HOW TO SHELL A MONOID

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Abstract. For a finitely generated submonoid Λ of $\mathbb{N}^d$, we consider minimal free resolutions of a field k as a module over the monoid algebra $k[\Lambda]$. Using a result of Laudal and Sletsjøe which intepretes the ranks of the free modules in the resolution as the homology of certain simplicial complexes associated to posets, we show how non-commutative Gröbner bases and the non-pure shellings of Björner and Wachs can be used to prove new results about these resolutions. In particular, when $k[\Lambda]$ is the coordinate ring of a normal affine toric variety, we give an explicit minimal free resolution for k as a $k[\Lambda]$-module.

I. Introduction.

Let Λ be a finitely generated submonoid of $\mathbb{N}^d$ and $k[\Lambda]$ its monoid algebra over a field k. If $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ is the minimal generating set of Λ, then

$$k[\Lambda] \quad \cong \quad k[x_1, \ldots, x_n]/I_\Lambda,$$

where I_Λ is the *toric ideal* generated by all binomials $x_{i_1} \cdots x_{i_r} - x_{j_1} \cdots x_{j_s}$ corresponding to additive relations $\alpha_{i_1} + \ldots + \alpha_{i_r} = \alpha_{j_1} + \ldots + \alpha_{j_s}$. This ideal is the kernel of the map $k[x_1, \ldots, x_n] \to k[z_1, \ldots, z_d]$ sending x_i to $\mathbf{z}^{\alpha_i} = z_1^{\alpha_{i1}} z_2^{\alpha_{i2}} \cdots z_d^{\alpha_{id}}$.

We are interested in the homology of the residue field k as a module over $k[\Lambda]$. It can be computed by the minimal free resolution of k which is graded by the monoid Λ. The generating function of the homology is the *multigraded Poincaré series*

$$P_k^{k[\Lambda]}(t, \mathbf{z}) \;=\; \sum_{\lambda \in \Lambda} \sum_i \dim_k \left(\operatorname{Tor}_i^{k[\Lambda]}(k,k)_\lambda \right) t^i \mathbf{z}^\lambda. \tag{1.1}$$

In [An] Anick disproved the conjecture that the Serre-Kaplansky problem on rationality has positive answer by constructing a local noetherian ring with transcendental Poincaré series. It is an open question whether $P_k^{k[\Lambda]}(t, \mathbf{z})$ is rational for a monoid algebra $k[\Lambda]$.

Computing the Betti numbers of a finite minimal free resolution by simplicial complexes has long tradition. For infinite resolutions this idea has been explored very little: Laudal and Sletsjoe [LS, 1.3] expressed the *Betti numbers* over $k[\Lambda]$ as

$$\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k,k)_\lambda \;=\; \dim_k \tilde{H}_{i-2}(\Delta(\lambda); k) \quad \text{for all } \lambda \in \Lambda \text{ and } i = 0, 1, 2, \ldots \tag{1.2}$$

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Here $\Delta(\lambda)$ denotes the simplicial complex of chains in the open interval $(0,\lambda)$ in the partial order on Λ defined by $\alpha \leq_\Lambda \beta$ if $\beta - \alpha \in \Lambda$. The simplicial complexes $\Delta(\lambda)$ are generally not pure, unless $\mathcal{A}$ happens to lie on an affine hyperplane in $\mathbb{R}^d$.

In this paper we introduce two ideas for computing the homology of k:

- We write the commutative ring $k[\Lambda]$ as a quotient of the free associative algebra $k\langle y_1,\ldots,y_n\rangle$ on n indeterminates. Let
$$k[\Lambda] \quad \cong \quad k\langle y_1,\ldots,y_n\rangle/J_\Lambda \quad \cong \quad k[x_1,\ldots,x_n]/I_\Lambda,$$
where J_Λ is the kernel of the map $k\langle y_1,\ldots,y_n\rangle \to k[z_1,\ldots,z_d]$ sending y_i to $\mathbf{z}^{\alpha_i}$. The set of all monomials in $k\langle y_1,\ldots,y_n\rangle$ which are mapped onto $\mathbf{z}^\lambda$ is called the *non-commutative fiber of* λ. The monomials in the non-commutative fiber of λ are in one-to-one correspondence with the facets (= maximal faces) in $\Delta(\lambda)$.
- In Theorem 3.5 we relate non-commutative Gröbner basis to a *non-pure shelling* for all complexes $\Delta(\lambda)$. By [BW] the existence of such a shelling implies that $\Delta(\lambda)$ is homotopy equivalent to a wedge of spheres of various dimensions.

