Therefore

$$\varphi(t) = \mathcal{M}_i \varphi(t') = \mathcal{M}_i \mathcal{M}_j \varphi(\tau) = \mathcal{M}_j \mathcal{M}_i \varphi(\tau) = \mathcal{M}_j \varphi(t'')$$

so that $\varphi(t)$ is well-defined and forced upon us by the assumptions. Thus we have proved existence and uniqueness.

We show that $\mathrm{Ker}(\varphi)$ is a zero-dimensional ideal. To see that it is an ideal, we let $f \in \mathrm{Ker}(\varphi)$, $g \in P$ and prove that $gf \in \mathrm{Ker}(\varphi)$. By linearity, we may assume that g is a term, and by induction we may assume that g is an indeterminate, say x_i. Then $\varphi(x_i f) = \mathcal{M}_i \varphi(f) = 0$. The ideal $\mathrm{Ker}(\varphi)$ is a proper ideal, since $\varphi(1) = w \neq 0$. It is zero-dimensional, because $P/\mathrm{Ker}(\varphi)$ is a finite-dimensional K-vector space.

Finally we prove c). We know that φ is surjective and $\mathrm{Ker}(\varphi)$ is an ideal, so that every $\mathcal{O}$ with the given property is such that $\overline{\mathcal{O}}$ is a basis of $P/\mathrm{Ker}(\varphi)$ as a K-vector space. This proves that $\varphi = \mathrm{NFV}_{\mathcal{O},I}$. Then we deduce from Proposition 2.5.b) that there exists an n-tuple of multiplication matrices, i.e. a unique n-tuple of pairwise commuting matrices $\mathcal{N}_1, \ldots, \mathcal{N}_n$ in the ring $\mathrm{Mat}_\mu(K)$, such that $\varphi(x_i f) = \mathcal{N}_i \cdot \varphi(f)$ for all $f \in P$ and all $i = 1, \ldots, n$. Since our matrices satisfy the same condition, we infer that $\mathcal{M}_i = \mathcal{N}_i$ for $i = 1, \ldots, n$, and the proof is complete. □

3. BORDER BASES AND MULTIPLICATION MATRICES

We are going to use some ideas developed in [5] to produce an extension of the notion of a Gröbner basis.

Let us recall a definition from [8].

DEFINITION 3.1. *A pair (g,t) is said to be a* **marked polynomial** *if g is a non-zero polynomial and $t \in \mathrm{Supp}(g)$. We also say that g is marked by t. Let $\mathcal{G} = (g_1,\dots,g_\nu)$ be a tuple of non-zero polynomials, $\mathcal{T} = (t_1,\dots,t_\nu)$ a tuple of power products. If $(g_1,t_1),\dots,(g_\nu,t_\nu)$ are marked polynomials, we say that $\mathcal{G}$ is* **marked by** *$\mathcal{T}$. The tuple $\mathcal{G}$ is said to be* **marked coherently** *by $\mathcal{T}$ if there exists a term ordering σ with the property that $t_i = \mathrm{LT}_\sigma(g_i)$ for every $i = 1,\dots,\nu$.*

In the case that $O = \{t_1,\dots,t_\mu\}$ is a complete set of estimable terms (see Definition 2.3) it is useful to define a related set of power products.

DEFINITION 3.2. *Let $O = \{t_1,\dots,t_\mu\}$ be a complete set of estimable terms. We define*

$$O^+ = \{t \in \mathbb{T}^n \setminus O \mid \exists\, i \in \{i,\dots,n\},\ t' \in O \text{ with } x_i t' = t\}$$

PROPOSITION 3.3. *Let I be a zero-dimensional proper ideal in P, let $O = \{t_1,\dots,t_\mu\}$ be a complete set of estimable terms such that $\overline{O} = (\bar{t}_1,\dots,\bar{t}_\mu)$ is a basis of P/I as a K-vector space, and let $O^+ = \{b_1,\dots,b_\nu\}$.*

a) For each $j = 1,\dots,\nu$ there exists a unique linear combination $\sum_{k=1}^{\mu} a_{kj} t_k$ with $a_{kj} \in K$ for $k \in \{1,\dots,\mu\}$, $j \in \{1,\dots,\nu\}$, and $g_j = b_j - \sum_{k=1}^{\mu} a_{kj} t_k \in I$.

b) The ideal I is generated by $\{g_1,\dots,g_\nu\}$.

