XIDEAL
Gröbner Bases for Exterior Algebra

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1 Description

The method of Gröbner bases in commutative polynomial rings introduced by Buchberger (e.g. [1]) is a well-known and very important tool in polynomial ideal theory, for example in solving the ideal membership problem. XIDEAL extends the method to exterior algebras using algorithms from [2].

There are two main departures from the commutative polynomial case. First, owing to the non-commutative product in exterior algebras, ideals are no longer automatically two-sided, and it is necessary to distinguish between left and right ideals. Secondly, because there are zero divisors, confluent reduction relations are no longer sufficient to solve the ideal membership problem: a unique normal form for every polynomial does not guarantee that all elements in the ideal reduce to zero. This leads to two possible definitions of Gröbner bases as pointed out by Stokes [3].

XIDEAL constructs Gröbner bases for solving the left ideal membership problem: Gröbner left ideal bases or GLIBs. For graded ideals, where each form is homogeneous in degree, the distinction between left and right ideals

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vanishes. Furthermore, if the generating forms are all homogeneous, then the Gröbner bases for the non-graded and graded ideals are identical. In this case, XIDEAL is able to save time by truncating the Gröbner basis at some maximum degree if desired.

XIDEAL uses the exterior calculus package EXCALC of E. Schrüfer [4] to provide the exterior algebra definitions. EXCALC must be loaded before XIDEAL will work. The basis 1-forms for the exterior algebra are automatically extracted from the input. Consequently, each expression must be written in terms of these 1-forms – p-form kernels with $p > 1$ are not allowed. Similarly all distinct 1-forms in the input are assumed to be linearly independent – if a dimension has been fixed (using the EXCALC `SPACEDIM` or `COFRAME` statements), then input containing more than this number of distinct 1-forms will generate an error. The exterior algebra can be based on either an abstract vector space or the cotangent space at some fixed point on a manifold. Any functions or 0-forms are treated as constant non-vanishing coefficients.

The term ordering used is the graded lexicographical ordering based on the prevailing EXCALC kernel ordering for the basis 1-forms. This puts highest degree first and then sorts terms of the same degree lexicographically. The EXCALC kernel ordering can be changed with the REDUCE `KORDER` or EXCALC `FORDER` statements.

2 Operators

XIDEAL

`XIDEAL` calculates a Gröbner left ideal basis in an exterior algebra. The syntax is

```
XIDEAL(S:list of forms[,R:integer]):list of forms.
```

`XIDEAL` calculates the Gröbner left ideal basis for the left ideal generated by `S` using graded lexicographical ordering based on the current kernel ordering. The resulting list can be used for subsequent reductions with `XMODULOP` as long as the kernel ordering is not changed. If the set of generators `S` is graded, an optional parameter `R` can be given, and `XIDEAL` produces a truncated basis suitable for reducing exterior forms of degree less than or equal to `R` in the left ideal. This can save time and space with large expressions, but the result cannot be used for exterior forms of degree greater than `R`. See also the switches `XSTATS` and `XFULLREDUCTION`.

XMODULO

`XMODULO` reduces exterior forms to their (unique) normal forms modulo a left ideal. The syntax is

```
XMODULO(F:form, S:list of forms):form
```

or

```
XMODULO(F:list of forms, S:list of forms):list of forms.
```

An alternative infix syntax is also available:

```
F XMODULO S.
```

`XMODULO(F,S)` first calculates a Gröbner basis for the left ideal generated by `S`, and then reduces `F`. `F` may be either a single exterior form, or a list of forms, and `S` is a list of forms. If `F` is a list of forms, each element is reduced, and any which vanish are deleted from the result. If this operator is used more than once, and `S` does not change between calls, then the Gröbner basis is not recalculated. If the set of generators `S` is graded, then a truncated Gröbner basis is calculated using the degree of `F` (or the maximal degree in `F`).

