Appeared in Computer Aided Geometric Design, vol. 9, pp. 25-50, 1992

Algorithm for Implicitizing Rational Parametric Surfaces

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Abstract: Many current geometric modeling systems use the rational parametric form to represent surfaces. Although the parametric representation is useful for tracing, rendering and surface fitting, many operations like surface intersection desire one of the surfaces to be represented implicitly. Moreover, the implicit representation can be used for testing whether a point lies on the surface boundary and to represent an object as a semi-algebraic set. Previously resultants and Gröbner basis have been used to implicitize parametric surfaces. In particular, different formulations of resultants have been used to implicitize tensor product surfaces and triangular patches and in many cases the resulting expression contains an extraneous factor. The separation of these extraneous factors can be a time consuming task involving multivariate factorization. Furthermore, these algorithms fail altogether if the given parametrization has base points.

In this paper we present an algorithm to implicitize parametric surfaces. One of the advantages of the algorithm is that we do not need multivariate factorization. If a parametrization has no base points, the implicit representation can be represented as a determinant of a matrix, otherwise we use perturbation techniques. In the latter case we make use of GCD operation to compute the implicit representation. We also describe an efficient implementation of the algorithm based on interpolation techniques.

Keywords: Rational Surfaces, Implicitize, Resultants, Surface Intersection, Base Points, Parametric Representation, Implicit Surfaces, Semi-algebraic Set

[1]This research was supported in part by David and Lucile Packard Fellowship and in part by National Science Foundation Presidential Young Investigator Award (# IRI-8958577).

1 Introduction

Currently most geometric modeling systems use a parametric form for representing surfaces. For computational reasons, they restrict attention to rational functions. A surface represented parametrically by rational functions is known as a *rational surface*. The parametrization of a rational surface represented in terms of homogeneous coordinates is:

$$(x, y, z, w) = (X(s,t), Y(s,t), Z(s,t), W(s,t)), \tag{1}$$

where $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are polynomials in the indeterminates s and t. Every rational surface can be represented implicitly as the zero set of an irreducible homogeneous polynomial $F(x, y, z, w) = 0$. Although the parametric formulation is useful for tracing, rendering and surface fitting, many operations like surface intersection desire one of the surfaces to be represented implicitly. Moreover, the implicit representation can be used for testing whether a point lies on the boundary and to represent an object as a semi-algebraic set. The process of converting from parametric to implicit is known as *implicitization*.

There are two known techniques for implicitization. Both these techniques reduce the problem of implicitizing rational surfaces to eliminating two variables from three parametric equations. The first technique involves the use of Elimination theory. In [Hoffmann '89, Chapter 5] the two variables are eliminated in succession by using the Sylvester resultant for two equations. The resulting expression does not correspond to the resultant of three parametric equations and contains an extraneous factor. The Dixon formulation for computing the resultant has been used to implicitize tensor product surfaces in [Sederberg et. al. '84]. However it is limited to tensor product surfaces and cannot be used for implicitizing other parametrizations like the triangular patches.

[Bajaj et. al. '88; Chionh '90] use Macaulay's formulation for computing the resultant of three parametric equations for implicitizing. Macaulay's formulation expresses the resultant as a ratio of two determinants. Many a time, both the determinants evaluate to zero, even if the resultant is not zero. To compute the resultant we need to perturb the equations and use limiting arguments. This corresponds to computing the characteristic polynomial of the two matrices and the resultant is expressed as the constant term of the ratio of two characteristic polynomials [Canny '90]. Perturbation corresponds to introducing an additional variable and thereby increasing the symbolic complexity of the resulting expression. Later on, we show that this technique is expensive in practice.

In general, it is believed that techniques based on Elimination theory can result in extraneous factors along with the implicit representation and their separation can be a difficult task [Hoffmann '90a, Hoffmann '90b]. Furthermore, these algorithms are not able to implicitize parametric surfaces like bicubic patches in a reasonable amount of time and space [Hoffmann '90b].

The second technique utilizes Gröbner bases. It computes a canonical representation of the ideal generated by the parametric equations, by defining a suitable ordering of the variables [Buchberger '89; Hoffmann '89, Chapter 7]. This technique is fairly expensive in practice and even for low degree parametrizations it may take a lot of time.

We analyze the problem of implicitization. In particular, we formulate the three parametric equations in such a manner that their resultant corresponds exactly to the implicit representation. These parametric equations are of the same degree and their resultant is expressed as the determinant of a matrix. Thus, we do not need to perturb the given equations or compute the characteristic polynomial of the given matrix. We consider two types of parametrizations: total degree bounded[1] and tensor product. In each case the implicit representation corresponds to the determinant of a matrix.

The algorithms mentioned above fail altogether when a given parametrization has base points in the parametric domain. A base point in the domain, say $s = s_0, t = t_0$, corresponds to a common solution of the following four equations

$$X(s,t) = 0, \quad Y(s,t) = 0, \quad Z(s,t) = 0, \quad W(s,t) = 0.$$

It turns out that many rational parametrizations have base points. All quadric surfaces, e.g. spheres, cylinders, cones are rational surfaces with base points. Moreover, all tensor product surfaces have base points at infinity. There is a special formulation of resultants [Dixon '08], which has been used to implicitize tensor product surfaces [Sederberg et. al. '84]. However this technique fails when there are base points in the affine domain or excess base points at infinity. In general, any faithful parametrization of a rational surface whose algebraic degree is not a perfect square, has base points. These base points blow up to rational curves on the surface (known as seam curves). Parametrizations with base points are analyzed in [Chionh '90; Manocha and Canny '91a] and techniques for implicitizing them are presented.

In this paper we present algorithms to implicitize parametric surfaces with and without base points. The strength of the algorithms lie in the fact that we do not use multivariate factorization. We also describe an efficient implementation based on interpolation techniques.

The rest of the paper is organized as follows. In Section 2 we specify our notation and present some background material from algebraic geometry. In Section 3 we analyze the problem of implicitization and show how we can use resultants or Gröbner bases on a parametrization without any base points to compute the implicit representation. In Section 4 we highlight many properties of rational surfaces with base points and show why the previous techniques of implicitization using resultants or Gröbner bases fail on such parametrizations. We also consider tensor product parametrizations as a special case of rational surface consisting of base points. We present algorithms for implicitizing parametric surfaces in Section 5 and discuss their implementation in Section 6.

Many of the results presented in Section 3 and 4 were also developed independently by [Chionh '90; Chionh and Goldman '91a; Chionh and Goldman '91b].

2 Background

A rational parametrization is a vector valued function of the form

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)). \tag{2}$$

[1] These parametrizations include the triangular patches.

We use lower case letters like s, t, x or y to denote scalar variables and upper case letters to represent scalar functions like $W(s,t)$ or $F(x,y,z)$ and homogeneous functions like $\overline{F}(x,y,w)$. Bold face upper case letters, like $\mathbf{F}(s,t)$, are used to represent vector valued functions and lower case bold face letters like $\mathbf{p}$ and $\mathbf{q}$ represent tuples like (s,t,u).

In (2), $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ are bivariate polynomials and assumed to have *power basis* representation. Else the algorithm can be easily modified to work directly on Bernstein polynomials. The degree of parametrization, (2) is the maximum of the degrees of $X(s,t)$, $Y(s,t)$, $Z(s,t)$ and $W(s,t)$ (represented in power basis formulation).

2.1 Algebraic Sets

Let us consider an algebraically closed field, C and define a polynomial ring

$$A = C[x_1, x_2, \ldots, x_m]$$

of m variables over C. All the polynomials used in this section are assumed to be defined over this ring.

Definition: The set of common zeros of a system of polynomials $F_1, \ldots, F_n$ in $x_1, \ldots, x_m$ is called an *algebraic set* and is denoted $V(F_1, \ldots, F_n) \subset C^m$. An algebraic set $V(F)$ defined by a single polynomial (which is not identically zero) is called a *hypersurface.* If F is linear, then $V(F)$ is called a *hyperplane.*

The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets [Hartshorne '77, Chapter 1].

If all the F_i are homogeneous, it is more convenient to work with the projective space P^{m-1}, formed by identifying points in C^m which are scalar multiples of each other. We use the same notation, $V(\overline{F}_1, \ldots, \overline{F}_n) \subset P^{m-1}$ for an algebraic set defined by homogeneous polynomials $\overline{F}_i$.

More details on rational surfaces, faithful parametrizations, algebraic plane curves are given in the appendix.

3 Implicitization

Given a rational surface

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where $\overline{X}(s,t,u)$, $\overline{Y}(s,t,u)$, $\overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials of degree n. $\overline{\mathbf{F}}$ is a vector valued function defined as

$$\overline{\mathbf{F}} : P^2 \rightarrow P^3,$$

where P is the complex projective space. Some analysis on the problem of implicitization is given in the appendix.

Let us consider the case when the parametrization, $\overline{\mathbf{F}}$, has *no base points* and the map $\overline{\mathbf{F}}$, is therefore, defined at all points in the domain. Using topological arguments it can be shown that the image of $\overline{\mathbf{F}}$ is the zero set of a homogeneous polynomial, $\overline{H}(x,y,z,w)$ [Manocha and Canny '91a]. Moreover

$$\overline{H}(x,y,z,w) = \overline{G}(x,y,z,w)^k,$$

where $\overline{G}(x,y,z,w)$ is an irreducible and homogeneous polynomial and k corresponds to the number of preimages of any generic point in $V(\overline{G}(x,y,z,w))$ with respect to $\overline{\mathbf{F}}$ (as defined in the appendix). If the parametrization is faithful $k=1$. The problem of implicitization corresponds to computing $\overline{G}(x,y,z,w)$.