PROOF. Remark 2.2 tells us that $P = V(O) \oplus I$, from which we deduce a). Let J be the ideal generated by $\{g_1,\dots,g_\nu\}$. To complete the proof we need to show that $J = I$. Since $J \subseteq I$ it suffices to show that $P = V(O) + J$, hence it suffices to show that $t \in V(O) + J$ for every term t. We know that $1 \in V(O)$, since O is a complete set of estimable terms, so we make induction on $\deg(t)$, and assume that the claim is proved up to degree $d-1$. Let t be a term of degree $d > 0$. There exists i such that $t = x_i t'$ for a suitable term t' of degree $d-1$. Then $t' = \sum_{i=1}^{\mu} a_i t_i + g$ with $a_i \in K$ for $i = 1,\dots,\mu$ and $g \in J$. Therefore $t = \sum_{i=1}^{\mu} a_i x_i t_i + x_i g$. Now, either $x_i t_i \in O$ or $x_i t_i \in O^+$. In the first case there is nothing to prove. If $x_i t_i \in O^+$, then $x_i t_i$ belongs to $V(O) + J$ by assumption, and the proof is complete. □

DEFINITION 3.4. *Following Proposition 3.3, we call G the set $\{g_1,\dots,g_\nu\}$ and let the elements of G be marked by the corresponding elements of O^+. Then the pair (G,O^+), is called the* **border basis** *of I with respect to O.*

EXAMPLE 3.5. *We continue the study of our running Example 2.4. In that case we have $O = \{1,\, x,\, y,\, x^2,\, y^2\}$. We get $O^+ = \{xy, x^3, y^3, x^2y, xy^2\}$ and we know already that $f_{xy} = xy - x - \frac{1}{2}y + x^2 - \frac{1}{2}y^2$, $f_{x^3} = x^3 - x$, $f_{y^3} = y^3 - y$. It is easy to check that $f_{x^2y} = x^2y - \frac{1}{2}y - \frac{1}{2}y^2$ and $f_{xy^2} = xy^2 - x - \frac{1}{2}y + x^2 - \frac{1}{2}y^2$. In conclusion, the set $F = \{f_{xy}, f_{x^3}, f_{y^3}, f_{x^2y}, f_{xy^2}\}$ with f_{xy} marked by xy, f_{x^3} by x^3, f_{y^3} by y^3, f_{x^2y} by x^2y, f_{xy^2} by xy^2, is the border basis of I with respect to O.*

We want to make an important connection between multiplication matrices and the notion of a border basis.

Let $O = \{t_1,\dots,t_\mu\}$ be a complete set of estimable terms and let $O^+ = \{b_1,\dots,b_\nu\}$. Let $G = \{g_1,\dots,g_\nu\}$ be polynomials marked by the corresponding elements of O^+ and such that $\mathrm{Supp}(g_k - b_k) \subseteq O$ for $k = 1,\dots,\nu$. We construct matrices $\mathcal{M}_1,\dots,\mathcal{M}_n \in \mathrm{Mat}_\mu(K)$ in the following way. If $x_k t_j \in O$, then the j-column of $\mathcal{M}_k$ is $(0,\dots,0,1,0,\dots,0)^{\mathrm{tr}}$, with 1 in the position corresponding to the power product $x_k t_j$. If $x_k t_j \in O^+$ there exists $i \in \{1,\dots,\nu\}$ such that $x_k t_j = b_i$, and in this case j-column of $\mathcal{M}_k$ contains the coefficients of $(t_1,\dots,t_\mu)$ in the representation of the polynomial $g_i - b_i$ as a linear combination of the elements in O.

DEFINITION 3.6. *The matrices $\mathcal{M}_1,\dots,\mathcal{M}_n$ constructed above are called* **the matrices associated to** *(G,O^+).*

THEOREM 3.7. *Let $\mathcal{O} = (t_1,\dots,t_\mu)$ be such that O is a complete set of estimable terms and let $O^+ = \{b_1,\dots,b_\nu\}$. Furthermore let $G = \{g_1,\dots,g_\nu\}$ be polynomials marked by the corresponding elements of O^+ so that $\mathrm{Supp}(g_k - b_k) \subseteq O$ for $k = 1,\dots,\nu$, and let I be the ideal generated by G. The following conditions are equivalent.*

i) The matrices $\mathcal{M}_1,\dots,\mathcal{M}_n$ associated to (G,O^+) are pairwise commuting.

ii) The pair (G,O^+) is the border basis of I with respect to O.

