SUM-OF-DIGITS FUNCTION FOR CERTAIN NON-STATIONARY BASES

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ABSTRACT. We introduce a new approach to the study of sum-of-digits functions for integral non-stationary bases and apply it to certain classical examples, namely to the Cantor and Ostrowski number systems.

1. INTRODUCTION

Let $\mathcal{Q} = (q_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $q_1 = 1$. Then each natural n can be represented in the form

$$n = \sum_{j=1}^{k} \varepsilon_j q_j \tag{1}$$

with k such that $q_k \leqslant n < q_{k+1}$ and with $\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1$ obtained by means of the *greedy algorithm*, i.e.

$$\varepsilon_k = \left[\frac{n}{q_k}\right], \ \varepsilon_j = \left[\frac{n - \sum_{i=j+1}^{k} \varepsilon_i q_i}{q_j}\right], \quad 1 \leqslant j < k.$$

(see, e.g., [1]). We call the set $\mathcal{Q} = (q_k)_1^{\infty}$ the *integral base* and $\varepsilon_k, \dots, \varepsilon_1$ the *digits*.

For an arbitrary integral base digits are not independent. For instance, obviously, $0 \leqslant \varepsilon_j < \frac{q_{j+1}}{q_j}$ for every base but if q_j are defined recursively and the length of the corresponding recurrence relation is greater than 1, then there exist some extra restrictions to the sequence of digits (see, e.g., [1]). Say, if $\mathcal{Q}$ is the sequence of Fibonacci numbers, then $\varepsilon_j \in \{0, 1\}$ and $\varepsilon_j \varepsilon_{j+1} = 0$ for each j.

Let a positive integer n be represented in the form (1). Put

$$s(n) = \sum_{j=1}^{k} \varepsilon_j,$$

and

$$S(N) = \sum_{n<N} s(n)$$

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for every positive integer N. We call $S(N)$ the *sum-of-digits function* for the base $\mathcal{Q}$.

The problem set up in the early 40's for $q_n = p^{n-1}$, $p \in \mathbb{N}$, consists in obtaining a precise asymptotic formula for $S(N)$. For the cited case Mirsky [3] (1949) proved that

$$S(N) = \frac{p-1}{2\log p} N\log N + O(N), \qquad N \to \infty$$

and showed also that the term $O(N)$ cannot be replaced by $o(N)$. However, in 1975 Delange [2] managed to obtain an explicit expression for $O(N)$ in this case. Namely,

$$S(N) = \frac{p-1}{2\log p} N\log N + NF\left(\frac{\log N}{\log p}\right),$$

where F is a certain continuous nowhere differentiable function of period 1 whose Fourier expansion was also computed in [2].

Later similar results have been established for the Fibonacci number system (Coquet and Van Den Bosch, 1986) and for more general bases $\mathcal{Q}$. However, most authors dealt with the q_k's satisfying one and the same recurrence relation for different k. Grabner and Tichy (see [5, 6]) treated the stationarity of a sequence q_n very widely (namely, in [6] they associated their base with an arbitrary $\beta > 1$ and its β-expansion). For such bases their results seem to be best possible. Kirschenhofer and Tichy [4] considered the case of Cantor number system (also called *mixed radix*), for which $q_n = a_1 \dots a_{n-1}$, where $a_k \geqslant 2$ for each k. They deduced asymptotic formulas for $S(N)$ in some specific cases, namely for a bounded or monotone sequence a_n.

In the present paper we provide a possible way to obtain asymptotic formulas for the first term of $S(N)$ for a wide class of integral bases. Since the preliminary purpose of this work was to obtain such a result for the case of Ostrowski expansions (see below), we will restrict ourselves to the cited case and to a simple case of the Cantor base as a framework.

2. General approach

Let $N = \sum_{k=1}^n \varepsilon_k q_k$ with $\varepsilon_n > 0$. In [5] it was shown that for an arbitrary base $\mathcal{Q}$

$$S(N) = \sum_{k=1}^{n} \varepsilon_k \left(S(q_k) + \frac{\varepsilon_k - 1}{2} q_k\right) + \sum_{k=1}^{n-1}\left(\sum_{k+1}^{n} \varepsilon_j\right) \varepsilon_k q_k.$$

Lemma 2.1. *The following estimate holds:*

$$\sum_{k=1}^{n-1}\left(\sum_{k+1}^{n} \varepsilon_j\right) \varepsilon_k q_k < N.$$