Although a rational surface indicates a map between projective space, for implicitization we restrict ourselves to the affine portion of the image [Manocha and Canny '91a]. Consider the following *parametric equations*:

$$\begin{array}{rcl} \overline{F}_1(s,t,u) & = & x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{F}_2(s,t,u) & = & y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{F}_3(s,t,u) & = & z\overline{W}(s,t,u) - \overline{Z}(s,t,u) = 0. \end{array} \quad (3)$$

Lets consider the algebraic set defined as the intersection of the three hypersurfaces $\overline{F}_1, \overline{F}_2$ and $\overline{F}_3$,

$$Q = V(\overline{F}_1, \overline{F}_2, \overline{F}_3) \subset P^2 \times C^3,$$

and let Π be the projection function

$$\Pi : P^2 \times C^3 \rightarrow C^3$$

such that

$$\Pi(s,t,u,x,y,z) = (x,y,z).$$

Theorem I: *If the given parametrization has no base points then Q consists of a single component. Moreover, that component can be represented as*

$$Q_1 = \{(s,t,u,x,y,z) | x = \frac{\overline{X}(s,t,u)}{\overline{W}(s,t,u)}, y = \frac{\overline{Y}(s,t,u)}{\overline{W}(s,t,u)}, z = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}\}.$$

Proof: It is easy to see that $Q_1 \subset Q$. Thus, Q_1 is a component of Q. Let us assume that Q consists of some other component, say P. Since $P \neq Q_1$, $\exists$ $\mathbf{p} = (s_1,t_1,u_1,x_1,y_1,z_1) \in P$ and $\mathbf{p} \notin Q_1$. There are two possibilities:

1. $\overline{W}(s_1,t_1,u_1) = 0$.

We know that $\mathbf{p} \in V(\overline{F}_1, \overline{F}_2, \overline{F}_3)$ and therefore

$$\overline{F}_1(s_1, t_1, u_1) = 0,$$

$$\Rightarrow \overline{X}(s_1, t_1, u_1) = x_1 \overline{W}(s_1, t_1, u_1) = 0.$$

Similarly, we can show that $\overline{Y}(s_1, t_1, u_1) = 0$ and $\overline{Z}(s_1, t_1, u_1) = 0$. This implies that (s_1, t_1, u_1) is a base point of $\overline{\mathbf{F}}$, which is contrary to our assumption.

2. $\overline{W}(s_1, t_1, u_1) \neq 0$.

   We know that $\mathbf{p} \in Q$ and therefore,

$$\overline{F}_1(s_1, t_1, u_1) = 0$$

$$\Rightarrow x\overline{W}(s_1, t_1, u_1) = \overline{X}(s_1, t_1, u_1)$$

$$\Rightarrow x_1 = \frac{\overline{X}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   Similarly we can show that

$$y_1 = \frac{\overline{Y}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}, \qquad z_1 = \frac{\overline{Z}(s_1, t_1, u_1)}{\overline{W}(s_1, t_1, u_1)}.$$

   This implies that $\mathbf{p} \in Q_1$.

Thus, all points in Q also lie in Q_1 and therefore,

$$Q = Q_1.$$

Thus, Q consist of one component.

Q.E.D.

Since Q is an irreducible algebraic set, each point in $\Pi(Q)$ lies in $\mathcal{Y}$. This follows from the representation of Q or Q_1 in Theorem I. Since Q and $\Pi(Q)$ are two dimensional algebraic sets, $\Pi(Q)$ correspond to the affine portion of the zero set of the implicit representation of $\overline{\mathbf{F}}(s, t, u)$. If the given parametrization is unfaithful, each point in $\Pi(Q)$ has more than one preimage with respect to $\overline{\mathbf{F}}$. In this case, $\Pi(Q)$ corresponds to an algebraic set of multiplicity greater than one. Thus,

$$\Pi(Q) = V(H(x, y, z)), \tag{4}$$

where $H(x, y, z) = G(x, y, z)^k$, $k \geq 1$. $k = 1$ if and only if $\overline{F}$ is a faithful parametrization. Using Bezout's theorem it can be shown that the algebraic degree of $H(x, y, z)$ is n^2, where n is the degree of the parametrization [Salmon '14]. The degree of $G(x, y, z)$ is n^2/k. Moreover, k corresponds to the number of points in the (s, t, u) plane, that are the preimages of an arbitrary point in $V(G(x, y, z))$.

3.1 Resultants

Given a set of m homogeneous equations in m variables, it is always possible to combine the equations to obtain from them a single equation $R = 0$, in which these variables do not appear. We are then said to have eliminated the variables and the quantity R is the *resultant* of the system of equations. In general, the resultant of any such system of equations is a function of the coefficients, whose vanishing is the necessary and sufficient condition for the given system to have a non trivial solution. There are different formulations of computing the resultant of a system of equations. Perhaps the most general one is given by [Macaulay '02]. If all the equations have the same degree, an improved version is given in [Macaulay '21].

Let us consider the system of equations (3), as polynomials in s, t, and u with coefficients being functions of x, y, and z. Since $\overline{\mathbf{F}}$ has no base points, the resultant of (3), say $R(x,y,z)$, is a nonzero polynomial in x, y and z. Lets consider $V(R(x,y,z))$ and let $(x_1,y_1,z_1) \in V(R(x,y,z))$. The fact that $R(x_1,y_1,z_1) = 0$ implies that $\exists\ s_1,t_1,u_1$ such that $(x_1,y_1,z_1,s_1,t_1,u_1) \in Q$. Thus,

$$\Pi(Q) = V(R(x,y,z)),$$

and from (4), it follows that

$$R(x,y,z) = gH(x,y,z),$$

where g is a scalar.

Given $R(x,y,z)$, $G(x,y,z)$ can be expressed as

$$G(x,y,z) = \frac{R(x,y,z)}{\mathrm{GCD}(R(x,y,z), R_z(x,y,z))}, \tag{5}$$

where $R_z(x,y,z)$ is the partial derivative of $R(x,y,z)$ with respect to z.

The resultant of three equal degree homogeneous equations can be expressed as a determinant of a matrix. Such formulations are given in [Salmon '1885; Dixon '08; Morley '25]. In particular the formulation in [Dixon '08] constructs a matrix of order $2n^2 - n$, where n is the degree of the equations, and its determinant correspond to the resultant. Some other cases where the resultant can be expressed as the determinant of a single matrix are given in [Morley and Coble '27]. Thus, we need not perturb the equations for computing the resultant. Later on we show that this formulation of resultants is an efficient and compact representation for the implicit representation.

Example I Let

$$\mathbf{F}(s,t) = (x,y,z) = (\frac{st+1}{t^2}, \frac{s}{t^2}, \frac{s^2}{t^2}).$$

After homogenizing we obtain the following system of equations

$$xt^2 - st = 0,$$

$$yt^2 - su = 0,$$

$$zt^2 - s^2 = 0.$$

Considering these equations as polynomials in s, t and u, the resultant is

$$R(x, y, z) = y^4 - 2xy^2z + x^2z^2 - z^3,$$

whose degree is 4, since $n = 2$. Since $\mathbf{F}$ is a faithful parametrization

$$R(x, y, z) = H(x, y, z) = G(x, y, z) = y^4 - 2xy^2z + x^2z^2 - z^3.$$

4 Base Points

A base point is a common solution of

$$\overline{X}(s, t, u) = 0, \quad \overline{Y}(s, t, u) = 0, \quad \overline{Z}(s, t, u) = 0, \quad \overline{W}(s, t, u) = 0.$$

The solution set of any of the polynomials, say $\overline{X}(s, t, u) = 0$, corresponds to an algebraic plane curve in the P^2 plane (denoted by homogeneous coordinates s, t and u). Each curve may have more than one component and the base point corresponds to the intersection of these curves. The *multiplicity* of each base point is equal to the minimum of the multiplicity of the curves at that point (as defined in the appendix). In other words, a base point has multiplicity k, if it is a k-fold point of $\overline{X}(s, t, u), \overline{Y}(s, t, u), \overline{Z}(s, t, u)$ and $\overline{W}(s, t, u)$. Let

$$\mathcal{S} = V(\overline{X}(s, t, u), \overline{Y}(s, t, u), \overline{Z}(s, t, u), \overline{W}(s, t, u))$$

be the set of base points. Since

$$\mathrm{GCD}(\overline{X}(s, t, u), \overline{Y}(s, t, u), \overline{Z}(s, t, u), \overline{W}(s, t, u)) = 1,$$

$\mathcal{S}$ is therefore, a finite set. Let $\mathbf{p} = (s_0, t_0, u_0) \in \mathcal{S}$. Moreover,

$$\overline{\mathbf{F}}(\mathbf{p}) = \overline{\mathbf{F}}(s_0, t_0, u_0) = (0, 0, 0, 0),$$

which does not correspond to any point in the image space. It has been known that base points blow up to rational curves on the surface (known as *seam curves*) [Clebsch '1869; Semple and Roth '85, Chapter 6; Snyder '70, page 281; Sederberg '90]. Furthermore, the degree of the seam curve is bounded by the multiplicity of the corresponding base point. The blow up can be explained in the following manner:

Consider the rational parametrization

$$\overline{\mathbf{F}}(s, t, u) = (x, y, z, w) = (su + 2tu + s^2, su + 3tu + t^2, su + tu + 2st, su + 4tu).$$

It follows that $\mathbf{p} = (0, 0, 1)$ is a base point of the given parametrization. Lets consider the first neighborhood of $\mathbf{p}$ in the domain. That can be obtained by substituting $s = mt$, $u = 1$ and taking the limit $t \to 0$. That is,

$$\mathbf{G}(m) = \lim_{t \to 0} \overline{\mathbf{F}}(mt, t, 1)$$

$$\Rightarrow \mathbf{G}(m) = \lim_{t\to 0}(mt + 2t + m^2t^2, mt + 3t + t^2, mt + t + 2mt^2, mt + 4t)$$
$$\Rightarrow \mathbf{G}(m) = \lim\ t \to 0(m + 2 + m^2t, m + 3 + t, m + 1 + 2mt, m + 4)$$
$$\Rightarrow \mathbf{G}(m) = (m + 2, m + 3, m + 1, m + 4),$$

which is the rational parametrization of a straight line. Thus, each direction in the first neighborhood of the base point gives rise to a different limit point in the image. For an arbitrary parametrization with a base point at the origin, this has been shown in Fig. I. For more on blowing up and its applications we recommend [Hartshorne '77, Chapter 1; Walker '50, Chapter 4]. If the parametric curves intersect tangentially at a base point, then the first neighborhood cannot be used for computing the seam curve.

