Applications of Gröbner Bases to Nonlinear Systems.

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Abstract. Some problems within nonlinear control theory are stated and solved using so called Gröbner bases from commutative algebra. The first problem is the one of stability and performance robustness of nonlinear systems subject to structured uncertainty. It is assumed that a Lyapunov function is given, and the quality of the solution depends on this Lyapunov function. It is also indicated how the robustness problem can be reinterpreted in different ways, e.g. in terms of controller design. The second problem is the one of finding a differential equation connecting the input and the output of a system starting from a state space description of it. In all cases the nonlinearities are required to be of polynomial type. The methods suggested are illustrated with concrete examples.

Keywords: Nonlinear control systems, commutative algebra, Lyapunov theory, stability robustness, performance robustness, state space equations, input-output equation

1 Introduction.

Lately there has been an increasing interest in applications of algebra (other than linear algebra!) to control theory. The main stream of algebraic control theory is probably differential algebra which has proved to be very powerful, see e.g. Fliess [5, 6] and Glad and Ljung [10]. In this paper we show how a method from commutative algebra, so called Gröbner bases, can be used in nonlinear control. For some other applications than those presented here, consult Forsman [7]. Let us first give the definitions of some concepts that are important in the paper.

Definition 1.1 The ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in k is denoted $k[x_1, \ldots, x_n]$. □

Definition 1.2 Let $F \subset k[x_1, \ldots, x_n]$. The *ideal generated by* F is the set of all polynomials that can be written as

$$\sum_{f \in F} \alpha_f f$$

for some polynomials $\alpha_f \in k[x_1, \ldots, x_n]$. This ideal is denoted (F). □

Definition 1.3 Consider $p \in k[x_1, \ldots, x_n]$ as a function $\mathbb{R}^n \to \mathbb{R}$. We say that p is *positive definite* if

$$p(0) = 0, \qquad \forall x \neq 0 : \quad p(x) > 0$$

□

2 Something about Gröbner Bases.

Gröbner bases are one of the most important tools in constructive commutative algebra today. There are many problems in pure mathematics that can be approached via Gröbner bases, see e.g. Robbiano [14]. Never the less it seems that the knowledge about this method is not yet very widespread within the field of control theory.

Gröbner bases are special kinds of generating sets for polynomial ideals, having some nice properties. For a detailed account of Gröbner bases see Pauer and Pfeifhofer [13]. In this paper we use Gröbner bases merely for eliminating variables in systems of polynomial equations, so instead of giving the exact definition of Gröbner base we state a theorem which plays a central rôle in elimination theory. To this end we give a definition.

Definition 2.1 A *ranking* of the variables $x_1, \ldots, x_n$ is simply a permutation of these symbols. If a occurs before b in the permutation we say that a has a *lower* rank than b, written $a \prec b$. If A and B are two sets of variables and any element of A has lower rank than all elements of B we write $A \prec B$. □

It is the following property that is interesting to us:

Theorem 2.1 *Let I be an ideal in $k[x_1, \ldots, x_n]$ and partition the variables $x_1, \ldots, x_n$ into two disjoint sets A and B. If $\mathbf{G}$ is a Gröbner base for I w.r.t. some ranking $A \prec B$, then*

$$k[A] \cap I = (k[A] \cap \mathbf{G}) \cup \{0\}$$

Proof. The proof can be found in for instance Buchberger [2]. □

Remark: For the information of those in the know we add that we are only using the *purely lexicographic* term ordering.

In words, theorem 2.1 states that with a proper choice of ranking the Gröbner base **G** of I will tell us if there are elements in I that are polynomials in the variables A only, and even more: **G** contains a generating set of the subideal in question!

What makes Gröbner bases really useful is that there exists an algorithm that computes the Gröbner base of any ideal, given a set of generators (see Buchberger [1]). This algorithm has been implemented on computers in several versions; for instance the symbolic algebra program Maple [3] has a Gröbner base package.

3 Stability and Performance Robustness.

There are several applications of Gröbner bases to problems that can be approached using Lyapunov theory. In [8] we presented a method for finding maximal levels of local Lyapunov functions. We will now see how similar ideas can be used in nonlinear robustness theory.