Moreover, if the two equivalent conditions are satisfied we have

a) The ideal I is zero-dimensional, and $\dim_K(P/I) = \mu$.

b) The tuple $\overline{O}$ is a basis of P/I as a K-vector space.

c) The matrices $\mathcal{M}_1,\dots,\mathcal{M}_n$ are the multiplication matrices of $\mathrm{NFV}_{\mathcal{O},I}$.

PROOF. The set O is a complete set of estimable terms, hence $1 \in O$ and we may assume that $t_1 = 1$. Let us prove that $i) \Rightarrow ii)$. We choose $w = e_1$, and if the matrices $\mathcal{M}_1,\dots,\mathcal{M}_n$ are pairwise commuting, we deduce from Theorem 2.9 that there exists a K-linear map $\varphi : P \longrightarrow K^\mu$ such that $\varphi(1) = e_1$ and $\varphi(x_i f) = \mathcal{M}_i \varphi(f)$. We claim that $\varphi(t_i) = e_i$ for $i = 1,\dots,\mu$. The claim is true for $i = 1$ by construction, and we make induction on the degree. So let $t_i \neq 1$. It is possible to write $t_i = x_k t_j$ for a suitable k and a term $t_j \in O$. Then $\varphi(t_i) = \mathcal{M}_k \varphi(t_j) = \mathcal{M}_k e_j$ where $e_j = (0,\dots,0,1,0,\dots,0)^{\mathrm{tr}}$, with 1 in position j. Therefore $\varphi(t_i)$ is the j column of $\mathcal{M}_k$, which is e_i by construction, and the claim is proved. Therefore $\overline{O}$ is a basis of P/I. We use Proposition 3.3 to see that (G,O^+) is the unique border basis of I with respect to O.

Now we prove that $ii) \Rightarrow i)$. If the pair (G,O^+) is the border basis of I with respect to O, then $\overline{O}$ is a basis of P/I, hence we may define $\mathrm{NFV}_{O,I} : P \longrightarrow K^\nu$ with the property that $\mathrm{NFV}_{O,I}(1) = e_1$. We apply Proposition 2.5 to this map and get a unique n-tuple of pairwise commuting matrices $\mathcal{N}_1,\dots,\mathcal{N}_n$ in the ring $\mathrm{Mat}_\mu(K)$, such that $\mathrm{NFV}_{O,I}(x_k f) = \mathcal{N}_k \cdot \mathrm{NFV}_{O,I}(f)$ for all $f \in P$ and all $k = 1,\dots,n$. Therefore $\mathrm{NFV}_{O,I}(x_k t_j) = \mathcal{N}_k \cdot \mathrm{NFV}_{O,I}(t_j) = \mathcal{N}_k \cdot e_j$, which is the j^{th} column of $\mathcal{N}_k$, and hence $\mathcal{N}_k = \mathcal{M}_k$ for every $k = 1,\dots,n$.

The proof of a), b), c) is already included in the arguments used to prove the equivalence of i) and ii). □

We know from Corollary 2.7 that the existence of the commuting matrices yields an algorithm, actually a formula, for the computation of $\mathrm{NF}_{(O,I)}(f)$. Can we achieve the same results by using the border basis (G, O^+) as a set of "rewriting rules"? We recall the fundamental theorem of [8]

Theorem 3.8. *A tuple $\mathcal{G}$ of marked polynomials is marked coherently if and only if every sequence of reduction steps associated to $\mathcal{G}$ terminates.*

We deduce from it that in the case of a non coherently marked border basis, there exist a polynomial f and an infinite sequence of reduction steps which starts at f. If we look at the multiplication matrices, we see that they encode the way to express the normal form of the product of a normal form by an indeterminate. This implies that in the case of a non coherently marked border basis F, we can use the polynomials in F as rewrite rules and obtain the desired normal form of f, but we must be careful and use them *only for rewriting elements in O^+*. To understand this remark better, let us see the following example. To learn more on that see [10].