Since $\overline{\mathbf{F}}$ is not defined at the base points, we modify its domain and define it as a mapping of the form

$$\overline{\mathbf{F}'} : P^2 \setminus \mathcal{S} \to P^3$$
$$\overline{\mathbf{F}'}(s, t, u) = \overline{\mathbf{F}}(s, t, u),$$

where $P^2 \setminus \mathcal{S}$ represents the difference of two sets. $P^2 \setminus \mathcal{S}$ is an open and irreducible set of dimension 2. Let $\mathcal{K}$ be the image of $\overline{\mathbf{F}'}$. We know that $\mathcal{K}$ is a 2-dimensional set and $\mathcal{K} \subset P^3$. In general, $\mathcal{K}$ is a proper subset of an algebraic set $V(\overline{H}(x, y, z, w))$. The problem of implicitization corresponds to computing $\overline{H}(x, y, z, w)$.

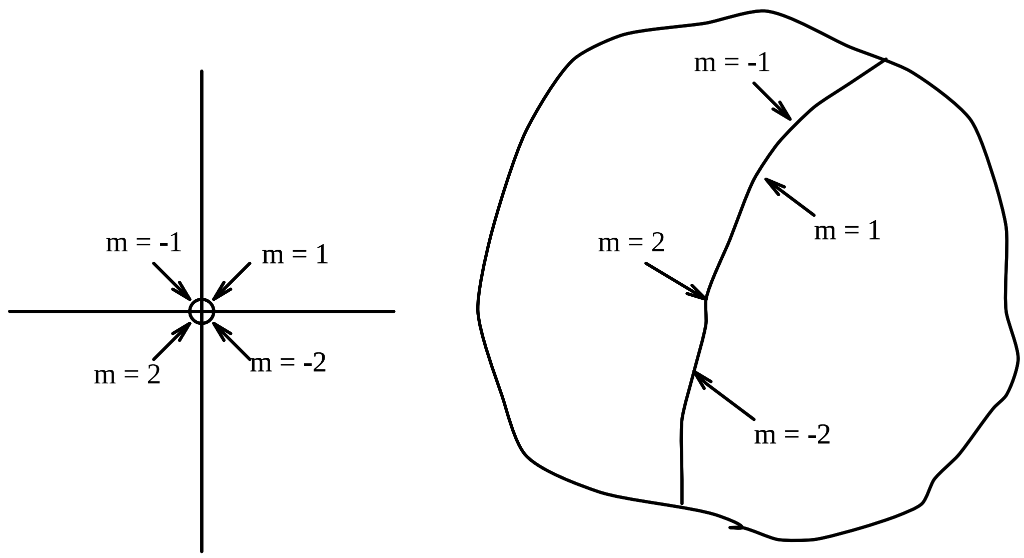

Fig. I
Blowing up of a base point at the origin.

The degree of the implicit representation is a function of the number of base points. If a parametrization of degree n has p simple base points, then the implicit representation has degree $n^2 - p$. This is assuming that the parametrization is faithful, else we divide it by a suitable k. This can be explained in the following manner [Salmon '14]:

The degree of the implicit equation of a parametric surface can be determined by counting the number of times it is intersected by a generic straight line. Let us assume that the parametrization is faithful. Such a line in the space can be represented as the intersection of following 2 planes:

$$a_1 x + a_2 y + a_3 z + a_4 = 0,$$
$$b_1 x + b_2 y + b_3 z + b_4 = 0.$$

To determine the number of intersections of this line with $\overline{\mathbf{F}}$, we substitute for x, y and z and obtain

$$\overline{F}_1(s,t,u) = a_1\overline{X}(s,t,u) + a_2\overline{Y}(s,t,u) + a_3\overline{Z}(s,t,u) + a_4\overline{W}(s,t,u) = 0,$$
$$\overline{F}_2(s,t,u) = b_1\overline{X}(s,t,u) + b_2\overline{Y}(s,t,u) + b_3\overline{Z}(s,t,u) + b_4\overline{W}(s,t,u) = 0.$$

These curves are each of degree n in s, t and u. According to Bezout's theorem the two curves intersect in n^2 points. If a parametrization has no base points, each such point of intersection in the (s,t,u) plane is a preimage of a point of intersection between the line and the surface. Thus, the surface has degree n^2. If $\overline{\mathbf{F}}$ has any base points, both $\overline{F}_1(s,t,u)$ and $\overline{F}_2(s,t,u)$ contain the base point. Thus, they intersect at that base point. The base points blow up to 1-dimensional curves on the surface $(V(\overline{H}(x,y,z,w)) \setminus \mathcal{K})$. Since there are a finite number of such curves on the surface, it is always possible to find a line which does not intersect the surface along any of these curves. In other words, the line does not intersect with any point in the set $V(\overline{H}(x,y,z,w)) \setminus \mathcal{K}$. For such a line the base point is not the preimage of any point of intersection on the surface, and the line intersects the surface at $n^2 - 1$ or less number of points. If there are p simple base points, then the degree of the implicit equation is $n^2 - p$. Base points of multiplicity k decrease the degree of the surface by at least k^2 [Snyder '70, page 281; Sederberg '90]. If $F_1(s,t,u)$ and $F_2(s,t,u)$ intersect tangentially along a k-fold base point, it decreases the degree by at least $k^2 + 1$.

4.0.1 Problem with Resultants

Given $\overline{\mathbf{F}}$, a parametrization with base points, we use resultants to compute the implicit equation. The corresponding parametric equations, (3), are

$$\begin{aligned}
\overline{F}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0,\\
\overline{F}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0,\\
\overline{F}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) = 0.
\end{aligned}$$

The resultant of these equations, by considering s, t and u as variables, is zero. This can be explained in the following manner:

Given $\mathbf{p} = (s_0, t_0, u_0)$, a base point in the parametrization. From the definition of a base point it follows that

$$\overline{F}_1(s_0,t_0,u_0) = 0 \quad \overline{F}_2(s_0,t_0,u_0) = 0, \quad \overline{F}_3(s_0,t_0,u_0) = 0.$$

Thus, the given system of equations, (3), has a non trivial solution (s_0, t_0, u_0). Moreover, this solution is independent of the coefficients, x, y and z. The resultant is therefore, identically zero.

4.0.2 Problem with Grőbner Bases

The Grőbner bases approach for implicitization is based on the fact that the implicit equation is contained in the ideal generated by the parametric equations [Buchberger '89; Hoffmann '89, Chapter 7]. Thus, by a suitable ordering of the variables, a polynomial in the Grőbner bases of the ideal corresponds to the implicit equation. When it comes to dealing with algebraic sets, Grőbner bases offer us the flexibility of working in the affine space.

The Grőbner bases approach fails whenever a parametrization has base points in the affine domain. All polynomial parametrizations fall into this category. If there are base points only at infinity, we can use the affine formulation of the given parametric equations for defining the generating set of the ideal and obtain the implicit equation by computing its Grőbner bases.

Theorem II: *Given a parametrization with base points in the affine domain. Let $\mathcal{I}$ be the ideal generated by the parametric equations. There is no polynomial in $\mathcal{I}$ which is independent of s and t, the variables used for defining the parametric domain.*

Proof: The ideal is of the form

$$\mathcal{I} = \{xW(s,t) - X(s,t), yW(s,t) - Y(s,t), zW(s,t) - Z(s,t)\}.$$

Let (s_0, t_0) be a base point in the affine domain.

Let us assume that there is a polynomial $F(x, y, z)$, which is independent of s and t in the ideal $\mathcal{I}$. Than $F(x, y, z)$ can be expressed as

$$\begin{aligned} F(x,y,z) &= A_1(x,y,z,s,t)(xW(s,t) - X(s,t)) + \\ &\quad A_2(x,y,z,s,t)(yW(s,t) - Y(s,t)) + \\ &\quad A_3(x,y,z,s,t)(zW(s,t) - Z(s,t)). \end{aligned}$$

Lets substitute $s = s_0, t = t_0$ in the above equation. Thus,

$$\begin{aligned} F(x,y,z) &= A_1(x,y,z,s_0,t_0)(xW(s_0,t_0) - X(s_0,t_0)) + \\ &\quad A_2(x,y,z,s_0,t_0)(yW(s_0,t_0) - Y(s_0,t_0)) + \\ &\quad A_3(x,y,z,s_0,t_0)(zW(s_0,t_0) - Z(s_0,t_0)), \\ \Rightarrow F(x,y,z) &= 0. \end{aligned}$$

Hence, the only polynomial independent of s and t, which lies in $\mathcal{I}$ is the zero polynomial. Thus, all other polynomials have a term containing s or t.

Q.E.D.

Given a parametrization with base points in the affine domain, the Grőbner bases of the ideal generated by the parametric equations does not contain the implicit equation.

4.1 Tensor Product Surfaces

In computer graphics and geometric modeling, most surfaces are represented as tensor product surfaces. A tensor product surface is a linear combination of bivariate basis functions, where each bivariate function is formed by taking every possible pair, one from one set of univariate functions and the other from another [Pratt '84]. Typical univariate functions are like the Bernstein polynomials, B-spline basis functions etc. Such a surface is represented as

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)) = \sum_j \sum_i \mathbf{A}_{ij} G_i(s) H_j(t),$$

where G_i and H_j are the univariate basis functions and $\mathbf{A}_{ij}$ are 4 dimensional vectors, used to represent the scalars for each component. Any component, say $X(s,t)$, can be represented in the power basis form as:

$$X(s,t) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j, \quad X_{mn} \neq 0,$$

which is of degree $m+n$ in s and t. However, the highest power of s in any monomial is m and that of t is n. After homogenizing we obtain

$$\overline{X}(s,t,u) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j u^{m+n-i-j}, \tag{6}$$

which is a homogeneous polynomial of degree $m+n$. Let

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)).$$

Lemma I: *Every tensor product surface of the form $\overline{\mathbf{F}}(s,t,u)$ has a base point of multiplicity n at $(1,0,0)$ and of multiplicity m at $(0,1,0)$.*

Proof Let us consider one of the components, say $\overline{X}(s,t,u)$. It follows from (6) that

$$\overline{X}(1,0,0) = 0, \quad \overline{X}(0,1,0) = 0.$$

Similarly $\overline{Y}$, $\overline{Z}$ and $\overline{W}$ vanish at these points, too. Thus, $(1,0,0)$ and $(0,1,0)$ are base points of $\overline{\mathbf{F}}$.