Proof. Using Abel transform, we get for $n \geqslant 3$ (for $n = 2$ the assertion is obvious):

$$\sum_{k=1}^{n-1}\left(\sum_{k+1}^{n} \varepsilon_j\right) \varepsilon_k q_k = \varepsilon_n(N - \varepsilon_n q_n) + \sum_{k=1}^{n-2}\left(\sum_{1}^{k} \varepsilon_j q_j\right)\varepsilon_{k+1} <$$

$$< \varepsilon_n q_n + \sum_{1}^{n-2} \varepsilon_{k+1} q_{k+1} = \varepsilon_n q_n + N - \varepsilon_n q_n - \varepsilon_1 < N.$$

Corollary 2.2. *We have*

$$S(N) = \sum_{k=1}^{n} \varepsilon_k S(q_k) + \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k^2 q_k + O(N). \tag{2}$$

Henceforth we will assume that there exists a sequence of finite recurrence relations between q_n of the form

$$q_{n+1} = a_n^{(1)} q_n + a_n^{(2)} q_{n-1} + \cdots + a_n^{(k)} q_{n-k+1}, \quad n \geqslant k,$$

where k may depend on n, and $a_n^{(1)} \geqslant \ldots \geqslant a_n^{(k)} \geqslant 1$. Put $S_k := S(q_k)$.

Lemma 2.3. *The sequence S_n satisfies the following recurrence relations:*

$$S_{n+1} = a_n^{(1)} S_n + a_n^{(2)} S_{n-1} + \cdots + a_n^{(k)} S_{n-k+1} + R_n, \tag{3}$$

where

$$R_n = \frac{1}{2} \sum_{j=1}^{k} a_n^{(j)} (a_n^{(j)} - 1)\, q_{n-j+1} + \sum_{j=2}^{k} \left(\sum_{i=1}^{j-1} a_n^{(i)} \right) a_n^{(j)} q_{n-j+1}. \tag{4}$$

Proof. By definition and by our assumption,

$$\begin{aligned} S_{n+1} = \sum_{m<q_{n+1}} s(m) &= \sum_{0}^{a_n^{(1)} q_n - 1} s(m) + \sum_{a_n^{(1)} q_n}^{a_n^{(1)} q_n + a_n^{(2)} q_{n-1} - 1} s(m) + \cdots \\ &+ \sum_{q_{n+1} - a_n^{(k)} q_{n-k+1}}^{q_{n+1}-1} s(m) = \sum_{j=1}^{k} S(a_n^{(j)} q_{n-j+1}) + a_n^{(2)} a_n^{(1)} q_{n-1} + \\ &+ a_n^{(3)} (a_n^{(1)} + a_n^{(2)}) q_{n-2} + \cdots + a_n^{(k)} \left(\sum_{j=1}^{k-1} a_n^{(j)} \right) q_{n-k+1} = \\ &= \sum_{j=1}^{k} a_n^{(j)} S_{n-j+1} + \frac{1}{2} \sum_{j=1}^{k} a_n^{(j)} (a_n^{(j)} - 1) q_{n-j+1} + \\ &+ \sum_{j=2}^{k} \left(\sum_{i=1}^{j-1} a_n^{(i)} \right) a_n^{(j)} q_{n-j+1}. \quad \square \end{aligned}$$

Then we need to represent the right side of (2) in such a form that it would explicitly depend on N. However, it is not our purpose to study the problem in question in such a general situation, and we will restrict ourselves to the study of simple and, at the same time, interesting examples. Note that, just as before, Abel transform will be our basic tool.

3. Examples

We will consider two basic examples, namely the Cantor and Ostrowski number systems.

The Cantor expansion. Let $a_1, a_2, \dots$ be a sequence of positive integers greater than 1. Let $q_1 = 1$, $q_n = a_1 \dots a_{n-1}$ for $n \geqslant 2$. Then expansion (1) for this case is called the *Cantor expansion.* Here $q_{n+1} = a_n q_n$, $S_{n+1} = a_n S_n + \frac{a_n - 1}{2} q_{n+1}$. Since $S_1 = 0$, it is easy to show that

$$S_n = \left(\sum_{k=1}^{n-1} \frac{a_k - 1}{2} \right) q_n$$

for $n \geqslant 2$. Then it follows from (2) that

$$\begin{aligned} S(N) &= \sum_1^n \left(\sum_{j=1}^{k-1} \frac{a_j - 1}{2} \right) \varepsilon_k q_k + \frac{1}{2} \sum_1^n \varepsilon_k^2 q_k + O(N) = \\ &= \left(\sum_{k=1}^{n-1} \frac{a_k - 1}{2} \right) N - \sum_{k=1}^{n-1} \left(\sum_1^k \varepsilon_j q_j \right) \frac{a_k - 1}{2} + \frac{1}{2} \sum_1^n \varepsilon_k^2 q_k + O(N). \end{aligned}$$