Example 3.9. *We go back to Example 2.8 and compute $\mathrm{NF}_{\mathcal{O},I}(x^6y^2)$ again, but this time we use the border basis F computed in Example 3.5. We proceed as follows. We write $x^6y^2 = x^3y^2 \cdot x^3$, we use the marked polynomial f_{x^3} to make a reduction step and get x^4y^2 which we write $x^4y^2 = xy^2 \cdot x^3$. We use f_{x^3} again and get $x^2y^2 = xy \cdot xy$. We use f_{xy} and get $xy(x+\frac{1}{2}y-x^2+\frac{1}{2}y^2) = x^2y+\frac{1}{2}xy^2-x^3y+\frac{1}{2}xy^3$. We use f_{x^3} and then f_{y^3} to get $x^2y+\frac{1}{2}xy^2-xy+\frac{1}{2}xy = -\frac{1}{2}xy+x^2y+\frac{1}{2}xy^2$. Finally we use f_{xy}, f_{x^2y}, f_{xy^2} and get $-\frac{1}{2}(-x^2+\frac{1}{2}y^2+x+\frac{1}{2}y)+\frac{1}{2}y^2+\frac{1}{2}y+\frac{1}{2}(x^2+\frac{1}{2}y^2+x+\frac{1}{2}y) = \frac{1}{2}y+\frac{1}{2}y^2$. Therefore $\mathrm{NF}_{\mathcal{O},I}(x^6y^2) = \frac{1}{2}y+\frac{1}{2}y^2$, which agrees with the value computed in Example 2.8.*

We could follow different paths, but suppose we try to reduce x^2y^2 in the following way. We reduce xy to $x+\frac{1}{2}y-x^2+\frac{1}{2}y^2$, hence we reduce x^2y^2 to $(x+\frac{1}{2}y-x^2+\frac{1}{2}y^2)^2$. This is equal to $x^4-x^2y^2+\frac{1}{4}y^4-2x^3-x^2y+xy^2+\frac{1}{2}y^3+x^2+xy+\frac{1}{4}y^2$, and we see the power product x^2y^2. Now it is easy to proceed with a strategy of rewriting that produces x^2y^2 periodically, and hence does not terminate.

4. AN INVERSE PROBLEM IN DESIGN OF EXPERIMENTS

We start this section by recalling some basic definitions in the theory of DoE.

Definition 4.1. *Let $E_1, E_2, \dots, E_n$ be finite subsets of K. Then the Cartesian product $\mathcal{D} = E_1 \times E_2 \times \cdots \times E_n \subset K^n$ is a finite set of points called a* **full factorial design**. *If moreover $\#(E_i) = l_i$ for $i = 1, \dots, n$, then we say that $E_1 \times E_2 \times \cdots \times E_n$ has* **levels** *$l_1, l_2, \dots, l_n$. An l^n-design is a full factorial design with $l_1 = l_2 = \cdots = l_n = l$.*

Let $E_1 = \{a_{11}, \dots a_{1\,l_1}\}, \dots, E_n = \{a_{n1}, \dots a_{n\,l_n}\}$ and let $\mathcal{D} = E_1 \times \cdots \times E_n$. Then let $f_k = \prod_{i=1}^{l_k}(x_k - a_{ki})$ for every $k = 1, \dots, n$. Then the defining ideal of $\mathcal{D}$ viewed as as an affine subvariety of $\mathbb{A}_n(K)$ is $I(\mathcal{D}) = (f_1, \dots, f_n)$. Moreover, the set

$$O(\mathcal{D}) = \{x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \mid 0 \le a_k \le l_k \text{ for } k = 1, \dots, n\}$$

is a complete set of estimable terms such that $\overline{O(\mathcal{D})}$ is a basis of $P/I(\mathcal{D})$ as a K-vector space.

Definition 4.2. *The n polynomials $f_1, \dots, f_n$ are called* **the canonical polynomials** *of $\mathcal{D}$. The set $O(D)$ is called* **the complete set of estimable terms** *of $\mathcal{D}$.*

Definition 4.3. *Let $\mathcal{D}$ be a full factorial design and let $\mathcal{F} \subset \mathcal{D}$. It is called a* **fractional factorial design** *or a* **fraction** *of $\mathcal{D}$. Its defining ideal is called $I(\mathcal{F})$.*

Now we are ready to state a main problem in DoE.

Problem 4.4. *Let $\mathcal{D}$ be a full factorial design, and let $O \subset O(\mathcal{D})$ be a complete set of estimable terms. What are the fractions $\mathcal{F}$ of $\mathcal{D}$ such that $\overline{O}$ is a basis of $P/I(\mathcal{F})$ as a K-vector space?*

Before we start to investigate the problem, we need a technical result which is a slight modification of Lemma 3.2. from [3], and whose proof is included here for the sake of completeness.