To analyze the multiplicity of $(1,0,0)$, we represent $\overline{X}(s,t,u)$ as

$$\overline{X}(s,t,u) = \overline{X}_0(t,u)s^{m+n} + \ldots + \overline{X}_{n-1}(t,u)s^{m+1} + \overline{X}_n(t,u)s^m + \ldots + \overline{X}_{m+n}(t,u),$$

where $\overline{X}_i(t,u)$ is a homogeneous polynomial of degree i in t and u. Since the highest degree of s in any term of $\overline{X}(s,t,u)$ is m, it follows that $\overline{X}_0(t,u)$, $\overline{X}_1(t,u)$, ..., $\overline{X}_{n-1}(t,u)$ vanish identically.

Thus, $(1,0,0)$ is a base point of multiplicity n. Similarly, $(0,1,0)$ is a base point of multiplicity m.

Q.E.D.

Since a tensor product surface always has base points at infinity, the resultant of the parametric equations, considering them as total degree bounded polynomials in s, t and u is identically zero. Dixon gave a special formulation of a resultant of three quantics in two independent variables, of the form $X(s,t)$, such that the vanishing of the resultant is a necessary and sufficient condition for the system to have a *non trivial* solution [Dixon '08]. The set of trivial solutions consists of

- $(1,0,0)$ of multiplicity n.
- $(0,1,0)$ of multiplicity m.

For total degree bounded polynomials, every base point in P^2 is a *non trivial* base point. We refer to this formulation for tensor product surfaces as *Dixon eliminant*[2]. The term resultant would be used for total degree bounded parametrizations. The Dixon's eliminant has been used by [Sederberg et. al. '84] to implicitize tensor product surfaces.

The degree of the homogeneous formulation of the parametric equations of $\mathbf{F}$ is $m+n$. Hence, the degree of the corresponding implicit equation can be at most $(m+n)^2$. However, base points of multiplicity n and m decrease the degree by n^2 and m^2, respectively, and the implicit representation is of degree $2mn$ or less. If the parametrization has base points in the affine domain or the curves $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{W}$ intersect tangentially at $(1,0,0)$ or $(0,1,0)$, the degree of the implicit equation is strictly less than $2mn$ and the Dixon's formulation for such equations is identically zero, too.

Let us see when do these curves, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{W}$, intersect tangentially along a base point at infinity. We will later on use these constraints for choosing a suitable perturbation. Consider the base point $(1,0,0)$. Let

$$\begin{aligned}
\overline{X}(s,t,u) &= \overline{X}_n(t,u)s^m + \ldots + \overline{X}_{m+n}(t,u),\\
\overline{Y}(s,t,u) &= \overline{Y}_n(t,u)s^m + \ldots + \overline{Y}_{m+n}(t,u),\\
\overline{Z}(s,t,u) &= \overline{Z}_n(t,u)s^m + \ldots + \overline{Z}_{m+n}(t,u),\\
\overline{W}(s,t,u) &= \overline{W}_n(t,u)s^m + \ldots + \overline{W}_{m+n}(t,u).
\end{aligned}$$

A line, $t/a = u/b$ is tangent to these curves at $(1,0,0)$ if and only if

$$\overline{X}_n(a,b) = 0, \quad \overline{Y}_n(a,b) = 0, \quad \overline{Z}_n(a,b) = 0, \quad \overline{W}_n(a,b) = 0.$$

Since $\overline{X}_n, \overline{Y}_n, \overline{Z}_n, \overline{W}_n$ are homogeneous polynomials in t and u this can happen if and only if

$$\overline{G}(t,u) = \text{ GCD}(\overline{X}_n(t,u), \overline{Y}_n(t,u), \overline{Z}_n(t,u), \overline{W}_n(t,u))$$

is a homogeneous polynomial of positive degree in t and u. Moreover, (a,b) is one of the roots of $\overline{G}(t,u)$. This constraint is equivalent to saying that

$$G(t) = \text{ GCD}(\sum_{j=0,n} X_{mj}t^j, \sum_{j=0,n} Y_{mj}t^j, \sum_{j=0,n} Z_{mj}t^j, \sum_{j=0,n} W_{mj}t^j)$$

is a polynomial of positive degree. $\sum_{j=0,n} X_{mj}t^j$ corresponds to the coefficient of s^m in $X(s,t)$. Similarly the curves intersect tangentially at $(0,1,0)$ if and only if

$$H(s) = \text{ GCD}(\sum_{i=0,m} X_{in}s^i, \sum_{i=0,m} Y_{in}s^i, \sum_{i=0,m} Z_{in}s^i, \sum_{j=0,m} W_{in}s^i)$$

is a polynomial of positive degree. In such cases Dixon's formulation can not be directly used for implicitizing. However, Gr̋obner bases can still be used since the base points are at infinity. Gr̋obner bases fail when the tensor product surface has base points in the affine domain.

[2]Dixon gave many other formulations for computing the resultant for a set of equations, including one for general polynomials. We are only referring to the one used for tensor product surfaces.

4.2 Implicitizing Parametric Surfaces with Base Points

Parametrizations with base points are analyzed in [Manocha and Canny '91a]. The technique for implicitization involves symbolic perturbation, computing the resultant or Grőbner bases of the perturbed system and using GCD of bivariate polynomials to compute the implicit representation.

Given a parametrization with base points, let us perturb one of the three parametric equations, (3), say $\overline{F}_3(s,t,u)$ and the resulting perturbed system is

$$\begin{aligned} \overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0, \end{aligned} \tag{7}$$

where $\overline{Z}_1(s,t,u)$ is a homogeneous polynomial of degree n such that

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

Almost any polynomial of degree n can be chosen for $\overline{Z}_1(s,t,u)$. We will denote this perturbed parametrization as $\overline{\mathbf{G}}$. The resulting parametrization has no base points.

It is still possible that for all choices of $\overline{Z}_1(s,t,u)$ the resultant of $\overline{G}_1(s,t,u)$, $\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$ is zero.

Theorem III: *Given a set of three equations of the form, $\overline{G}_1(s,t,u)$, $\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$, where $\overline{Z}_1(s,t,u)$ is chosen such that*

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

The necessary and sufficient condition that the resultant of $\overline{G}_i$'s does not vanish is that

$$\overline{P}(s,t,u) = GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u))$$

is a constant.

Proof: [Manocha and Canny '91a]

Q.E.D.

To circumvent this problem of vanishing resultant in certain cases we perform a change of coordinates and let the new parametrization be

$$\overline{\mathbf{F}}'(s,t,u) = (x', y', z', w') = (x, y + kz, z, w)$$

$$= (\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

where k is a scalar. The corresponding parametric equations are

$$\begin{aligned} \overline{G}_1'(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0, \\ \overline{G}_2'(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) - k\overline{Z}(s,t,u) = 0, \\ \overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0. \end{aligned}$$

Since $GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1$, for any generic k,

$$GCD(\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1,$$

too. We compute the implicit representation in terms of x', y' and z' and substitute them to obtain an implicit equation in terms of x, y and z.

Thus, it is reasonable to assume that the resultant of $\overline{G}_i$'s is nonzero. Let us express it as a polynomial in λ

$$\begin{aligned} R(x,y,z,\lambda) &= \lambda^i P_i(x,y,z) + \ldots + \lambda^d P_d(x,y,z), \\ &= \lambda^i S(x,y,z,\lambda) \end{aligned} \tag{8}$$

where $i > 0$, $S(x,y,z,0) \neq 0$. In fact i corresponds to the number of base points in the given parametrization (counted properly) [Manocha and Canny '91a].

Theorem IV: *$P_i(x,y,z)$ can be expressed as a polynomial of the form*

$$P_i(x,y,z) = H(x,y,z)F(x,y),$$

where $H(x,y,z)$ is the implicit representation and $F(x,y)$ is only a polynomial in x and y and correspond exactly to the projections of the seam curves on the $X-Y$ plane.

Proof: [Manocha and Canny '91a]

Q.E.D.

To compute $H(x,y,z)$ from $P_i(x,y,z)$, we choose 2 generic values of z, say z_1 and z_2, and $GCD(P_i(x,y,z_1), P_i(x,y,z_2))$ would correspond to $F(x,y)$. Thus, the implicit equation can be represented as

$$H(x,y,z) = \frac{P_i(x,y,z)}{GCD(P_i(x,y,z_1), P_i(x,y,z_2))}.$$

A simple consequence of previous theorems is

Theorem V: *Every parametrization has an implicit representation, which has the same coefficient field as the parametric equations.*

Proof: [Manocha and Canny '91a]

Q.E.D.

Furthermore, the extraneous factor in the lowest degree term of the resultant, $F(x,y)$, can be used to compute the rational parametrizations of seam curves [Manocha and Canny '91a].

5 Algorithm

In the previous sections we have highlighted the technique used for implicitizing rational parametric surfaces. These techniques consist of computing the resultants or Gröbner bases of the parametric equations. If the parametrization has base points, the parametric equations are perturbed and we compute the resultant or Grőbner bases of the perturbed system. The implicit equation is extracted from the lowest degree term by making use of the GCD operation. Although the Grőbner bases provide us the additional flexibility of working in the affine space, we chose resultants for their computational efficiency. In general the running time complexity of the Grőbner bases algorithm can be doubly exponential in the number of variables as compared to the singly exponential complexity of the resultants [Canny '88]. As a matter of fact, multipolynomial resultant algorithms provide the most efficient methods (as far as asymptotic complexity is concerned) for solving system of polynomial equations or eliminating the variables [Bajaj et. al. '88]. In our applications the number of variables is fixed and any argument based on the asymptotic complexity may not have much significance. Our motivation for choosing resultants stemmed from the fact that most implicit representations are dense polynomials in x, y and z. Grőbner bases seem to perform well on sparse systems and for dense polynomials resultants are considered faster in practice. Moreover, algorithms for resultants deal with matrices and determinants and are relatively simple to implement as compared to Grőbner bases.

