Informal Notes (July 1997)

How to define singular solutions of a differential equation

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1. Classical views

The general solution of an ordinary differential equation of first order

$$\phi\,(x, y, y') = 0,$$

where ϕ is analytic, is usually understood as a solution depending on an arbitrary constant. Besides, there are at least three definitions which have been used in classical books to define a singular solution of $\phi\,(x, y, y') = 0$:

1 (Murphy 1960) a solution of $\phi\,(x, y, y') = 0$ that can not be obtained by assigning a numerical value to the arbitrary constant in the general solution. (Murphy 1960)

2 () an envelope of the family of curves given by the general solution of the differential equation.

3 (Ince 1956) a solution consisting of singular points, that is, a common solution to

$$\phi\,(x, y, y') = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y'}\,(x, y, y') = 0$$

These definitions are more or less easily lifted for higher order ordinary differential equations.

We are going to study a bunch of examples, where ϕ is polynomial, to show that these definitions are not equivalent.

Note that for first order differential equations, if the family of curves given by the general solution has an envelope, Definition 2 of a singular solution, then this envelope is a singular solution according to Definition 3. Indeed, at all of its point, $\phi(x, y, y')$ has a multiple root in y'. The converse is not true as shown in the following example.

EXAMPLE: Consider the differential equation

$$\phi(x, y, y') \equiv {y'}^2 - y^3 = 0.$$

According to the third definition, if there is any singular solution it satisfies

$$\phi(x, y, y') = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y'}(x, y, y') \equiv 2\,y' = 0$$

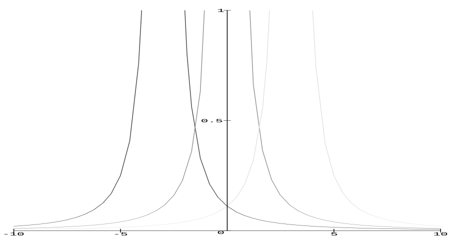
and therefore $y = 0$. $y(x) \equiv 0$ is in fact a solution that is singular according to Definition 3.

The general solution can be given by

$$\tilde{y}(x) = \frac{4}{(x-a)^2} \quad \text{where } a \text{ is an arbitrary constant .}$$

These solutions are represented in Figure 1.

Figure 1. ${y'}^2 - y^3 = 0$



We see that $y(x) \equiv 0$ is not a envelope and therefore not a singular solution according to Definition 2.

It is not either a singular solution according to Definition 1 if we consider that a can be made infinite.

The next example illustrates that an envelope is not always a singular solution according to the Definition 1.

EXAMPLE: Consider the differential equation

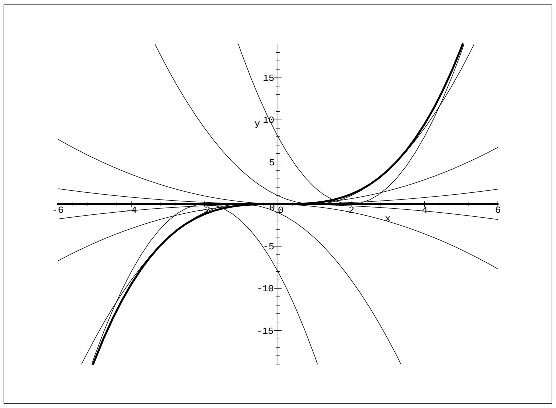
$$\phi(x, y, y') \equiv {y'}^3 - 4\,x\,y\,y' + 8\,y^2 = 0.$$

The general solution can be given by

$$\tilde{y}(x) = a\,(x-a)^2 \text{ where } a \text{ is an arbitrary constant .}$$

They are represented in Figure 2.

Figure 2. $y'^3 - 4\,x\,y\,y' + 8\,y^2 = 0$



The associated family of curves has two envelopes

$$\bar{y}(x) \equiv 0 \quad \text{and} \quad \hat{y}(x) = \frac{4}{27}x^3.$$

Both are singular solutions according to Definition 2 and therefore according to Definition 3. Nonetheless, $\bar{y}(x) \equiv 0$ can be obtained from the general solution for $a = 0$ and is therefore not a singular solution according to Definition 3.

One might still think that the singular solutions of Definition 1 is a subset of the singular solutions of Definition 3, when one is allowed to take infinite arbitrary constant. It is sort of true, but Definition 1 is difficult - if not impossible - to lift to higher order differential equation.

Example: Consider the differential equation

$$\phi(x, y, y') \equiv y'' - 3\,y\,y' = 0.$$

This equation has no singular solution according to Definition 3.