Say that Λ *supports a poset* if there is a term order $\prec$ on $k[x_1,\ldots,x_n]$ and a partial order P on $\{x_1,\ldots,x_n\}$ such that the initial ideal $in_\prec(I_\Lambda)$ is the Stanley-Reisner ideal of P. This means that $in_\prec(I_\Lambda)$ is generated by the products x_ix_j, where x_i, x_j are incomparable in P. The resulting reduced Gröbner basis of I_Λ consists of elements of the form $x_ix_j - x_{\nu_1}x_{\nu_2}\cdots x_{\nu_r}$. It is shown in Section 4 that many monoids arising in geometry do support a poset. For instance, all affine normal toric surfaces and many projective toric surfaces have this property. Higher-dimensional monoid algebras supporting posets include Veronese rings, Segre rings, and toric algebras with straightening law. When Λ supports a poset we obtain a rational formula for $P_k^{k[\Lambda]}(t,\mathbf{z})$ in terms of a Hilbert function. This formula is well known to hold in the special case when $k[\Lambda]$ is graded in the sense of Remark 2.1.

The rationality of the Poincaré series for normal toric surfaces was first proved in [LS]. Our shellings give explicit formulas. Furthermore, in Section 5 we construct the minimal free resolution of k over $k[\Lambda]$ in this case.

II. The bar complex and normal toric surfaces.

Our point of departure in this project was to understand the articles of Laudal [Lau] and Laudal-Sletsjoe [LS], in which the formula (1.2) was derived and used to prove rationality of the Poincaré series for normal affine toric surfaces. We start out by giving a short proof of the formula (1.2):

Consider the *bar resolution* $\mathbb{B}$ of k over $k[\Lambda]$; see [Ma, Ch. 10,§2]. This is an infinite resolution which is far from being minimal. The i-term B_i in $\mathbb{B}$ is the free $k[\Lambda]$-module with basis $(\Lambda_+)^i$, the set of ordered i-tuples of non-zero elements of Λ. We compute the Betti numbers $\dim_k \mathrm{Tor}_*^{k[\Lambda]}(k,k)_\lambda$ by tensoring the bar complex $\mathbb{B}$ with k and then taking homology. The tensored bar complex looks like

$$\mathbb{B}\otimes k : \ \ldots \to B_i \otimes k \xrightarrow{d_i \otimes k} B_{i-1}\otimes k \to \ldots \to B_0 \otimes k = k \tag{2.1}$$

where $B_i \otimes k$ is the k-vector space with basis $(\Lambda_+)^i$. Write the basis elements in the form $[\lambda_1|\lambda_2|\cdots|\lambda_{i-1}|\lambda_i]$ with $\lambda_j \in \Lambda_+$. The differential acts by the rule

$$(d_i \otimes k)[\lambda_1|\cdots|\lambda_i] \quad = \quad \sum\nolimits_{1\leq j\leq i-1} (-1)^j \cdot [\lambda_1|\cdots|\lambda_j + \lambda_{j+1}|\cdots|\lambda_i]. \tag{2.2}$$

If $i = 1$ then this means $(d_1 \otimes k)[\lambda] = 0$ for any $\lambda \in \Lambda_+$. The differential (2.2) preserves the sum $\lambda = \lambda_1 + \cdots + \lambda_i$ of the entries in each bracket, that is, the complex (2.1) is the direct sum of its finite-dimensional graded components $(\mathbb{B} \otimes k)_\lambda$. If we identify $[\lambda_1|\lambda_2|\cdots|\lambda_i]$ with the chain $\lambda_1 \leq \lambda_1 + \lambda_2 \leq \cdots \leq \lambda_1 + \cdots + \lambda_{i-1}$ in the open interval $(0, \lambda)$ of Λ, then the differential (2.2) becomes precisely the boundary map in the simplicial complex $\Delta(\lambda)$ of chains in $(0, \lambda)$. We conclude that the reduced homology of $\Delta(\lambda)$ in dimension $i - 2$ equals the i-th homology of $(\mathbb{B} \otimes k)_\lambda$. The latter is the k-vector space $\mathrm{Tor}_i^{k[\Lambda]}(k, k)_\lambda$. This completes our proof of (1.2) $\square$

The following class of monoids corresponds to projective toric varieties.

Remark 2.1. *For a submonoid Λ of $\mathbb{N}^d$ the following conditions are equivalent:*

(1) *The ideal I_Λ is homogeneous with respect to the usual grading $deg(x_i) = 1$.*
(2) *The monoid algebra $k[\Lambda]$ can be graded by setting $deg(x_i) = 1$.*
(3) *The minimal generators of Λ lie on a common affine hyperplane in $\mathbb{R}^d$.*
(4) *The simplicial complex $\Delta(\lambda)$ is pure (= all facets have the same dimension) for all $\lambda \in \Lambda$.*

We call a monoid algebra *graded* if it satisfies the equivalent conditions in Remark 2.1. An interesting class of graded monoid algebras are those for which the residue field has a linear resolution, that is, all the entries in the maps of the minimal free resolution are linear. Such an algebra is called a *Koszul algebra.*