5.1 Checking for Base Points

Given a parametrization of degree n,

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

the base points correspond to common roots of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$. In general 4 such polynomials have no roots in common. Consider the intersection of this surface with a line represented as the intersection of the following 2 planes

$$a_1x + a_2y + a_3z + a_4w = 0$$

$$b_1x + b_2y + b_3z + b_4w = 0,$$

where a_i's and b_i's can be considered as symbolic variables. To determine the number of intersections of the line with the given surface, we substitute for x, y, z and w and obtain

$$\overline{F}_1(s,t,u) = a_1\overline{X}(s,t,u) + a_2\overline{Y}(s,t,u) + a_3\overline{Z}(s,t,u) + a_4\overline{W}(s,t,u) = 0$$

$$\overline{F}_2(s,t,u) = b_1\overline{X}(s,t,u) + b_2\overline{Y}(s,t,u) + b_3\overline{Z}(s,t,u) + b_4\overline{W}(s,t,u) = 0.$$

The given parametrization has base points, if and only if, $\overline{F}_1(s,t,u)$ and $\overline{F}_2(s,t,u)$ have common roots for all values of a_i's and b_i's. The common roots correspond to the base points. They can be determined by computing the u-resultant of $\overline{F}_1$ and $\overline{F}_2$ [Waerden '50]. The u-resultant is the resultant of

$$\begin{aligned} \overline{F}_1(s,t,u) &= 0 \\ \overline{F}_2(s,t,u) &= 0 \\ \alpha s + \beta t + \gamma u &= 0. \end{aligned} \qquad (9)$$

The u-resultant is a homogeneous polynomial of degree n^2 in α, β and γ and it decomposes into linear factors of the form $(s_1\alpha + t_1\beta + u_1\gamma)$, where s_1, t_1 and u_1 are functions of a_i's and b_i's. The given parametrization has a base point if and only if there is a factor the form $(s'\alpha + t'\beta + u'\gamma)$, where s', t' and u' are scalar constants and independent of a_i's and b_i's. In this case (s', t', u') is a base point of the given parametrization. The number of base points (counted properly with respect to multiplicity) is given by the number of factors of the form $(s'\alpha + t'\beta + u'\gamma)$. If there are n^2 base points, the given parametrization is a degenerate parametrization and its image is not a 2-dimensional surface.

The 3 equations (9) are of degree n, n and 1. Their resultant can be expressed as the determinant of a matrix [Morley and Coble '27]. We prefer this formulation of resultant over Macaulay's formulation [Macaulay '02], since it involves computing a single determinant and we do not have to perturb the given equations.

The u-resultant of $\overline{F}_1$ and $\overline{F}_2$ is a homogeneous polynomial of degree $n^2 + 2n$ in 11 variables (a_i's, b_i's, α, β and γ). In practice, this computation can be very time consuming even for low degree parametrizations. Furthermore, we need to factorize the resultant into linear factors and thereby adding to the complexity of the computation. For practical applications we present a probabilistic algorithm.

Consider 4 generic combinations of $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ of the form

$$\overline{G}_i(s,t,u) = x_i\overline{X}(s,t,u) + y_i\overline{Y}(s,t,u) + z_i\overline{Z}(s,t,u) + w_i\overline{W}(s,t,u), \quad 1 \leq i \leq 4,$$

where x_i's, y_i's, z_i's and w_i's are random numbers. We use these combinations for computing the following u-resultants

$$\begin{aligned}\overline{R}_1(\alpha,\beta,\gamma) &= \text{Resultant}(\overline{G}_1(s,t,u), \overline{G}_2(s,t,u), \alpha s + \beta t + \gamma u),\\ \overline{R}_2(\alpha,\beta,\gamma) &= \text{Resultant}(\overline{G}_3(s,t,u), \overline{G}_4(s,t,u), \alpha s + \beta t + \gamma u).\end{aligned}$$

$\overline{R}_1(\alpha,\beta,\gamma)$ and $\overline{R}_2(\alpha,\beta,\gamma)$ are homogeneous polynomials of degree n^2 in α, β and γ. Let

$$\overline{P}(\alpha,\beta,\gamma) = \text{GCD}(\overline{R}_1(\alpha,\beta,\gamma), \overline{R}_2(\alpha,\beta,\gamma)). \tag{10}$$

Let d be the degree of $\overline{P}(\alpha,\beta,\gamma)$. If $d = 0$, the given parametrization has no base points else for each base point (s', t', u'), $(s'\alpha + t'\beta + u'\gamma) \mid \overline{P}(\alpha,\beta,\gamma)$. For almost all combinations, $\overline{G}_1, \ldots, \overline{G}_4$, each linear factor of $\overline{P}(\alpha,\beta,\gamma)$ corresponds to a base point. Therefore d corresponds to the number of base points in the given parametrization and the degree of the implicit representation is $n^2 - d$. To reduce the symbolic complexity of the computation, we may specialize α, β or γ to some constants (one or two at a time).

Our implicitization algorithm only needs to know whether the given parametrization has any base points (and not the actual number of base points) and in such cases it perturbs the parametric equations. A simple algorithm to check for the existence of base points is obtained in the following manner. Consider any three generic combinations, $\overline{G}_1(s,t,u)$, $\overline{G}_2(s,t,u)$ and $\overline{G}_3(s,t,u)$, and let R be their resultant. For almost all such combinations, R is zero if and only if the given parametrization has base points. The resultant can be expressed as the determinant of a matrix and all entries of the matrix are numeric. The computation in this case is purely numeric and very fast in practice.

5.1.1 Tensor Product Surfaces

A tensor product surface can either have base points in the affine domain or excess base points at infinity. Given a tensor product surface

$$\mathbf{F}(s,t) = (X(s,t), Y(s,t), Z(s,t), W(s,t)),$$

where any component, say $X(s,t)$, is of the form

$$X(s,t) = \sum_{i=0,m} \sum_{j=0,n} X_{ij} s^i t^j, \quad X_{mn} \neq 0.$$

According to Lemma IV, it has a base point of multiplicity n at $(s,t,u) = (1,0,0)$ and of multiplicity m at $(s,t,u) = (0,1,0)$.

Let us homogenize the given parametrization and take its 4 generic combinations to compute $\overline{P}(\alpha, \beta, \gamma)$, (10). Since the parametrization has base points at infinity, $\overline{P}(\alpha, \beta, \gamma)$ can be expressed as

$$\overline{P}(\alpha, \beta, \gamma) = \alpha^{n^2} \ \beta^{m^2} \ \overline{Q}(\alpha, \beta, \gamma),$$

where $\overline{Q}(\alpha, \beta, \gamma)$ is a homogeneous polynomial. Let d be the degree of $\overline{Q}(\alpha, \beta, \gamma)$. The given parametrization has a base point in the affine domain or an excess base point at infinity, if and only if $d > 0$. Only in such cases the Dixon eliminant fails to compute the implicit representation and we need to perturb the parametric equations to compute the implicit representation. The degree of the implicit representation is $2mn - d$.

The technique used to check for the existence of base points is obtained by considering three generic combinations of X, Y, Z and W, say $G_1(s,t)$, $G_2(s,t)$ and $G_3(s,t)$. Let R be the dixon eliminant of G_1, G_2 and G_3. For almost all combinations, R is zero if and only if the given parametrization has base points in the affine domain or excess base points at infinity.

5.2 Form of Implicit Representation

In this section, we present a simple algorithm to verify whether the implicit representation, $H(x,y,z)$ is independent of z. This algorithm is required before considering the efficient perturbation and thereby perturbing the parametric equation containing the z variable (as shown in (7)).

The implicit representation is independent of z, if any line represented as the intersection of the following two planes

$$w_1 x = x_1 w$$

$$w_1 y = y_1 w,$$

where $(x_1, y_1, w_1) = (\overline{X}(s_1, t_1, u_1), \overline{Y}(s_1, t_1, u_1), \overline{W}(s_1, t_1, u_1))$ lies on the surface. In other words the given line intersects the surface at an infinite number of points. This should hold for all (s_1, t_1, u_1), where (s_1, t_1, u_1) correspond to a point in the domain. A simple probabilistic algorithm to check for this property is given below.

Let (s_1, t_1, u_1) correspond to a random point in the domain and consider the equations

$$\overline{F}_1(s,t,u) = \overline{X}(s_1,t_1,u_1)\overline{W}(s,t,u) - \overline{W}(s_1,t_1,u_1)\overline{X}(s,t,u) = 0$$

$$\overline{F}_2(s,t,u) = \overline{Y}(s_1,t_1,u_1)\overline{W}(s,t,u) - \overline{W}(s_1,t_1,u_1)\overline{Y}(s,t,u) = 0$$

and

$$\overline{G}(s,t,u) = \text{ GCD}(\overline{F}_1(s,t,u), \overline{F}_2(s,t,u)).$$

If $\overline{G}(s,t,u)$ is a constant, then the implicit representation of $\overline{\mathbf{F}}$ is not independent of z. For almost all choices of (s_1, t_1, u_1), the fact that $\overline{G}(s,t,u)$ is a polynomial of positive degree implies that the implicit equation is independent of z.

5.3 Choice of Perturbing Polynomial

If a parametrization has base points, we perturb the polynomials (as shown in (7)) to compute the implicit representation. The only constraint on the perturbing polynomial $Z_1(s,t,u)$ is imposed by theorem III, i.e.

$$V(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u), \overline{Z}_1(s,t,u)) = \phi.$$

To simplify the symbolic complexity of the resulting computation we choose a perturbing polynomial of the form

$$\overline{Z}_1(s,t,u) = a_1 s^n + a_2 t^n + a_3 u^n,$$

where a_1, a_2 and a_3 are random numbers. For almost all choices of a_1, a_2 and a_3 this polynomial will satisfy the constraint of the theorem III. Else the resultant of the perturbed system is identically zero.