Lemma 4.5. *Let $\mathcal{D}$ be a full factorial design and denote by $I(\mathcal{D}) = (f_1, \dots, f_n)$, where $f_1, \dots, f_n$ are its canonical polynomials. Denote by $\overline{K}$ the algebraic closure of K and let I be a proper ideal of $\overline{K}[X]$ such that $I(\mathcal{D}) \subset I$. Then*

- *a) I is a radical ideal, which defines a Fraction of $\mathcal{D}$.*
- *b) I can be defined by polynomials in $K[X]$.*
- *c) The polynomials of every border basis of I are elements of $K[X]$.*

Proof. Let $\mathbb{A}^n_{\overline{K}}$ be the affine space of dimension n over $\overline{K}$. Let $\mathcal{F} \subset \mathbb{A}^n_{\overline{K}}$ be the set of zeroes of I. From the fact that $I(\mathcal{D}) \subset I$ it follows that $\mathcal{F} \subseteq \mathcal{D}$. By localizing the ring $A = \overline{K}[X]/I$ at the maximal ideals M corresponding to the points of $\mathcal{D}$, we have that either $IA_M = (1)$ or $IA_M = MA_M$. Therefore I is a radical ideal, hence it is the defining ideal of $\mathcal{F}$.

Finally, condition $c)$ follows from Proposition 3.3. $\square$

We observe that $O = \{t_1, \dots, t_\mu\}$ is given, which implies that $O^+ = \{b_1, \dots, b_\nu\}$ is also given. We know that O is a complete set of estimable terms, hence $1 \in O$, and we may assume that $t_1 = 1$. Moreover, $\mathcal{D}$ is fixed, which implies that its canonical polynomials $f_1, \dots, f_n$ are given. A fraction $\mathcal{F}$ is a solution of Problem 4.4 if and only if $\overline{O}$ is a basis of $P/I(\mathcal{F})$ and $f_i \in I(\mathcal{F})$ for $i = 1, \dots, n$. Proposition 3.3 shows that, given O, ideals I such that $\overline{O}$ is a basis of P/I are in one to one correspondence to border bases $(g_1, \dots, g_\nu)$ marked by $(b_1, \dots, b_\nu)$. This fact suggests the following algorithm

THEOREM 4.6. *Let $\mathcal{D}$ be a full factorial design with levels $l_1, l_2, \dots, l_n$, let $f_1, \dots, f_n$ be the canonical polynomials of $\mathcal{D}$, and let $O \subset O(\mathcal{D})$ be a complete set of estimable terms. Consider the following sequence of instructions.*

1) *Subdivide the set O^+ into two subsets, O_1^+, O_2^+, where $O_1^+ = O^+ \cap \{x_1^{l_1}, \dots, x_n^{l_n}\}$, $O_2^+ = O^+ \setminus O_1^+$.*
2) *Subdivide the set $F = \{f_1, \dots, f_n\}$ into two subsets, F_1, F_2, where $F_1 = \{f_i \in F \mid x_i^{l_i} \in O^+\}$, $F_2 = F \backslash F_1$, and f_i is marked by $x_i^{l_i}$ for every $f_i \in F_1$.*
3) *Let $\eta = \#(O_2^+)$, and for every power product $b_k \in O_2^+$ let $g_k = b_k + \sum_{j=1}^{\mu} a_{kj} t_j$ be polynomials in the ring $K[A][x_1, \dots, x_n]$, where A is a list of $\mu \times \eta$ independent new indeterminates a_{kj}. Mark g_k by b_k for $k = 1, \dots, \eta$, and call G_2 the set of these marked polynomials.*
4) *Let $G = F_1 \cup G_2$ and construct $\mathcal{M}_1, \dots, \mathcal{M}_n$, the matrices associated to (G, O^+).*
5) *Impose that $\mathcal{M}_1, \dots, \mathcal{M}_n$ are pairwise commuting.*
6) *Let $w = (1, 0 \dots, 0)^{\mathrm{tr}}$ and impose that the equations $f_k(\mathcal{M}_1, \dots, \mathcal{M}_n) \cdot w = (0, 0, \dots, 0)^{\mathrm{tr}}$ hold true for every $f_k \in F_2$.*
7) *Construct the ideal $\mathcal{I}(O)$ in $K[A]$ generated by the polynomials arising from Steps 5) and 6).*