By virtue of the fact that $\sum_{k=1}^n \varepsilon_k^2 q_k \leqslant \max(a_1, \dots, a_n) \cdot N$, and of the inequalities

$$\sum_{k=1}^{n-1} a_k \sum_1^k \varepsilon_j q_j < \sum_{k=1}^{n-1} a_k q_{k+1} < \max(a_1, \dots, a_{n-1}) \cdot N,$$

the following assertion is valid.

Theorem 3.1. *If the condition*

$$\max(a_1, a_2, \dots, a_n) = o \left(\sum_{k=1}^n a_k \right), \qquad n \to \infty \tag{$*$}$$

is satisfied, then

$$S(N) = \left(\frac{1}{2} \sum_{k=1}^{n-1} (a_k - 1) \right) \cdot N + O(\max(a_1, \dots, a_n) \cdot N).$$

Remark. This theorem generalizes the corresponding result from the work [4] obtained for the bounded sequences $\{a_n\}$.

The Ostrowski expansion. Let α be be an irrational number in the interval $(0, 1)$ and let $\alpha = [a_1, a_2, \dots]$ be its continued fraction expansion. The numerator and denominator of the n'th convergent p_n/q_n satisfy the well-known relations:

$$\begin{aligned} &q_1 = 1, \quad q_2 = a_1, \quad q_{n+1} = a_n q_n + q_{n-1}, \quad n \geqslant 2, \\ &p_1 = 0, \quad p_2 = 1, \quad p_{n+1} = a_n p_n + p_{n-1}, \quad n \geqslant 2. \end{aligned}$$

Put $\mathcal{Q} = (q_n)_{n=1}^{\infty}$. Then in expansion (1) the digits satisfy the following Markov restrictions (see, e.g., [1]):

$$0 \leqslant \varepsilon_1 \leqslant a_1 - 1, \quad 0 \leqslant \varepsilon_n \leqslant a_n, \quad \varepsilon_n = a_n \Rightarrow \varepsilon_{n-1} = 0, \qquad n \geqslant 2.$$

This representation of positive integers was introduced by Ostrowski [9] and since then the Ostrowski expansions have been used for different purposes. Their main application was the study of the *discrepancy* of the sequence $(\{n\alpha\})_1^{\infty}$ (see, e.g., [8] and references therein). Recently this expansion was applied to the arithmetic expression for the adic realization of an irrational rotation of the circle (see [7]). More precisely, being completed in the natural topologies, the Ostrowski expansion yields the metric and topological isomorphism between the rotation through α and so-called *adic* transformation on the Markov compactum of admissible sequences of digits $\varepsilon_1, \varepsilon_2, \ldots$ which have been described above. For more details see [7].

So, it follows from equations (2)–(4) that for the Ostrowski base

$$S_{n+1} = a_n S_n + S_{n-1} + R_n, \qquad n \geqslant 2, \tag{5}$$
$$S_1 = 0, \quad S_2 = \frac{1}{2}\, a_1(a_1 - 1),$$

where

$$R_n = a_n \left(\frac{a_n - 1}{2}\, q_n + q_{n-1} \right). \tag{6}$$

To solve the system (5), we use the explicit formula for the particular solution of a nonhomogenous difference equation (see [10], Chapter V, equation (30)). In this framework for the case in question we take p_n and q_n as the base for the two-dimensional space of solutions of the corresponding homogenous equation. A particular solution $\widetilde{S}_n$ of system (5) may be chosen in such a way:

$$\widetilde{S}_n = \sum_{k=1}^{n-1} \frac{\begin{vmatrix} p_k & q_k \\ p_n & q_n \end{vmatrix}}{\begin{vmatrix} p_k & q_k \\ p_{k+1} & q_{k+1} \end{vmatrix}} R_k = \sum_{k=1}^{n-1} (-1)^k (p_k q_n - p_n q_k) R_k.$$