For tensor product surfaces, $\mathbf{F}(s,t)$, where the highest degree of s in any monomial is m and the highest degree of t in any monomial is n, we choose a perturbation polynomial of the form

$$Z_1(s,t) = a_1 s^m t^n + a_2 s^m + a_3 t^n + a_4,$$

where a_1, a_2, a_3 and a_4 are random numbers. In this case we compute the Dixon eliminant of the perturbed system and extract the implicit equation from its lowest degree term after expressing it as a polynomial in the perturbing variable.

5.4 Algorithm

Given a parametrization

$$\overline{\mathbf{F}}(s,t,u) = (x,y,z,w) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$

an algorithm for implicitization is

1. Let
$$\overline{P}(s,t,u) = \text{ GCD}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)).$$
If $\overline{P}(s,t,u)$, the common factor, is a polynomial of positive degree, cancel out the common factor from each component of the parametrization.

2. Check whether the given parametrization has base points.

3. Check whether the implicit representation, $H(x,y,z)$, is independent of z. If it is independent of z, interchange $\overline{Y}(s,t,u)$ and $\overline{Z}(s,t,u)$, such that the new parametrization is equivalent to
$$(x,y,z,w) = (\overline{X}(s,t,u), \overline{Z}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u)),$$
and its implicit equation will be a function of x and z only. Substitute y for z and the resulting representation would correspond to the implicit representation of $\overline{\mathbf{F}}$.

4. If the parametrization has no base points, compute the resultant of the parametric equations. Let the resultant be $H(x,y,z)$ and the implicit representation, $G(x,y,z)$, can be computed as
$$G(x,y,z) = \frac{H(x,y,z)}{\text{GCD}(H(x,y,z), H_z(x,y,z))}.$$

5. Let $k = \frac{\overline{Z}(s,t,u)}{\overline{W}(s,t,u)}$. If k is a constant, the implicit representation is given by $z = k$.

6. Let
$$\overline{P}(s,t,u) = \text{ GCD}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{W}(s,t,u)).$$
If $\overline{P}(s,t,u)$ is a polynomial of positive degree perform a coordinate transformation (follows from Theorem III) and let the new parametrization be
$$\overline{\mathbf{F}}'(s,t,u) = (x',y',z',w') = (x, y+kz, z, w) =$$
$$(\overline{X}(s,t,u), \overline{Y}(s,t,u) + k\overline{Z}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)),$$
where k is a random number. Rest of the algorithm is used to implicitize $\overline{\mathbf{F}}'$. Given $G(x',y',z')$, the implicit representation of $\overline{\mathbf{F}}'$, the implicit representation of $\overline{\mathbf{F}}$ corresponds to $G(x, y-kz, z)$.

7. Compute the resultant of the following system of equations
$$\begin{aligned}
\overline{G}_1(s,t,u) &= x\overline{W}(s,t,u) - \overline{X}(s,t,u) = 0,\\
\overline{G}_2(s,t,u) &= y\overline{W}(s,t,u) - \overline{Y}(s,t,u) = 0,\\
\overline{G}_3(s,t,u) &= z\overline{W}(s,t,u) - \overline{Z}(s,t,u) + \lambda\overline{Z}_1(s,t,u) = 0,
\end{aligned}$$
where $\overline{Z}_1(s,t,u)$ is a perturbing polynomial. Express the resultant as a polynomial in λ and let $P_i(x,y,z)$ correspond to its lowest degree term.

8. Choose 2 generic values of z, $z = z_1$ and $z = z_2$, and let

$$H(x, y, z) = \frac{P_i(x, y, z)}{\mathrm{GCD}(P_i(x, y, z_1), P_i(x, y, z_2))}.$$

The implicit representation is computed as

$$G(x, y, z) = \frac{H(x, y, z)}{\mathrm{GCD}(H(x, y, z), H_z(x, y, z))}.$$

6 Implementation

The algorithm in the previous section can be easily implemented on top of any computer algebra system. The main operations are computing the matrix entries for resultant formulation, expanding symbolic and numeric determinants, GCD of multivariate polynomials. Most computer algebra systems support these operations. However when it comes to practice, expanding a symbolic determinant becomes a time consuming task. Consider the problem of implicitizing bicubic tensor product Bézier surfaces (with no base points in the affine domain or excess base points at infinity). The implicit representation corresponds to the determinant of a matrix of order 18, where each matrix entry is a linear polynomial in x, y and z. However the computer algebra systems on commonly available workstations (Sun-3's, Sun-4's) are not able to compute such determinants in a reasonable amount of time and space. Some experiments with the implementations of Grőbner bases and resultants in Macsyma 414.62 on a Symbolics lisp machine (with 16MB main memory and 120MB virtual memory) are described in [Hoffmann '90b]. For many cases of bicubic surfaces, these systems are unable to implicitize such surfaces and fail due to insufficient virtual memory. Only a new algorithm for basis conversion is able to implicitize such surfaces, however it takes about 10^5 seconds, which would be considered impractical for most applications [Hoffmann '90b].

Let us analyze some properties of the implicit representation. A polynomial equation of degree d in 3 variables can have up to M monomials, where

$$M = \binom{d+3}{d}.$$

For a polynomial of degree 18 that comes to 1330 terms. In practice the implicit representations are dense polynomials. This is not difficult to see, since a coordinate transformation of the form

$$\begin{aligned} x &= \alpha_1\overline{x} + \alpha_2\overline{y} + \alpha_3\overline{z} \\ y &= \beta_1\overline{x} + \beta_2\overline{y} + \beta_3\overline{z} \\ z &= \gamma_1\overline{x} + \gamma_2\overline{y} + \gamma_3\overline{z} \end{aligned}$$

would result in a very dense polynomial in $\overline{x}, \overline{y}$ and $\overline{z}$ for almost all choices of α_i, β_i and γ_i. Furthermore the coefficients of the implicit representation are much larger than those of the parametrization. If the absolute magnitude of the coefficients of parametric equations is $|N|$, the coefficients of the implicit representation can have magnitude of the order of $|N|^d$. This follows from the algorithms used for implicitization. Each entry in the matrix has coefficient size equivalent to that of the parametrization and the order of the matrix is equal to the degree of the implicit representation (for tensor product surfaces). For general parametrizations, the order of the matrix is $2n^2 - n$, whereas the implicit equation has degree at most n^2. This property can have significant impact on the numerical stability of the algorithms, when the coefficients of the parametrization and implicit representation are floating point numbers. Since the accuracy of the solution obtained with floating point arithmetic is not completely understood, we restrict ourselves to exact arithmetic.

There are many reasons for the failure and bad performance of implicitization algorithms implemented within the framework of computer algebra systems.

- Most computer algebra systems use sparse representation for multivariate polynomials and the computations become slow whenever the polynomials generated are dense.

- The algorithms used are symbolic in nature and large intermediate expressions may be generated. Their implementations in lisp-like environments may require a large amount of virtual memory and thereby slowing down the computations.

- These systems use exact arithmetic and represent the coefficients of intermediate expressions as *bignums*. As a result the cost of arithmetic operations is quadratic in the coefficient size. The coefficient size is proportional to the degree of the polynomial expressions being generated and tends to grow exponentially with the degree.

The bottleneck in our algorithm is the symbolic expansion of determinants. The rest of the computations are fast enough in the computer algebra systems. We therefore implemented our algorithm within the framework of a computer algebra system and used a separate implementation for determinant expansion. The main idea is to guess the form of the determinant, say a polynomial of degree d in 3 or 4 variables, corresponding to unperturbed and perturbed parametric equations, respectively, and use Vandermonde interpolation for computing the coefficients [Ben-Or and Tiwari '88]. The resulting problem is equivalent to that of interpolating a univariate polynomial which has the same number of coefficients as the multivariate polynomial. As a result, the algorithm only involves numeric computations and no intermediate symbolic expressions are generated. Since the implicit representations are dense polynomials, we use a dense representation for the resultant. To circumvent the problem of coefficient growth and its impact on arithmetic computation, we chose to work in finite fields. The order of the finite fields is about 2^{31} and thereby making the best use of hardware implementations of 32 bit integer arithmetic (available on most workstations). The main problem is in getting a tight bound on the coefficients of the implicit equations. Since the resultant of the parametric equations (perturbed or unperturbed) corresponds to a determinant, we can use Hadamard's bound for computing a bound on the coefficients of the resultant. However, the bound so obtained is rather loose and we use a probabilistic algorithm for computing the coefficients of the implicit equation. The algorithm involves computing the coefficients modulo various primes and use chinese remainder theorem to compute the corresponding bignums. If for two successive prime sequences (the second one contains one more prime than the first sequence) the same bignums are obtained, then those bignums correspond to the coefficients of the implicit equation with high probability. More details of interpolation based algorithms to compute resultants and the probabilistic algorithm for computing the coefficients of the resultant are given in [Manocha and Canny '91b]. The complexity of the algorithm is output sensitive and is given by $O(|C|M(logM)^2)$, where $|C|$ corresponds to the size of resultant coefficients, and M is the number of terms that can be present in the resultant [Manocha and Canny '91b]. If M is small, a simpler algorithm of complexity $O(|C|M^2)$ may be used.

6.1 Interpolation

Lets consider the case when we do not perturb the given equations and each entry of matrix is a linear polynomial of the form $a_1x + a_2y + a_3z + a_4$. Let $D(x_1, y_1, z_1)$ be the determinant of the matrix, when x, y and z are specialized to x_1, y_1 and z_1, respectively. We choose 3 distinct primes, say p_1, p_2 and p_3 and compute M determinants of the form $D(p_1, p_2, p_3)$, $D({p_1}^2, {p_2}^2, {p_3}^2)$, $\ldots$, $D({p_1}^M, {p_2}^M, {p_3}^M)$, where M corresponds to the number of monomials in the implicit representation. Given M, we use algorithms for Vandermonde interpolation to compute the coefficients of the polynomial $H(x, y, z)$ and thereby reducing the problem to interpolating a univariate polynomial. If we are computing the resultant of perturbed parametric equations, then the resultant is a polynomial of the form $R(x, y, z, \lambda)$. We therefore, use 4 distinct primes and their powers for interpolation. In this case each entry of the matrix is of the form

$$a_1x + a_2y + a_3z + \lambda(a_4 + a_5x + a_6y) + a_7.$$

All the computations are performed over a finite field.