This is an algorithm which returns an ideal $\mathcal{I}(O)$ in $K[A]$ with the following properties

a) *The ideal $\mathcal{I}(O)$ is zero-dimensional and defined over the field K.*
b) *The zeros of $\mathcal{I}(O)$ are in one-to-one correspondence with the solutions of Problem 4.4*

PROOF. Let $N = \mu \times \eta$. By considering Step 5), we see that every zero of $\mathcal{I}(O)$ in $\mathbb{A}_{\overline{K}}^N$ corresponds to an n-tuple of commuting matrices $\mathcal{M}_1, \dots, \mathcal{M}_n$. By Theorem 3.7 we deduce that it corresponds to the unique border basis of an ideal I with respect to O. By considering Step 6), we see that $I(\mathcal{D}) \subset I$. Now Lemma 4.5.c implies that the polynomials of a border basis of I are in P, and hence we have proved that all the zeros of $\mathcal{I}(O)$ are in $\mathbb{A}_K^N$, i.e. that $\mathcal{I}(O)$ is defined over K. We use Lemma 4.5.a to see that I is radical, hence it is the ideal of a finite subset of points of $\mathcal{D}$. But there is only a finite number of such subsets, which proves that $I(\mathcal{D}) \subset I$ is zero-dimensional. This finishes the proof of a). But the same argument shows that there is a one-to-one correspondence between the zeros of $\mathcal{I}(O)$ and the fractions $\mathcal{F}$ of $\mathcal{D}$ such that $\overline{O}$ is a basis of $P/I(\mathcal{F})$. This proves b), and the proof is complete. □

REMARK 4.7. *It should be noted that the set of solutions of Problem 4.4 is always non-empty, since it contains the so-called distracted fractions (see [11]).*

REMARK 4.8. *If we are not satisfied with the representation of a fraction $\mathcal{F}$ via its border basis, we may perform the extra step of solving the corresponding system of polynomial equations and actually find the points $p_1, \dots, p_\mu \in \mathbb{A}_K^n$ such that $\mathcal{F} = \{p_1, \dots, p_\mu\}$.*

5. TWO EXAMPLES

5.1 Example 1

Let $\mathcal{D} \subset \mathbb{Q}^2$ be the full factorial design whose canonical polynomials are $f_1 = x^3 - x$ and $f_2 = y^2 - 1$. Then we have $F = \{f_1, f_2\}$ and $O(\mathcal{D}) = \{1, x, y, x^2, xy, x^2y\}$. The set $O = \{1, x, y\} \subset O(\mathcal{D})$ is a complete set of estimable terms.

We apply the algorithm described in Theorem 4.6 to this example. We subdivide F in the two subsets $F_1 = \{y^2 - 1\}$ and $F_2 = \{x^3 - x\}$, and mark $y^2 - 1$ by y^2. We subdivide the set $O^+ = \{x^2, xy, y^2\}$ in the two subsets $O_1^+ = \{y^2\}$ and $O_2^+ = \{x^2, xy\}$. We introduce new independent indeterminates $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$, and construct the polynomials $g_1 = x^2 + a_{11} + a_{12}x + a_{13}y$, $g_2 = xy + a_{21} + a_{22}x + a_{23}y$ in $K[A][x, y]$ and we mark them by x^2 and xy respectively. We obtain $G = \{f_2, g_1, g_2\}$.

The matrices associated to (G, O^+) are

$$\mathcal{M}_x = \begin{pmatrix} 0 & -a_{11} & -a_{21} \\ 1 & -a_{12} & -a_{22} \\ 0 & -a_{13} & -a_{23} \end{pmatrix} \qquad \mathcal{M}_y = \begin{pmatrix} 0 & -a_{21} & 1 \\ 0 & -a_{22} & 0 \\ 1 & -a_{23} & 0 \end{pmatrix}$$

We impose that $\mathcal{M}_x$ and $\mathcal{M}_y$ commute by imposing that the matrix $\mathcal{M}x\mathcal{M}y - \mathcal{M}y\mathcal{M}x$ is the zero matrix, that is

$$\begin{pmatrix} 0 & a_{12}a_{21} - a_{11}a_{22} - a_{21}a_{23} - a_{13} & a_{2,1}a_{22} - a_{23} \\ 0 & a_{22}a_{23} - a_{21} & -a_{22}^2 + 1 \\ 0 & a_{13}a_{22} - a_{12}a_{23} + a_{23}^2 + a_{11} & -a_{22}a_{23} + a_{21} \end{pmatrix} \quad (1)$$

is the zero matrix.