Consider the dynamical system

$$\dot{x} = f(x, \eta), \qquad x = (x_1, \ldots, x_n) \tag{1}$$

where $\eta = (\eta_1, \ldots, \eta_q)$ are real, time-invariant parameters. We assume that each component of f is polynomial in both x and η. This restriction is not as severe as one might at first believe, see e.g. Fliess [5].

Let V be a local Lyapunov function for the above system. Thus $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite radially unbounded function such that there exists a neighbourhood C of the origin such that

$$\forall x \in C \setminus \{0\} : \qquad (\nabla V \cdot f)(x) < 0 \tag{2}$$

This condition will of course depend on η. We suppose that we have a nominal value $\eta = \eta^0$ and that (2) holds for η^0. From now on we will use the abbreviation

$$Q = \nabla V \cdot f = \dot{V} \tag{3}$$

Definition 3.1 The regions Ω and B_d are defined by

$$\Omega(\eta) = \{ x \in \mathbb{R}^n \mid Q(x, \eta) < 0 \} \cup \{0\}$$

$$B_d = \{ x \in \mathbb{R}^n \mid V(x) \leq d \}$$

where Q is defined by equation (3). □

It is a classical result (see e.g. Hahn [11]) that the stability of (1) is guaranteed within B_d if

$$B_d \subseteq \Omega(\eta) \tag{4}$$

In Forsman and Glad [8] the problem of finding the largest d such that (4) holds for η^0 was considered. Here, instead we keep d fixed and vary η.

Within this framework we may state and solve different robustness problems in nonlinear systems theory. When doing this we start from a given $\delta > 0$ and given nominal parameters η^0 such that $B_\delta \subseteq \Omega(\eta^0)$.

First we consider the *stability robustness* problem. This can be formulated as follows:

For what values of η do we have $B_\delta \subseteq \Omega(\eta)$?

In words: for what values of the parameters η can we guarantee stability within B_δ using the Lyapunov function V? If we call this set M we have

$$M = \{ \eta \in \mathbb{R}^q \mid B_\delta \subset \Omega(\eta) \}$$

This problem can be approached with Gröbner bases, in the following manner. Consider the ideal $J = (V(x) - \delta, Q(x, \eta))$. We are interested in finding out: for what $\eta \in \mathbb{R}^q$ does J have real zeroes, considered as an ideal in $k[x_1, \ldots, x_n]$? This problem typically leads to inequalities in η, so in order to solve it we look for the critical cases, i.e. the boundaries of the wanted regions. This means that we are looking for η such that

$$\partial B_\delta \cap \partial\Omega(\eta) \neq \emptyset \tag{5}$$

Having done this we obtain a hypersurface ∂M in parameter space and we only need to know in which component of the region bounded by ∂M that (5) is satisfied. But this is easy, since we know that η^0 belongs to this component. To find the critical η we use the geometrical fact that when contact occurs between Ω and B_δ the gradients of their respective defining polynomial are parallel. This gives us n new equations while introducing one new variable, the Lagrange multiplier λ. Altogether we obtain $n+2$ polynomials in $n+q+1$ variables (the x, the η and λ):

$$V - \delta,\ Q,\ \frac{\partial V}{\partial x_1} - \lambda\frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} - \lambda\frac{\partial Q}{\partial x_n} \tag{6}$$

where Q is defined by equation (3). The ideal generated by the polynomials (6) typically contains an element in η only. (Confer the linear case, where we generically can eliminate $n+1$ variables from $n+2$ equations.) The Gröbner base for (6) w.r.t. a ranking

$$\{ x_1, \ldots, x_n, \lambda \} \succ \{ \eta_1, \ldots, \eta_q \} \tag{7}$$

gives us a generator for the wanted subideal.

Let us have a look on some examples to see how these ideas work in practice.