Our algorithm requires the value of M. Since it is difficult to compute the actual value, we use an upper bound corresponding to the number of monomials that the polynomials of degree d in 3 or 4 variables can have. Though there are many algorithms available for sparse interpolation, in practice they either require a tight bound on the number of monomials that the polynomials may have or actually figure out the actual monomials first and than compute their coefficients [Ben-Or and Tiwari '88; Kaltofen and Lakshman '88]. In the former case, the problem is similar to that of ours and in the latter case, these algorithms seem to take more time in figuring out the actual monomials present in the polynomial and as a result may be slower as compared to the dense interpolation algorithms. Furthermore, the implicit equations and resultants of perturbed systems are generally dense polynomials.

If the resultant or Dixon eliminant of the unperturbed parametric equations does not vanish,

$$M = \begin{pmatrix} d+3 \\ d \end{pmatrix},$$

where d is the degree of the implicit representation ($d = 2mn$ for tensor product surfaces of the form $s^m t^n$ and $d = n^2$ for triangular patches of degree n). If the parametrization has base points then the determinant of the perturbed system is a polynomial of degree $2d$ (where $d = 2mn$ or $d = n^2$ depending on the case) in x, y, z and λ. In general such a polynomial can have up to

$$\binom{2d+4}{4}$$

monomials. However, we can use some properties of the resultant of the perturbed parametric equations to improve this bound.

The resultant is a polynomial of degree d in the coefficients of each equation. As a result the sum of the degrees of z and λ in any monomial does not exceed d. According to (8), the resultant can be expressed as

$$R(x,y,z,\lambda) = \lambda^i P_i(x,y,z) + \lambda^{i+1} P_{i+1}(x,y,z) + \ldots + \lambda^d P_d(x,y,z),$$

where i correspond to the total number of base points in the parametrization and $P_j(x,y,z)$ is a polynomial of the form

$$P_j(x,y,z) = Q_d(x,y) + zQ_{d-1}(x,y) + \ldots + z^{d-j} Q_j(x,y),$$

and $Q_k(x,y)$ is a polynomial of degree k in x and y. Thus,

$$\begin{aligned} M &= \sum_{j=i}^{d} \left(\sum_{k=j}^{d} \binom{k+2}{2} \right) \\ &= \sum_{j=i}^{d} \left(\binom{d+3}{3} - \binom{j+2}{3} \right). \end{aligned}$$

Some deterministic and probabilistic algorithms to compute i are presented in Section 5.1. We make use of (8) to present a simple and probabilistic algorithm.

Let $(x,y,z) = (x_1,y_1,z_1)$ correspond to a random point in space and compute the determinant $R(x_1,y_1,z_1,\lambda)$, which is a polynomial of degree d in λ. We can use Vandermonde interpolation to compute it. For almost all choices of (x_1,y_1,z_1) the lowest degree of λ in $R(x_1,y_1,z_1,\lambda)$ corresponds to i.

6.2 Performance

The symbolic determinant computation has been implemented in C++ on a Sun-4 (a 10 MIPS machine) and IBM RS/6000 (a 34 MIPS machine). The rest of the algorithm has been implemented on top of Mathematica. The bottleneck of the computation is the determinant computation. In Table I and II its performance for different parametrizations is given. The timings correspond to a single iteration of the finite field computation. The total number of iterations is a function of the coefficient size of the output. Since the coefficient size is proportional to the degree of the implicit representation, more iterations are needed for higher degree implicit equations. We use finite fields of order 2^{31} and therefore, the number of iterations is bounded by $k+1$, where k is the minimum integer satisfying the relation

$$|N| < 2^{30k}$$

and $|N|$ is the magnitude of the coefficient of maximum size of the resultant.

Parametrization	Implicit Degree	M	Sun-4	IBM RS/6000
s^2+t^2	4	10	1 sec.	1 sec.
s^3+t^3	9	220	6 sec.	3 sec.
s^2t^2	8	165	4 sec.	2 sec.
s^3t^3	18	1330	100 sec.	23 sec.
s^3t^4	24	2925	430 sec.	118 sec.

Table 1: The performance of implicitization algorithm for parametrizations without base points (a single iteration over a finite field)

Parametrization	Base Points	Implicit Degree	M	Sun-4	IBM RS/6000
s^2+t^2	2	2	74	2 sec.	1 sec.
s^3+t^3	3	6	1064	52 sec.	16 sec.
st^3	2	4	295	4 sec.	2 sec.
s^2t^2	4	4	510	13 sec.	4 sec.
s^3t^3	3	15	15300	4700 sec.	1180 sec.

Table 2: The performance of implicitization algorithm for parametrizations with base points (a single iteration over a finite field)

Thus, we see that the algorithm does not perform well for bicubic parametrizations with base points. In general $3-4$ iterations are needed and even on a fast machine like IBM RS/6000 that amounts to 4500 seconds. The main problem is in the perturbation technique, which increases the symbolic complexity of the resultant.

7 Conclusion

In this paper we analyzed the problem of implicitizing rational parametric surfaces. If a parametrization has no base points, the resultant or Dixon eliminant of the parametric equation corresponds exactly to the implicit equation. The base points decrease the degree of the implicit equations and blow up to rational curves on the algebraic surface. We use symbolic perturbation to compute the implicit equations of parametrizations with base points. The strength of our technique lies in the fact that we use GCD of bivariate polynomials to extract the implicit equation from the lowest degree term of the resultant of the perturbed system. Although, similar analysis hold for Gröbner bases, we recommend resultants for efficiency reasons.

Algorithms implemented in the framework of computer algebra systems (available on commonly available workstations) are unable to implicitize parametric surfaces like bicubic patches. We present an algorithm for computing symbolic determinants and as a result achieve a significance performance improvement as compared to the previous implementations of the implicitization algorithms. If a parametrization has no base points (or a tensor product surface has no base points in the affine domain or excess base points at infinity), the algorithm is very fast on machines like IBM RS/6000. Furthermore, the algorithm can be easily parallelized and thereby achieve speedups proportional to the level of parallelism. However, perturbation increases the symbolic complexity of the resultants and the resulting algorithm to implicitize parametric surfaces with base points is slow. We therefore, need efficient algorithms to implicitize parametric surfaces with base points.

For surface intersection, the implicit form has been represented as an unexpanded matrix determinant. More details are given in [Manocha and Canny '91c].

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9 Appendix

9.1 Rational Surface

In geometric modeling a surface parametrization, (2), indicates a mapping of the form

$$\mathbf{F} : R^2 \rightarrow R^3.$$

In fact the domain is often restricted to a finite interval, of the form $[a_1, b_1] \times [a_2, b_2]$ or a triangle. Since the field of real numbers is not algebraically closed, it is often useful to extend this definition to its algebraic closure, C, the set of complex numbers. Hence we consider the parametrization as a vector valued function

$$\mathbf{F} : C^2 \rightarrow C^3.$$

Till now we have viewed our surface as a geometric object in *affine* space. However, there are a lot of advantages in considering the object in *projective* space. Projective n-dimensional space consists of the affine n-dimensional space plus the points at *infinity.*

Let $s = s_0, t = t_0$ be a solution of $W(s,t) = 0$. Since (2) is a homogeneous representation of the surface, $\mathbf{F}(s_0, t_0)$ is a point at infinity. Furthermore, the parameters s and t can correspond to infinity as well. For example all tensor product surfaces have base points at infinity. Thus, $\mathbf{F}(s,t)$ should be regarded as the following function:

$$\mathbf{F} : \quad P^2 \rightarrow P^3$$

where P denotes the complex projective space. A parameter value in the domain, P^2, is represented by the tuple (s,t,u) and $u = 0$ corresponds to the parameter values at infinity. The rational surface $\mathbf{F}(s,t)$ should be interpreted as a representation of the form

$$\overline{\mathbf{F}}(s,t,u) = (\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) \tag{11}$$

where $\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u)$ and $\overline{W}(s,t,u)$ are homogeneous polynomials in s, t and u and each polynomial has the same degree. Moreover,

$$\mathrm{GCD}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u)) = 1.$$

9.2 Algebraic Plane Curves

In this section we present some results on algebraic plane curves. A plane curve of degree n, can be represented by an equation $F(x, y) = 0$, where F is a polynomial of degree n. The corresponding homogeneous representation is $\overline{F}(x, y, w) = 0$. Any point on the curve in general is a *simple point*. A few points on the curve are *multiple points* [Semple and Roth '85, Chapter 2].

Definition: A *multiple point* of order k (or k-fold point, $k > 1$) of a degree n curve, is a point $\mathbf{p}$ of the curve such that a generic line through $\mathbf{p}$ meets the curve in only $n - k$ further points.