We impose that $(\mathcal{M}_x^3 - \mathcal{M}_x) \cdot (1, 0, \dots, 0)^{\mathrm{tr}} = 0$ and get the following polynomial equations

$$\begin{aligned} & a_{11}a_{12} + a_{13}a_{21} = 0 \\ & a_{12}^2 + a_{13}a_{22} - a_{11} - 1 = 0 \\ & a_{12}a_{13} + a_{13}a_{23} = 0 \end{aligned} \qquad (2)$$

We let $\mathcal{I}(O)$ be the ideal in $K[A]$ generated by the entries of the matrix (1) and the polynomials in (2). A small computation carried on with CoCoA shows that $\mathcal{I}(O)$ is a zero-dimensional, radical ideal and its multiplicity is 18.

In conclusion, out of the $20 = \binom{6}{3}$ triples of points in $\mathcal{D}$, there are 18 triples which solve Problem 4.4.

5.2 Example 2

Let $\mathcal{D} \subset \mathbb{Q}^2$ be the full 3^2 factorial design whose canonical polynomials are $f_1 = x^3 - x$ and $f_2 = y^3 - y$. In this case $F = \{f_1, f_2\}$ and $O(\mathcal{D}) = \{1, x, y, x^2, xy, y^2, x^2y, xy^2\}$. Let $O = \{1, x, y, x^2, y^2\} \subset O(\mathcal{D})$. It is a complete set of estimable terms.

We get $O^+ = \{x^3, y^3, xy, x^2y, xy^2\}$ and we easily check that $F_1 = F = \{f_1, f_2\}$ and $F_2 = \emptyset$; we mark (f_1, f_2) by (x^3, y^3). We have $O_1^+ = \{x^3, y^3\}$ and $O_2^+ = \{xy, x^2y, xy^2\}$. We introduce new independent indeterminates

$$a_{11}, \dots, a_{15},\ a_{21}, \dots, a_{25},\ a_{31}, \dots, a_{35}$$

and construct the polynomials

$$\begin{aligned} g_1 &= xy \; + a_{11} + a_{12}x + a_{13}y + a_{14}x^2 + a_{15}y^2 \\ g_2 &= x^2y + a_{21} + a_{22}x + a_{23}y + a_{24}x^2 + a_{25}y^2 \\ g_3 &= xy^2 + a_{31} + a_{32}x + a_{33}y + a_{34}x^2 + a_{35}y^2 \end{aligned}$$

in $K[A][x,y]$. We mark them by xy, x^2y and xy^2 respectively. We obtain $G = F_1 \cup G = \{f_1, f_2, g_1, g_2, g_3\}$.

The matrices associated to (G, O^+) are

$$\mathcal{M}_x = \begin{pmatrix} 0 & 0 & -a_{11} & 0 & -a_{31} \\ 1 & 0 & -a_{12} & 1 & -a_{32} \\ 0 & 0 & -a_{13} & 0 & -a_{33} \\ 0 & 1 & -a_{14} & 0 & -a_{34} \\ 0 & 0 & -a_{15} & 0 & -a_{35} \end{pmatrix} \quad \mathcal{M}_y = \begin{pmatrix} 0 & -a_{11} & 0 & -a_{21} & 0 \\ 0 & -a_{12} & 0 & -a_{22} & 0 \\ 1 & -a_{13} & 0 & -a_{23} & 1 \\ 0 & -a_{14} & 0 & -a_{24} & 0 \\ 0 & -a_{15} & 1 & -a_{25} & 0 \end{pmatrix}$$

We impose that $\mathcal{M}_x$ and $\mathcal{M}_y$ commute by imposing that the matrix $\mathcal{M}x\mathcal{M}y - \mathcal{M}y\mathcal{M}x$ is the zero matrix, and get