It nonetheless admits the two solutions

$$\tilde{y}(x) = a\,\tan(a\,x + b) \quad \text{and} \quad \hat{y}(x) = c$$

where a, b, c are arbitrary constants. and one is tempted to say that $\hat{y}(x) = c$ is a singular solution according to the first definition.

But consider the following fact. Given c and a point x_0, chose $\bar{a}$ and $\bar{b}$ in $\mathbf{C}$ such that

$$\left\{\begin{array}{rcl} \bar{a}\,\tan(\bar{a}\,x_0+\bar{b}) &=& c \\ 1+\tan(\bar{a}\,x_0+\bar{b}) &=& 0. \end{array}\right.$$

You can take for instance $\bar{a} = i\,c$ and $\bar{b} = i\,c\,x_0 + \arctan(-i)$, where $\sqrt{-1} = \pm i$. You can then check that the coefficient of the Taylor series of $\bar{a}\,\tan(\bar{a}x+\bar{b})$ at $x = x_0$ all vanish except for the constant term c. Therefore, in the neighbouhood of $x = x_0$, the solution $\bar{a}\,\tan(\bar{a}x+\bar{b})$ is very (exponentially) closed to the solution $\hat{y}(x) = c$. In that sense $\hat{y}(x) = c$ is a limiting case of $\tilde{y}(x) = a\,\tan(a\,x+b)$.

2. Algebraic standpoint

General, singular and essential singular solutions can be defined in terms of differential ideals. Instead going through these definitions, we give here the interpretation that can be made out of it.

The singular solutions of the differential equations

$$p(x, y, y', \ldots, y^{(k)}) = 0,$$

where p is a polynomial are the solutions that also satisfy

$$\frac{\partial p}{\partial y^{(k)}}(x, y, y', \ldots, y^{(k)}) = 0.$$

This amounts thus to Definition 1 of the previous section.

The non-singular solutions are the ones that do not make vanish this second equation. They are naturally part of the general solution.

For a first order differential equation, a singular solution belongs to the general solution if it satisfies all the algebraic differential equations that are satisfied by the non-singular solutions. The singular solutions that do not belong to the general solution are *essential*.

Example: Consider the differential equation

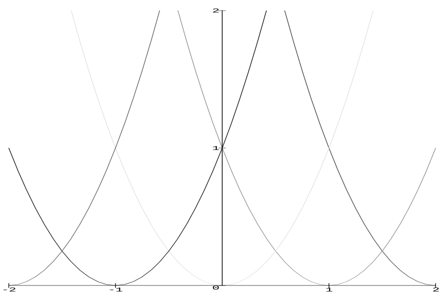
$$\phi(x, y, y') \equiv {y'}^2 - 4\,y = 0.$$

If there is any singular solution, it satisfies $\frac{\partial\phi}{\partial y'} \equiv 2y' = 0$. We can can thus check that $y(x) \equiv 0$ is the only singular solution.

The general solution can be given as $y(x) = (x-a)^2$. All these solutions are non singular.

The point here is that the differential equation $y'' - 2 = 0$ is satisfied by the general solution $y(x) = (x-a)^2$ but not by the singular solution $y(x) = 0$. This latter singular solution therefore does not belong to the general solution. We say it is an essential singular solution. You can check on Figure 3 that it is a envelope of the curves given by the general solution.

Figure 3. $y'^2 - 4\,y = 0$



EXERCISE: Show that the singular solution of $y'^2 - y^3 = 0$ belongs to the general solution of the differential equation.

Ritt showed that the singular solutions that belong to the general solution can be approximated at any order by non-singular solutions. This was illustrated in the last example of previous section. In that respect, the singular solutions of the intuitive Definition 1, of previous section, correspond to the essential singular solutions defined here.

As for essential singular solutions of first order differential equations, Hambuger(Hamburger 1893) proved that they are envelopes of non-singular solutions.

Higher order differential equations still need more investigations about enveloping properties of their singular solutions. Still, the general solution of a differential equation of any order consists of the non-singular solutions and their limits, in a sense made precise by Ritt† and illustrated in the last example of previous section. The essential singular solutions are the solutions that can not be obtained as a limit of another solutions.

The Rosenfeld-Gröbner algorithm‡ provides the proper reduction process to determine the existence and the equations of the singular solutions. The algorithm is implemented in the *Diffalg* library for Maple V. This latter algorithm is extended by the algorithm in (Hubert 1997) to determine the essential singular solutions.

† see the definition of *adherence* in (Ritt 1950)
‡ (Boulier 1994)

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