Let us investigate the behavior of a curve at a multiple point. We assume that the point under consideration is the origin, i.e. $\mathbf{p} = (0, 0, 1)$. The curve can be represented as

$$\overline{F}(x, y, w) = \overline{U}_0(x, y)w^n + \overline{U}_1(x, y)w^{n-1} + \ldots + \overline{U}_{n-1}(x, y)w + \overline{U}_n(x, y) = 0,$$

where $\overline{U}_i(x, y)$ is a homogeneous polynomial of degree i in x and y. A generic line through the origin can be represented in the form $x/a = y/b$. The point of this line whose coordinates are $(ka, kb, 1)$, where k is a scalar, lies on the curve if k is any of the roots of the equation

$$\overline{U}_0(a, b) + k\overline{U}_1(a, b) + k^2\overline{U}_2(a, b) + \ldots + k^i\overline{U}_i(a, b) + \ldots + k^n\overline{U}_n(a, b) = 0. \qquad (12)$$

To make the curve have a k-fold point at the origin corresponds to making the equation, (12), have k nonzero roots for every value of the ratio a/b. This can happen, if and only if $\overline{U}_0(x, y)$, $\overline{U}_1(x, y)$, $\ldots$, $\overline{U}_{k-1}(x, y)$ vanish identically. A line corresponds to a tangent at $\mathbf{p}$, if it has $k + 1$ of its intersections with the curve at $\mathbf{p}$ and the $n - k - 1$ intersections at other points on the curve. All lines of the form $x/a' = y/b'$, where $\overline{U}_k(a', b') = 0$ are tangent to the curve at $\mathbf{p}$. There can be at most k such lines.

A simple version of *Bezout's theorem* is used for determining the number of intersections between a curve of degree m and that of degree n. It is assumed that the curves have no component in common. That is:

Two curves of degree m and n intersect at mn points, counted properly with respect to multiplicity.

9.3 Faithful and Unfaithful parametrizations

In many cases a rational curve or surface can be identically described by a lower degree rational parametrization. Such curves or surfaces have *unfaithful* parametrizations. In particular, a surface parametrization is *faithful* if there is a one to one relationship between the points on the surface and the parameter values, except for a finite number of points and curves on the surface. Another popular terminology for faithful and unfaithful parametrizations are proper and improper parametrizations, respectively.

Consider the following affine parametrization of the unit sphere

$$(x, y, z, w) = (1 - s^2 - t^2, 2s, 2t, 1 + s^2 + t^2).$$

Since the preimage of $(x, y, z, w) = (0, 0, 1, 1)$ consists of a unique point in the parametric domain ($s = 0, t = 1$), the given parametrization is faithful. If we reparametrize by substituting $s = uv$ and $t = u^2$, we obtain

$$(x, y, z, w) = (1 - u^2v^2 - u^2, 2uv, 2u^2, 1 + u^2v^2 + u^4),$$

which is an unfaithful parametrization. There are two points in the preimage of $(0, 0, 1, 1)$, $(u, v) = (1, 0)$ and $(u, v) = (-1, 0)$.

Every rational surface can be represented by a faithful parametrization [Castelnuovo '1894]. However no algorithms are known at the moment for computing the faithful parametrization of an unfaithfully parametrized rational surface. Resultants and Gröbner bases have been used to decide whether a given parametrization is faithful [Bajaj et. al. '88; Hoffmann '89, Chapter 7]. Our implicitization algorithm also determines whether a given parametrization is faithful or not.

9.4 Implicitization

Given a rational surface

$$\overline{\mathbf{F}}(s, t, u) = (x, y, z, w) = (\overline{X}(s, t, u), \overline{Y}(s, t, u), \overline{Z}(s, t, u), \overline{W}(s, t, u)),$$

where $\overline{X}(s, t, u)$, $\overline{Y}(s, t, u)$, $\overline{Z}(s, t, u)$ and $\overline{W}(s, t, u)$ are homogeneous polynomials of degree n. $\overline{\mathbf{F}}$ is a vector valued function defined as

$$\overline{\mathbf{F}} : P^2 \to P^3$$

Let the image of $\overline{\mathbf{F}}$ be $\mathcal{Y}$, where $\mathcal{Y} \subset P^3$. We assume that $\mathcal{Y}$ has dimension 2. For example, $\overline{\mathbf{F}}$ is not a parametrization of the form

$$\overline{\mathbf{F}}(s, t, u) = (x, y, z, w) = (s + t, s + t, s + t, u),$$

whose image is the 1-dimensional line, $x = y = z$.

A parametrization of the form $\overline{\mathbf{F}}$ corresponds to the plane representation of a rational surface in P^3. To a generic point $\mathbf{p}$ of the plane, there corresponds a unique point of $\mathcal{Y}$. The only exceptions are the base points, which blow up to rational curves on the surface. However, there are a finite number of such base points in the parametric domain. At times, the preimage of a point in $\mathcal{Y}$ consists of a curve in the domain. Let $\mathcal{Z}$ be one such curve of the plane, where $\mathcal{Z} \subset P^2$. Such curves are known as the *fundamental curves* [Semple and Roth '85, Chapter 6]. If $\overline{\mathbf{F}}$ is a faithful parametrization, then there are open sets $\mathcal{U} \subseteq P^2$ and $\mathcal{V} \subseteq \mathcal{Y}$ such that $\mathcal{U}$ isomorphic to $\mathcal{V}$. Moreover, $\mathcal{U}$ and $\mathcal{V}$ are dense in P^2 and $\mathcal{Y}$, respectively (A subset $\mathcal{A}$ is said to be dense in its superset $\mathcal{B}$, if it has the same dimension as that of $\mathcal{B}$). In fact

$$\overline{\mathbf{F}}(\mathcal{U}) = \mathcal{V}.$$

Thus, P^2 and $\mathcal{Y}$ are birationally equivalent [Hartshorne '77, Chapter 1, Corollary 4.5]. For more on birational maps we recommend [Hartshorne '77, Chapter 1; Mumford '76, Chapter 8]. As a result there exists an inverse rational function (with respect to $\overline{\mathbf{F}}$)

$$\overline{\mathbf{F}}^{-1} : \mathcal{Y} \to P^2,$$

which can be represented as

$$\overline{\mathbf{F}}^{-1}(x, y, z, w) = (s, t, u) = (\overline{S}(x, y, z, w), \overline{T}(x, y, z, w), \overline{U}(x, y, z, w)), \tag{13}$$

where $\overline{S}, \overline{T}$ and $\overline{U}$ are homogeneous polynomials in x, y, z and w. Furthermore, each of them has the same degree. Just like $\overline{\mathbf{F}}$ is not defined at the base points, $\overline{\mathbf{F}}^{-1}$ is not defined at those values of (x, y, z, w), which correspond to the images of fundamental curves (like $\overline{\mathbf{F}}(\mathcal{Z})$)). To show the birational equivalence, we can remove such points and their images from P^2 and $\mathcal{Y}$ and the birational map is, therefore, defined between the corresponding open sets.

Given $\overline{\mathbf{F}}$, resultants and Gr̋obner bases have been used to compute $\overline{\mathbf{F}}^{-1}$ in [Bajaj et. al. '88] and [Hoffmann '89, Chapter 7], respectively. It is possible that there is more than one faithful rational parametrization of a given rational surface. All such parametrizations of a given surface are birationally equivalent.

Let $\overline{\mathbf{F}}$ be an unfaithful parametrization. Then $\overline{\mathbf{F}}$ has a corresponding faithful parametrization of the form

$$\overline{\mathbf{G}} : P^2 \to P^3$$

$$\overline{\mathbf{G}}(p, q, r) = (x, y, z, w) = (\overline{X}_1(p, q, r), \overline{Y}_1(p, q, r), \overline{Z}_1(p, q, r), \overline{W}_1(p, q, r)).$$

The image of $\overline{\mathbf{G}}$ is $\mathcal{Y}$. Since $\overline{\mathbf{G}}$ is a faithful parametrization, it has an inverse map

$$\overline{\mathbf{G}}^{-1} : \mathcal{Y} \to P^3$$

$$\overline{\mathbf{G}}^{-1}(x, y, z, w) = (p, q, r) = (\overline{P}(x, y, z, w), \overline{Q}(x, y, z, w), \overline{R}(x, y, z, w)),$$

where $\overline{P}$, $\overline{Q}$ and $\overline{R}$ are homogeneous polynomials of same degree. Moreover,

$$(p, q, r) = \overline{\mathbf{G}}^{-1}(x, y, z, w) = \overline{\mathbf{G}}^{-1}(\overline{\mathbf{F}}(s, t, u)).$$

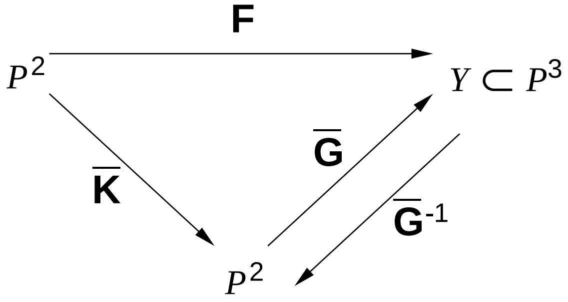

Fig. II
The relationship between the various functions

Let $\overline{\mathbf{G}}^{-1}\overline{\mathbf{F}}$ be $\overline{\mathbf{K}}$ and

$$(p,q,r) = \overline{\mathbf{K}}(s,t,u) = (\overline{P}_1(s,t,u), \overline{Q}_1(s,t,u), \overline{R}_1(s,t,u)),$$

where $\overline{P}_1(s,t,u) = \overline{P}(\overline{X}(s,t,u), \overline{Y}(s,t,u), \overline{Z}(s,t,u), \overline{W}(s,t,u))$. We can similarly define $\overline{Q}_1$ and $\overline{R}_1$. $\overline{\mathbf{K}}$ is a rational map of the form

$$\overline{\mathbf{K}} : P^2 \to P^2.$$

It is possible that $\overline{\mathbf{F}}$ has base points, while $\overline{\mathbf{G}}$ has no base points. For example, consider the faithful parametrization of a plane

$$\overline{\mathbf{G}}(p,q,r) = (x,y,z,w) = (p+r, 2p+q, q-3r, q+r).$$

$\overline{\mathbf{G}}$ has no base points. Substitute

$$(p,q,r) = \overline{\mathbf{K}}(s,t,u) = (st+t^2, su+tu, s^2+su),$$

and the resulting unfaithful parametrization is

$$\begin{aligned} \overline{\mathbf{F}}(s,t,u) &= (x,y,z,w) \\ &= (s^2+t^2+st+su, 2t^2+2st+su+tu, -3s^2-2su+tu, s^2+tu+2su). \end{aligned}$$

$(s,t,u) = (0,0,1)$ is a base point of $\overline{\mathbf{F}}$.