$$\begin{array}{l} a_{11}a_{13} + a_{15}a_{31} + a_{21} \\ a_{11}a_{12} + a_{14}a_{21} + a_{31} \\ a_{11}a_{23} + a_{25}a_{31} + a_{11} \\ a_{11}a_{32} + a_{21}a_{34} + a_{11} \\ a_{12}a_{13} + a_{15}a_{32} - a_{11} - a_{14} + a_{22} \\ a_{12}^2 + a_{14}a_{22} + a_{32} \\ a_{12}a_{23} + a_{25}a_{32} + a_{12} - a_{21} - a_{24} \\ a_{12}a_{32} + a_{22}a_{34} + a_{12} \\ a_{13}^2 + a_{15}a_{33} + a_{23} \\ a_{12}a_{13} + a_{14}a_{23} - a_{11} - a_{15} + a_{33} \\ a_{13}a_{23} + a_{25}a_{33} + a_{13} \\ a_{13}a_{32} + a_{23}a_{34} + a_{13} - a_{31} - a_{35} \\ a_{13}a_{14} + a_{15}a_{34} - a_{12} + a_{24} \\ a_{12}a_{14} + a_{14}a_{24} + a_{34} \\ a_{14}a_{23} + a_{25}a_{34} + a_{14} - a_{22} \\ a_{14}a_{32} + a_{24}a_{34} + a_{14} \\ a_{13}a_{15} + a_{15}a_{35} + a_{25} \\ a_{12}a_{15} + a_{14}a_{25} - a_{13} + a_{35} \\ a_{15}a_{23} + a_{25}a_{35} + a_{15} \\ a_{15}a_{32} + a_{25}a_{34} + a_{15} - a_{33} \end{array} \qquad (3)$$

There is no Step 6) to be performed since $F_2 = \emptyset$. Moreover $\mathcal{I}(O)$ is generated by the polynomials (3). A computation carried on with CoCoA shows that $\mathcal{I}(O)$ is zero-dimensional, radical and its multiplicity is 81.

In conclusion, out of the $126 = \binom{9}{5}$ five-tuples of points in $\mathcal{D}$, there are 81 five-tuples which solve Problem 4.4.

One of the solutions of $\mathcal{I}(O)$ is the point $p \in \mathbb{Q}^{15}$ whose coordinates are

$$\begin{array}{lllll} a_{11} = 0 & a_{12} = -1 & a_{13} = -\frac{1}{2} & a_{14} = 1 & a_{15} = -\frac{1}{2} \\ a_{21} = 0 & a_{22} = \;\; 0 & a_{23} = -\frac{1}{2} & a_{24} = 0 & a_{25} = -\frac{1}{2} \\ a_{31} = 0 & a_{32} = -1 & a_{33} = -\frac{1}{2} & a_{34} = 1 & a_{35} = -\frac{1}{2} \end{array}$$

By substituting those values in $G \subset \mathbb{Q}(A)[x,y]$ we have the border basis $G_p \subset \mathbb{Q}[x,y]$ whose elements are

$$\begin{array}{l} x^3 - x, \\ y^3 - y, \\ xy \;\; - x - \frac{1}{2}y + x^2 - \frac{1}{2}y^2, \\ x^2y - \frac{1}{2}y - \frac{1}{2}y^2, \\ xy^2 - x - \frac{1}{2}y + x^2 - \frac{1}{2}y^2 \end{array}$$

marked by x^3, y^3, xy, x^2y, xy^2 respectively.

The fraction $\mathcal{F}_p$ of $\mathcal{D}$ defined by G_p is

$$\{(0,0), (0,-1), (1,0), (1,1), (-1,1)\}$$

We have already met this fraction in Example 2.4.

5.3 A Final Remark

A natural question is how many of the 81 fractions we have found in Example 5.2 can be obtained via Gröbner basis techniques, as explained in [3]. We have seen in Example 2.4 that at least $\mathcal{F}_p$ cannot be found with such methods.

Indeed, reasoning as in Example 2.4, it is easy to see that the fractions which correspond to solutions of $\mathcal{I}(O)$ and have the property that a_{14}, a_{15} are both different from zero, cannot be found with Gröbner basis techniques. The number of such solutions is 36.

All other fractions can be found with Gröbner basis methods. Let us show how by examining $\mathcal{G} = (f_1, f_2, g_1, g_2, g_3)$. The fact that $\mathcal{G}$ is not marked coherently by the tuple $(x^3, y^3, xy, x^2y, xy^2)$ is an obstruction to the applicability of Gröbner methods. This obstruction can be removed if the coefficient of either x^2 or y^2 in g_1 is zero. If any of these conditions is met, then $\mathcal{G}$ is marked coherently by $(x^3, y^3, xy, x^2y, xy^2)$, the hypothesis of Corollary 2.14 of [3] hold and we can thus proceed with the Gröbner methods explained there.

If we do that we find the remaining 45 solutions.

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