We put $\alpha_m := |q_m \alpha - p_m| = (-1)^{m+1}(q_m \alpha - p_m)$ for each positive m and represent p_n in the last-mentioned formula as $q_n \alpha + (-1)^n \alpha_n$. Thus, we have

$$\widetilde{S}_n = \left(\sum_{k=1}^{n-1} (-1)^k (p_k - q_k \alpha) R_k \right) q_n + O(q_n),$$

as

$$\alpha_n \sum_{k=1}^{n-1} q_k R_k < \frac{1}{2} \alpha_n \sum_{k=1}^{n-1} a_k q_k q_{k+1} < \frac{1}{2} \alpha_n \sum_{k=1}^{n-1} q_{k+1}^2 <$$
$$< \alpha_n (q_n^2 + q_{n-1}^2) < 2\alpha_n q_n^2 < 2q_n.$$

Finally, $S_n = \widetilde{S}_n + b_1 q_n + b_2 p_n$ for some suitable constants b_1, b_2. Our aim is to show that as in the case of the Cantor expansions,

$$C_n \asymp \sum_1^{n-1} a_k,$$

where $C_n := \sum_{k=1}^{n-1} R_k \alpha_k$, from which it will follow that

$$S_n = C_n q_n + O(q_n), \quad n \to \infty.$$

Lemma 3.2. *The following estimates hold:*

$$\frac{1}{16} \sum_{k=1}^{n-1} a_k < C_n < \frac{1}{2} \sum_{k=1}^{n-1} a_k.$$

Proof. 1. By formula (6), $R_k = \frac{1}{2} a_k (q_{k+1} - q_k + q_{k-1})$. Then $R_k \alpha_k < \frac{1}{2} a_k q_{k+1} \alpha_k < \frac{1}{2} a_k$, since it is known that

$$q_{k+1}\alpha_k + q_k \alpha_{k+1} = 1. \tag{7}$$

2. To prove the lower estimate, it suffices to show that

$$R_j \alpha_j + R_{j-1} \alpha_{j-1} > \frac{1}{16}(a_j + a_{j-1})$$

for each $j \geqslant 2$. We have

$$R_j \alpha_j = \frac{a_j \alpha_j}{2}(q_{j+1} - q_j + q_{j-1}) > \frac{a_j \alpha_j}{2}(q_{j+1} - q_j).$$

Consider the following cases:

(i) $a_j \geqslant 2, a_{j-1} \geqslant 2$. Here $q_{j+1} - q_j = (a_j - 1) q_j + q_{j-1} \geqslant \frac{1}{2} q_{j+1}$, whence $R_j \alpha_j > \frac{a_j \alpha_j}{4} q_{j+1} > \frac{1}{8} a_j$, as $q_{j+1} \alpha_j > \frac{1}{2}$ by (7). Analogously, $R_{j-1} \alpha_{j-1} > \frac{1}{8} a_{j-1}$.

(ii) $a_j \geqslant 2, a_{j-1} = 1$. For this case $R_j \alpha_j + R_{j-1} \alpha_{j-1} > R_j \alpha_j > \frac{1}{8} a_j \geqslant \frac{a_j + a_{j-1}}{16}$.

(iii) $a_j = 1, a_{j-1} \geqslant 2$. This case is similar to the previous one.

(iv) $a_j = a_{j-1} = 1$. Here $q_{j+1} = q_j + q_{j-1} = 2q_{j-1} + q_{j-2} < 3q_{j-1}$, and $R_j \alpha_j = q_{j-1} \alpha_j > \frac{1}{3} q_{j+1} \alpha_j > \frac{1}{6} = \frac{1}{6} a_j$, whence, finally, $R_j \alpha_j + R_{j-1} \alpha_{j-1} > R_j \alpha_j > \frac{1}{6} a_j = \frac{1}{12}(a_j + a_{j-1})$.

The proof is complete.

So, we have obtained the required asymptotic equality $C_n \asymp \sum_{k=1}^{n-1} a_k$, and by virtue of this,

$$S_n = C_n q_n + O(q_n), \quad n \to \infty.$$

Now we return to estimating $S(N)$ itself. So, it follows from (2) that

$$S(N) = \sum_{k=1}^{n} C_k \varepsilon_k q_k + \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k^2 q_k + O(N).$$

Assume that the condition $(*)$ from Theorem 3.1 is satisfied for the case in question. Then, using Abel transform, we obtain

$$S(N) = C_n \cdot N + \sum_{k=1}^{n-1}(C_k - C_{k+1}) \sum_{1}^{k} \varepsilon_j q_j + O(\max(a_1, \dots, a_n) \cdot N) =$$
$$= C_n \cdot N + O(\max(a_1, \dots, a_n) \cdot N)$$

by Lemma 3.2. Thus, we have proved the main result of the present work.

