Testing emptiness of real hypersurfaces, real algebraic sets and semi-algebraic sets

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1 Introduction

New methods different from cylindrical algebraic decomposition and based on the critical point method have been proposed in the last years for testing emptiness of semi-algebraic sets [4, 1, 2]. They are all based on a black box : RealSoving1: testing emptiness for real algebraic sets which have a finite number of complex zeroes. RealSolving1 consists of two steps: 1a) compute a Groebner basis and check if there are a finite number of complex solutions, 1b) compute the number of real solutions, starting from this Groebner basis. In the geometric situations which are considered here, such a finite number of solutions is not guaranteed, it is needed to use infinitesimal deformations to ensure a finite number of solutions.

We are trying to adapt these methods to make them feasible in practice.

2 Testing the emptiness on a hypersurface defined by a single equation

2.1 Theoretically good algorithm

The hypersurface is deformed into a smooth bounded hypersurface with only a finite number of critical points in the X_1 direction. The original hypersurface is non empty over the reals if and only if one of these critical points is real and bounded.

In this theoretical algorithm, it is needed to double the degree and to add an infinitely big and an infinitely small element. This does not affect the asymptotic complexity but is bad in practice.

This computation can be done using RealSolving1b) with infinitesimal coefficients since the equations of the critical points are automatically a Groebner basis given the deformation we consider (see [1, 4]).

We propose several improvements of this method. The basic deformation we shall consider here (modified from the one in [4, 1]) is the following: $f(X_1, \ldots, X_k)$ is replaced by $f_1 = \epsilon(X_1^{n_1} + \ldots + X_k^{n_k} + 1) + (1-\epsilon)f$ (resp. $f_2 = -\epsilon(X_1^N + \ldots + X_k^N + 1) + (1+\epsilon)f$) where n_i is an even number greater than the degree of f in X_i. The hypersurface f is non empty if and only if there are bounded zeros on f_1 or f_2. Note that the degrees of f_1 and f_2 are very close to the degree of f (while in the original deformation in [4, 1] the degree was doubled).

2.2 Dealing efficiently with the projection on X_1

We present first an idea of [3].

In order to avoid to add an infinitely big element to be in a bounded situation, we consider separately the original hypersurface f and its image g by an inversion by the unit sphere. Unbounded connected components not containing the origin give bounded sets in this transformation

If X_1 has a finite number of critical points in the X_1 direction (this is known computing a Groebner basis of the equation and of $k-1$ of its partial derivatives), we check if there are real solutions among this finite number of points. So we use the two steps of RealSolving1 If there are, we can conclude that the hypersurface is non empty. If we do not have a finite number of critical points, we make the deformations f_1 and f_2 described above, proceed in the same way and test if we obtain points which are bounded. We use only the step b of RealSolving1. If we found no real point with this process, we invert and do the same test for g (note that now the degrees have doubled). Now se are looking for critical points which are not infinitesimally close to the origin. If we found no real point this way, we can conclude that the hypersurface is empty.

Note that we need no Groebner basis computation with infinitesimals.

This strategy can be improved as follows. Note that an unbounded connected component having no critical point of X_1 cuts the hyperplane $X_1 = 0$.

So, consider successively $f(0, 0, \ldots, 0, O, X_k) = 0$ (a finite nuber of points), the critical points of X_{k-1} on $f(0, 0, \ldots, X_{k-1}, X_k)$ ...... the critical points of X_2 on $f(0, X_2, \ldots, X_{k-1}, X_k)$ the critical points of X_1 on $f(X_1, X_2, \ldots, X_{k-1}, X_k)$, stopping as soon as a real critical point is obtained.

The computations are done first with the original equations, then if X_{k-1} (resp. $\ldots, X_2, X_1$ has not a finite number of critical points, with deformations as f_1 and f_2 above.

We use again Groebner Basis computations over the integers, and RealSolving1 b) over the integers and with an infinitesimal.

If no point have been found this way, it is needed to proceed with the inversion.

Another possibility is to look at zeroes at infinity [2].

2.3 Dealing efficiently with the distance function

Another approach consists in finding the minimum of the distance function.

If the distance function has a finite number M of minima and maxima on $Z(f)$, $Z(f)$ is empty over the reals if and only if M has a real point

If the distance fucntions has an infinite number of minima and maxima, several deformations can be used to go to a situation where the distance function has a finite number of minima and maxima.

If we make a deformation $f-\epsilon$, it is needed to make Groebner basis computations in ϵ, if the deformation is f_1 (resp. f_2) this step is not needed.

3 Testing the emptiness of a system of equations

The algorithm with the best asymptotic complexity appears in [1, 4]. The first subroutine is to test emptiness of a real algebraic set. The sum of squares of the equations is taken, which defines a singular hypersurface. Now, we are back to the case of a hypersurface. We can apply one of the algorithms above. The degree has doubled and since the hypersurface is singular it is needed to use an infinitesimal deformation.

3.1 Minimizing the distance function

When we study the zero set of equations $f_1, \ldots, f_m$ which have only transverse intersections, we can consider the k equations expressing that $(x_1, \ldots, x_k)$ is a common zero of $f_1, \ldots, f_m$, and that the vector $(x_1, \ldots, x_k)$ is orthogonal to a basis of $k-m$ vectors of the tangent $k-m$ plane at $(x_1, \ldots, x_k)$ (this is expressed in terms of jacobian minors and can be in some cases more efficient than doubling the degree). In most cases, this defines a finite number of points M on $Z(f_1, \ldots, f_m)$ minimizing the distance funtion to the origin. The set $Z(f_1, \ldots, f_m)$ has no real point if and only if the set M has no real point.

When the geometric hypothesis above are not fulfilled, it is needed to define a deformation of $f_1, \ldots, f_m$ using one single infinitesimal to get this good geometric situation and to apply the same algorithm to this deformation.

3.2 Triangular systems

It is possible to put the algebraic set under the form of a finite union of solutions of a triangular system such that every such set is non empty (in the complexes) and has an obvious complex dimension.

A first step to test reality consists in testing the reality of the last hypersurface (the one with less variables). Then, one can try to go up in the triangular set .

3.3 Parametric rational univariate representation

¿From a triangular system of dimension m it is posible to compute at least in principle a parametric univariate representation consisting of an algebraic hypersurface in $m+1$ variables and $k-m$ equations of degree one. The algebraic set has real solutions if only if the corresponding hypersurface has real solutions.

4 General semi-algebraic sets

We remind the theoretically good algorithm of [1, 4]. For testing non-emptiness of a semi-algebraic set, several algebraic sets are considered. In order to get a good theoretical complexity in terms of the number of equations, the polynomials are first put in general position which requires the use of extra infinitesimals. But this step may be not needed in practice. Then there are several reduction steps to the case of an hypersurface.

In this theoretical algorithm, each reduction step needs the addition of one or two extra infinitesimals which does not affect the asymptotic complexity but is bad in practice.

We do not know precisely now how to modify this method to make it efficient.

5 Conclusion

The main tools needed for implementing these algorithms are now available in the FRISCO framework: efficient Groebner basis computations and efficient real solving for a finite number of points as well as efficient infinitesimal arithmetic.

It remains to test the different strategies proposed and improve them in practice.

References

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