Gröbner Basis Example

by
Harvey J. Greenberg
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This is taken from Thomas [1] (see also Ziegler [2]):

$$\min\ x_1 + 3x_2 + 14x_3 + 17x_4 : x \in \mathbf{Z}^4_+,$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A Gröbner basis for the coefficient matrix (A) and objective vector (c) is given by these three vectors:

$$h^1 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \qquad h^2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \qquad h^3 = \begin{pmatrix} -1 \\ -2 \\ -1 \\ 0 \end{pmatrix}.$$

It is easy to verify the first two conditions: $Ah^i = 0$ and $ch^i < 0$ for $i = 1, 2, 3$.

These change vectors also have the property (as they must) that if x is any non-optimal solution for ILP(b), there exists h^i such that $x + h^i \geq 0$. Here are a few points for various right-hand sides to illustrate that this is so.

For $x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$, choose $h = h^2$ and the improved point is

$$x + h = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

For $x = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, choose $h = h^1$ and the improved point is

$$x + h = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

For $x = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 9 \\ 14 \end{pmatrix}$, choose $h = h^3$ and the improved point is

$$x + h = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

Another example is the knapsack problem:

$$\min\ x_1 + x_2 : x \in \mathbf{Z}^3_+, 10x_1 + 11x_2 + x_3 = b.$$

The optimal solution is unique for $b = 0, 10, 20, \ldots$: $x^*(b) = (\frac{b}{10}, 0, 0)$ (= LP relaxation solution). For $b = 10k + r$ with $0 < r < 10$, there are alternative solutions: $X^*(b) = \{(k - h, h, r - h) : 0 \leq h \leq r\}$. A non-optimal solution, therefore, must have $x_3 > 10$, and an improvement is to shift one unit to either x_1 or x_2 — i.e., a Gröbner basis for this system is simply one vector:

$$\mathcal{G}_c = \left\{ \begin{pmatrix} 1 \\ 0 \\ -10 \end{pmatrix} \right\}.$$

References

[1] R.R. Thomas. A geometric Buchberger algorithm for integer programming. *Mathematics of Operations Research*, 20:864–884, 1995.

[2] G.M. Ziegler. Gröbner bases and integer programming. In A.M. Cohen, H. Cuypers, and H. Sterk, editors, *Some Tapas of Computer Algebra*, chapter 7, pages 168–183. Springer-Verlag, Berlin, FRG, 1999.