Example 3.1 Consider

$$f(x) = \begin{pmatrix} -x_1 + \alpha x_1^2 x_2 \\ -x_2 \end{pmatrix}, \qquad V = 3x_1^2 + 4x_1x_2 + 4x_2^2$$

and let $\delta = 1$. Thus $\eta = \alpha$. In [8] we showed that $\alpha_0 = 2$ is a valid choice of nominal parameter, i.e. $B_1 \subset \Omega(2)$. The level curves of the Lyapunov function are ellipses in $\mathbb{R}^2$. We wish to decide for what values of α the the elliptic region

$$B_1 = \{ x \in \mathbb{R}^2 \mid V(x) \le 1 \}$$

in the phase plane is guaranteed to be stable, using the given Lyapunov function. To do this we compute a Gröbner base for the ideal generated by the equations (6), which in this case are four:

$$\begin{gathered} 3\,x_1^2 + 4\,x_2\,x_1 + 4\,x_2^2 - 1, \\ 6\,\alpha\,x_1^3 x_2 - 6\,x_1^2 - 8\,x_2\,x_1 + 4\,\alpha\,x_1^2 x_2^2 - 8\,x_2^2, \\ 6x_1 + 4x_2 + 12\lambda x_1 - 18\lambda\alpha x_1^2 x_2 + 8\lambda x_2 - 8\lambda\alpha x_1 x_2^2, \\ 4x_1 + 8x_2 + 8\lambda x_1 - 8\lambda\alpha x_1^2 x_2 - 6\lambda\alpha x_1^3 + 16\lambda x_2 \end{gathered} \tag{8}$$

If we use the ranking

$$x_2 \succ x_1 \succ \lambda \succ \alpha$$

then this Gröbner base contains the polynomial

$$p = 2\,\alpha^3 - 147\,\alpha^2 + 1536\,\alpha + 8192 \tag{9}$$

which has three real roots:

$$-3.84\ldots,\ 17.9\ldots,\ 59.4\ldots$$

Figure 1 shows B_1 together with $\Omega(\alpha)$ for the three α:s above. This picture explains the third root (59.4) which might at first seem confusing. We conclude that B_1 is stable for $\alpha \in (-3.84, 17.9) = M$. □

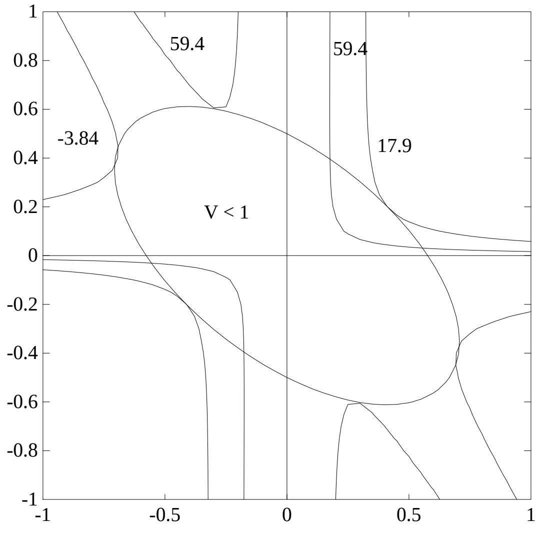

Figure 1: State space. The curve $V = 1$ and three level curves of Q.

The case $q > 1$ is of course more complicated. We will indicate in an example how our approach works in this situation.

Example 3.2 Let

$$f(x) = \begin{pmatrix} -x_1 + \alpha x_1^2 x_2 \\ -\beta x_2 \end{pmatrix}, \qquad V = x_1^2 + x_2^2$$

and $\delta = 1$. In this case $\eta = (\alpha, \beta)$ and $\eta^0 = (2, 1)$. The four polynomials (6) are

$$\begin{gathered} x_1^2 + x_2^2 - 1, \quad 2\,x_1(\alpha\,x_1^2 x_2 - x_1) - 2\,\beta\,x_2^2, \\ 2x_1 + 4\lambda x_1 - 6\lambda\alpha x_1^2 x_2, \quad 2x_2 - 2\lambda\alpha x_1^3 + 4\lambda\beta x_2 \end{gathered} \tag{10}$$

The Gröbner base for this ideal with respect to the ranking

$$x_2 \succ x_1 \succ \lambda \succ \beta \succ \alpha \tag{11}$$

contains the polynomial

$$\begin{aligned} p = {} & 4\,\beta^3\alpha^2 + 27\,\beta\,\alpha^4 - 16\,\beta^4 - 132\,\beta^2\alpha^2 + 64\,\beta^3 \\ & -132\,\beta\,\alpha^2 - 96\,\beta^2 + 4\,\alpha^2 + 64\,\beta - 16 \end{aligned} \tag{12}$$

So, in the parameter space, the region for which

$$B_1 \subseteq \Omega(\alpha, \beta)$$

is bounded by the curve

$$\partial M = \{ (\alpha, \beta) \in \mathbb{R}^2 \mid p(\alpha, \beta) = 0 \} \tag{13}$$

where p is as in equation (12). Figure 2 is a plot of this ∂M in parameter space. Since p is even in α we only show the positive α-axis. Figure 3 shows B_1 and $\Omega(\eta)$ for the values of η that are marked in figure 2.