XMODULOP

`XMODULOP` reduces exterior forms to their (not necessarily unique) normal forms modulo a set of exterior polynomials. The syntax is

```
XMODULOP(F:form, S:list of forms):form
```

or

```
XMODULOP(F:list of forms, S:list of forms):list of forms.
```

An alternative infix syntax is also available:

```
F XMODULOP S.
```

`XMODULOP(F,S)` reduces `F` with respect to the set of exterior polynomials `S`, which is not necessarily a Gröbner basis. `F` may be either a single exterior form, or a list of forms, and `S` is a list of forms. This operator can be used in conjunction with `XIDEAL` to produce the same effect as `XMODULO`: for a single form `F` in an ideal generated by the graded set `S`, `F XMODULO S` is equivalent to `F XMODULOP XIDEAL(S,EXDEGREE F)`.

3 Switches

XFULLREDUCE

`ON XFULLREDUCE` allows `XIDEAL` and `XMODULO` to calculate reduced (but not necessarily normed) Gröbner bases, which speeds up subsequent reductions, and guarantees a unique form (up to scaling) for the Gröbner basis. `OFF XFULLREDUCE` turns of this feature, which may speed up calculation of the Gröbner basis. `XFULLREDUCE` is `ON` by default.

XSTATS

`ON XSTATS` produces counting and timing information. As `XIDEAL` is running, a hash mark (`#`) is printed for each form taken from the input list, followed by a sequences of carets (`^`) and dollar signs (`$`). Each caret represents a new basis element obtained by a simple wedge product, and each dollar sign represents a new basis element obtained from an S-polynomial. At the end, a table is printed summarising the calculation. `XSTATS` is `OFF` by default.

4 Examples

Suppose EXCALC and XIDEAL have been loaded, the switches are at their default settings, and the following exterior variables have been declared:

```
pform x=0,y=0,z=0,t=0,f(i)=1,h=0,hx=0,ht=0;
```

In a commutative polynomial ring, a single polynomial is its own Gröbner basis. This is no longer true for exterior algebras because of the presence of zero divisors, and can lead to some surprising reductions:

```
xideal {d x^d y - d z^d t};
```

```
         {d T^d Z + d X^d Y,

          d X^d Y^d Z,

          d T^d X^d Y}

f(3)^f(4)^f(5)^f(6)
  xmodulo {f(1)^f(2) + f(3)^f(4) + f(5)^f(6)};

     0
```

The heat equation, $h_{xx} = h_t$ can be represented by the following exterior differential system.

```
S := {d h - ht*d t - hx*d x,
      d ht^d t + d hx^d x,
      d hx^d t - ht*d x^d t};
```

`XMODULO` can be used to check that the exterior differential system is closed under exterior differentiation.

```
d S xmodulo S;

     {}
```

Non-graded left and right ideals are no longer the same:

```
d t^(d z+d x^d y) xmodulo {d z+d x^d y};

     0

(d z+d x^d y)^d t xmodulo {d z+d x^d y};

      - 2*d t^d z
```

Higher order forms can now reduce lower order ones:

```
d x xmodulo {d y^d z + d x,d x^d y + d z};

     0
```

Any form containing a 0-form term generates the whole ideal:

```
xideal {1 + f(1) + f(1)^f(2) + f(2)^f(3)^f(4)};

     {1}
```

References

[1] B. Buchberger, *Gröbner Bases: an algorithmic method in polynomial ideal theory,* in *Multidimensional Systems Theory* ed. N.K. Bose (Reidel, Dordrecht, 1985) chapter 6.

[2] D. Hartley and P.A. Tuckey, *A Direct Characterisation of Gröbner Bases in Clifford and Grassmann Algebras,* Preprint MPI-Ph/93–96 (1993).

[3] T. Stokes, *Gröbner bases in exterior algebra,* J. Automated Reasoning **6**(1990)233–250.

[4] E. Schrüfer, *EXCALC, a system for doing calculations in the calculus of modern differential geometry, User's manual,* (The Rand Corporation, Santa Monica, 1986).