Theorem 3.3.

(1) *If an irrational number $\alpha = [a_1, a_2, \dots]$ satisfies the condition $(*)$, then the following asymptotic formula for $S(N)$ is valid:*

$$S(N) = \left(\sum_{k=1}^{n-1} R_k \alpha_k\right) \cdot N + O(\max_{1 \leqslant k \leqslant n} a_k \cdot N), \tag{8}$$

where

$$R_k = \frac{a_k}{2}(q_{k+1} - q_k + q_{k-1}), \quad \alpha_k = |q_k \alpha - p_k|, \quad k \in \mathbb{N}.$$

(2) *The first term of asymptotic formula (8) grows as $\left(\sum_{k=1}^{n-1} a_k\right) \cdot N$.*

The following propositions answer the question about the behavior of $S(N)$ in the case, when the condition $(*)$ is possibly not satisfied.

Proposition 3.4. *Both for the Cantor and for Ostrowski expansions*

$$\left(\sum_{1}^{n-1} a_k\right) \cdot N = O(S(N))$$

for any $N \in \mathbb{N}$ and for any parameters $(a_1, a_2, \dots)$.

Proof. It follows from relation (2) that it suffices to show that

$$\left(\sum_{1}^{n-1} a_k\right) \cdot N = O\left(\sum_{1}^{n} C_k \varepsilon_k q_k\right),$$

which in turn immediately follows from the inequalities

$$\sum_{1}^{n} C_k \varepsilon_k q_k \geqslant C_n \cdot N > \frac{1}{16}\left(\sum_{1}^{n-1} a_k\right) \cdot N$$

which are valid for both expansions.

Proposition 3.5.

(1) *If a sequence* $(a_1, a_2, \dots)$ *is such that* $a_n = O(\sum_1^{n-1} a_k)$, *then for both expansions*

$$S(N) \asymp \left(\sum_1^{n-1} a_k\right) \cdot N.$$

(2) *If, on the contrary, a sequence* $(a_1, \dots)$ *satisfies the condition* $a_n \gg \sum_1^{n-1} a_k$, *then for each of the two expansions in question there exists* N *such that*

$$S(N) \gg \left(\sum_1^{n-1} a_k\right) \cdot N.$$

Proof. (1) As it has been shown above, for both cases

$$\sum_1^n C_k \varepsilon_k q_k < C_n \cdot N < \frac{1}{2}\left(\sum_1^{n-1} a_k\right) \cdot N,$$

and by our assumption,

$$\sum_1^n \varepsilon_k^2 q_k \leqslant \max(a_1, \dots, a_n) = O\left(\left(\sum_1^{n-1} a_k\right) \cdot N\right).$$

(2) Let N be such that $\varepsilon_1 = a_1 - 1, \varepsilon_2 = a_2 - 1, \dots, \varepsilon_n = a_n - 1$. Here

$$\sum_1^n \varepsilon_k^2 q_k > \varepsilon_n^2 q_n = (a_n - 1)^2 q_n > \frac{1}{4} a_n^2 q_n > \frac{1}{8} a_n q_{n+1} >$$
$$> \frac{1}{8} a_n \cdot N \gg \left(\sum_1^{n-1} a_k\right) \cdot N. \quad \square$$

Concluding remarks. The author hopes that the approach described above may be also applied to the class of bases $\mathcal{Q}$ considered in [1], i.e. to those with

$$q_{n+1} = a_n q_n + b_1 q_{n-1} + \dots + b_{s-1} q_{n-s+1}, \quad n \geqslant s,$$

where $s, b_1, \dots, b_{s-1}$ are constants while a_n's may differ (and, as usual, $a_n \geqslant b_1 \geqslant \dots \geqslant b_{s-1}$ for each n). The main technical difficulty here consists in how to represent the solution of the corresponding nonhomogenous discrete equation for S_n in such a form that one could prove a result similar to Lemma 3.2. We plan to return to this question in subsequent works.

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