The above analysis is quite incomplete. For example it is not immediately obvious that negative values of β should be excluded. To show that we have to employ more sofisticated arguments from real algebraic geometry. □

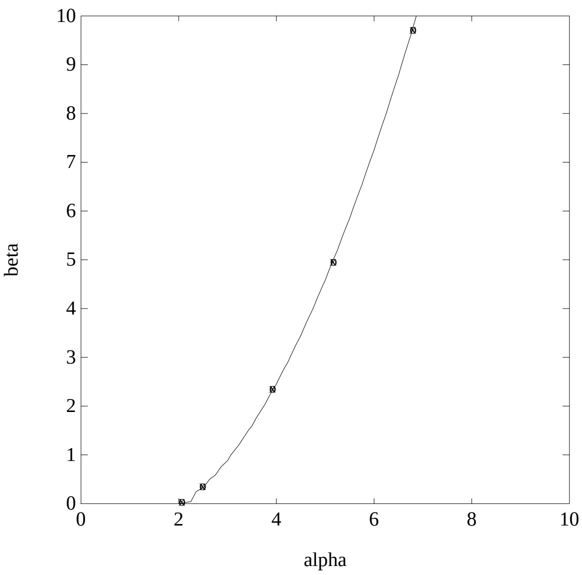

Figure 2: The zero set of p in parameter space.

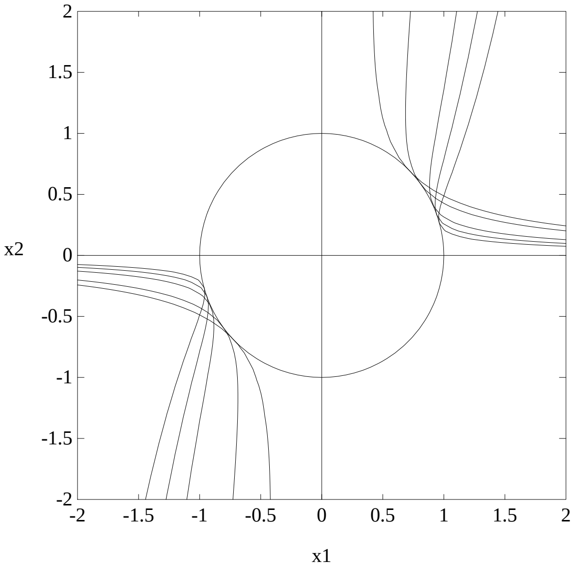

Figure 3: State space. The unit circle and level curves of Q for some values of η.

To state the next problem we need a definition.

Definition 3.2 Let $a \in k[x_1, \ldots, x_n]$ be positive definite. We define a region $\Omega_a(\eta)$ as

$$\Omega_a(\eta) \;=\; \{\, x \in \mathbb{R}^n \mid Q(x,\eta) \le -a(x) \,\}$$

where Q is defined by equation (3). $\square$

Now, one way of formulating the problem of *performance robustness* is the following:

Given a, for which η do we have $B_\delta \subseteq \Omega_a(\eta)$?

As earlier, any trajectory starting in B_δ remains there; but not only that, we also have a measure of *how fast* it approaches the origin.

In this problem the polynomials (6) are modified to

$$\begin{gathered} V-\delta, \quad Q+a, \\ \frac{\partial V}{\partial x_1} - \lambda \frac{\partial}{\partial x_1}(Q+a), \ldots, \frac{\partial V}{\partial x_n} - \lambda \frac{\partial}{\partial x_n}(Q+a) \end{gathered} \tag{14}$$

Let us now consider a concrete example.

Example 3.3 Take the same system and Lyapunov function as in example 3.1 and choose $a = x_1^2 + x_2^2$. The Gröbner base for the polynomials (14) w.r.t. a ranking of the type (7) will then contain the polynomial

$$\begin{aligned} p \;=\; \alpha^5 - \frac{6695}{112}\alpha^4 + \frac{5248}{9}\alpha^3 + \frac{1266523}{672}\alpha^2 \\ -\frac{10459}{168}\alpha + \frac{5491}{2016} \end{aligned} \tag{15}$$

which has roots

$$-2.569\ldots,\ 16.01\ldots,\ 46.31\ldots$$

Figure 4 below shows B_1 together with $\Omega_a(-2.56)$ and $\Omega_a(16)$. $\square$

We conclude by just remarking that measures of performance other than $\nabla V \cdot f$ are possible, for instance $x \cdot f$. Such a measure has both advantages and disadvantages but we will not go into a detailed analysis of these matters here.

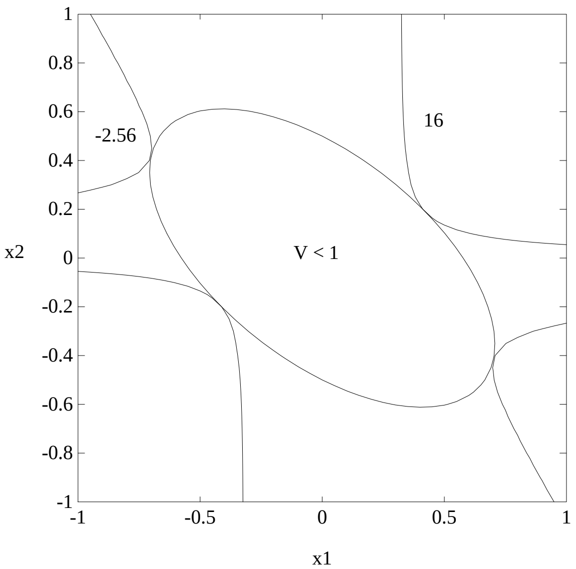

Figure 4: State space. B_1 and two level curves of $Q+a$.

4 A Note on Controller Design.

It is interesting to note that the methods described in the previous section also provide a means for designing feedback controllers for nonlinear systems. Just consider the η as controller parameters, for instance feedback gains, and ask for what values of η that the controller satisfies certain perfomance criteria.

Let us reinterpret the examples of the previous section in this spirit!

Example 4.1 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1 x_2 u \\ \dot{x}_2 &= -x_2 \\ y &= x_1 \end{aligned}$$

subject to the output feedback law $u = -\alpha y$. This corresponds exactly to examples 3.1 and 3.3. $\square$

Example 4.2 Example 3.2 can be thought of as the system

$$\dot{x}_1 = -x_1 x_2 u_1, \quad \dot{x}_2 = u_2$$

when governed by the control law $u_1 = -\alpha x_1$ $u_2 = -\beta x_2$. $\square$

5 Bounds on Control Signals.

There is yet another interesting application of the ideas presented in the two preceding sections. Here we will just sketch it briefly. Suppose that we are interested in choosing a state feedback law that makes the state tend to zero, but are faced with the problem of bounded control signals; i.e. only u:s such that

$$\forall t: \qquad |u(t)| < K \tag{16}$$

are allowed, for some $K > 0$. This is very common in practice, due to saturation in actuators etc. Using the techniques of section 3 we are able to solve the following problem: for what initial values x^0 can we guarantee that the control signal satisfies (16)?

Namely, if the control law is rational in x, i.e. $u \in k(x)$ then equation (16) gives a polynomial $f \sim u^2 - K^2$. Define a region

$$R_K = \{ x \in \mathbb{R}^n \,|\, f(x, K) < 0 \} \tag{17}$$

and check for what d (or K) we have $B_d \subset R_K$.

6 State Space to Input-Output Conversion.

In this section we examine the problem of going from a state space description of a nonlinear system to an input-output relation. If all nonlinearities are polynomial this problem is best formulated in the language of differential algebra (see e.g. Kolchin [12]) and it has been addressed earlier from a differential algebraic point of view, see e.g. Glad [9] and Diop [4]. We shall briefly recall the definitions needed here (slightly simplified for understandability).

Definition 6.1 A (ordinary) *differential ring* is a ring on which a derivation is defined. The derivation is supposed to be w.r.t. the implicit variable t. □

Notice that the derivation operator is not an element of the ring as is the case with so called Weyl algebras. The ring of differential polynomials in the variables $z = (z_1, \ldots, z_r)$ over k is denoted $k\{z_1, \ldots, z_r\}$.

Definition 6.2 A *differential ideal* is an ideal that is closed under derivation. □

The differential ideal generated by a set F of differential polynomials is denoted $[F]$.

We now introduce the notation $x = (x_1, \ldots, x_n)$.

Definition 6.3 A *state space description* of a polynomial system is a system of differential equations

$$\dot{x}_1 = f_1(x, u), \; \ldots, \; \dot{x}_n = f_n(x, u)$$
$$y = h(x, u)$$

where $h \in k[x, u]$, $\forall i: \; f_i \in k[x, u]$ □

Thus every polynomial system on state space form corresponds to a differential ideal in $k\{x, u, y\}$

$$I = [\varphi_1, \ldots, \varphi_n, y - h] \tag{18}$$

where

$$\varphi_i = \dot{x}_i - f_i(x, u), \qquad i = 1, \ldots, n \tag{19}$$

The problem of conversion from state space to input-output form is simply:

Let I be the differential ideal defined by equation (18). Find a generator for the differential ideal $I \cap k\{u, y\}$.

Mathematically this can be put: find the contraction of I to $k\{u, y\}$. Gröbner bases provide an algorithm for doing this.

The basic idea is to consider the equations involving derivatives as purely algebraic ones. For instance we may regard y and $\dot{y}$ as two different variables and $\dot{y}^2 - 2yu$ as an element in $k[y, \dot{y}, u]$. To obtain an input-output relation we differentiate the output polynomial $y - h$ with respect to t a number of times and, for every $\dot{x}_i$ that appears, we *replace it according to its state space equation*: i.e. $\dot{x}_1$ is replaced by $f_1(x, u)$ etc. In this manner we obtain a number of polynomials ψ_i

$$\frac{d^i}{dt^i}(y - h) \quad \rightarrow \quad \psi_i \tag{20}$$

in u and its derivatives, y and its derivatives and x. After n derivations we have $n+1$ polynomials, and we may eliminate the variables x using Gröbner bases.

Example 6.1 Suppose that we want to find a differential relation between u and y starting from the state space description

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + u \\ \dot{x}_2 &= -2x_2 + x_1^2 \\ y &= x_1 \end{aligned}$$

By differentiating w.r.t. t and replacing $\dot{x}_i \rightarrow f_i$ we get

$$\begin{gathered} \psi_0 = y - x_1, \quad \psi_1 = \dot{y} - (u - x_1 x_2) \\ \psi_2 = \ddot{y} - (\dot{u} - (u - x_1 x_2)\, x_2 - x_1\, (x_1^2 - 2x_2)) \end{gathered} \tag{21}$$

Substitute

$$y \rightarrow y_0, \; \dot{y} \rightarrow y_1, \; \ddot{y} \rightarrow y_2, \; u \rightarrow u_0, \; \dot{u} \rightarrow u_1$$

in the ψ_i and compute the Gröbner base $\mathbf{G}$ for

$$\begin{gathered} (y_0 - x_1, \; y_1 - u_0 + x_1 x_2, \\ y_2 - u_1 + u_0 x_2 - x_1 x_2^2 + x_1^3 - 2x_2 x_1) \end{gathered}$$

w.r.t. the ranking

$$x_2 \succ x_1 \succ y_2 \succ y_1 \succ y_0 \succ u_1 \succ u_0$$

Then the last element of $\mathbf{G}$ is

$$(y_2 + 2y_1 - 2u_0 - u_1)y_0 + y_0^4 - y_1^2 + u_0 y_1 \qquad (22)$$

which means that the input-output relation is

$$(\ddot{y} + 2\dot{y} - 2u - \dot{u})\,y + y^4 - \dot{y}^2 + u\dot{y} \;=\; 0$$

$\square$

It should be noted that the above approach also works for discrete time systems, MIMO systems and for systems with implicit output equations.

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