Corollary 2.2. *A graded monoid algebra $k[\Lambda]$ is Koszul if and only if the simplicial complex $\Delta(\lambda)$ is Cohen-Macaulay for every $\lambda \in \Lambda$.*

Proof. Being Koszul means $\mathrm{Tor}_i^{k[\Lambda]}(k, k)_\lambda = 0$ unless i equals the degree of λ. $\square$

All the graded monoids appearing in Section 4 have shellable complexes $\Delta(\lambda)$ and are hence Koszul since pure shellable complexes are Cohen-Macaulay (see e.g. [BW, Cor 12.6]). However, the results in this paper are not restricted to the graded case. They are even more interesting for non-graded monoids.

We give a preview for the case of a 2-dimensional normal submonoid Λ of $\mathbb{N}^2$, studied in [LS]. Order the unique set of minimal generators $\alpha_1, \ldots, \alpha_n$ of Λ counter-clockwise so that

$$det(\alpha_i, \alpha_{i+1}) \quad = \quad 1 \qquad \text{for } i = 1, 2, \ldots, n-1. \tag{2.3}$$

Let $\prec$ be the purely lexicographic term order on $k[x_1, \ldots, x_n]$. Then the reduced Gröbner basis of I_Λ with respect to $\prec$ consists of $\binom{n-1}{2}$ binomials $x_i x_j - x_r^{i_r} x_{r+1}^{i_{r+1}}$ where $1 \leq i < r < j < n$. The initial ideal defined by this term order equals

$$in_\prec(I_\Lambda) \quad = \quad \langle\, x_i x_j \,:\, 1 \leq i < j - 1 \leq n - 1 \,\rangle. \tag{2.4}$$

The key observation is that $in_\prec(I_\Lambda)$ is the Stanley-Reisner ideal for a poset P on $\{x_1, x_2, \ldots, x_n\}$. Namely, P is the "zig-zag poset" having covering relations $x_i <_P x_{i+1}$ if i is odd and $x_i >_P x_{i+1}$ if i is even.

Consider the non-commutative polynomial ring $k\langle y_1, \ldots, y_n\rangle$, and reorder its variables according to any linear extension of P, say,

$$y_1 < y_3 < y_5 < \cdots < y_2 < y_4 < y_6 < \cdots . \tag{2.5}$$

Let $<$ denote the lexicographic order on monomials in $k\langle y_1, \ldots, y_n\rangle$ induced by (2.5). Define a term order $\lessdot$ on $k\langle y_1, \ldots, y_n\rangle$ by setting $y_{i_1} \cdots y_{i_r} \lessdot y_{j_1} \cdots y_{j_s}$ if $x_{i_1} \cdots x_{i_r} \prec x_{j_1} \cdots x_{j_s}$ or if $(x_{i_1} \cdots x_{i_r} = x_{j_1} \cdots x_{j_s})$ and $(y_{i_1} \cdots y_{i_r} < y_{j_1} \cdots y_{j_s})$. This defines a linear ordering on the facets of $\Delta(\lambda)$ for each $\lambda \in \Lambda$. We shall see that this ordering defines a (non-pure) shelling of $\Delta(\lambda)$ and hence determines the homology of $\Delta(\lambda)$ in the most explicit manner. The algebraic property responsible for our shelling is that (2.4) lifts to a quadratic initial ideal of the non-commutative ideal J_Λ. More precisely, $in_\lessdot(J_\Lambda)$ is generated by $\{\, y_i y_j \,:\, |i-j| \geq 2 \text{ or } j \text{ odd} \,\}$.

Example 2.3. In Theorems 3.8 and 5.1 we shall derive formulas for the Poincaré series of $k[\Lambda]$. These results are well-known when Λ is graded, since in this case $k[\Lambda]$ is a Koszul algebra and the complexes $\Delta(\lambda)$ are Cohen-Macaulay. However even for affine toric surfaces the complexes $\Delta(\lambda)$ generally have homology in more than one dimension. (The appearance of homology in more than one dimension is an obstacle to use counting the Euler characteristic of the minimal free resolution in order to compute the Poincaré series; see also Remark 3.9).

We present an example: Let $n = 4$ and Λ the monoid generated by $(1,0), (1,1), (2,3), (5,8)$. The non-commutative Gröbner basis for J_Λ constructed above equals

$$\begin{aligned} &y_1y_3 \to y_2y_2y_2,\ y_3y_1 \to y_2y_2y_2,\ y_1y_4 \to y_3y_3y_2y_2,\ y_4y_1 \to y_3y_3y_2y_2, \\ &y_2y_4 \to y_3y_3y_3,\ y_4y_2 \to y_3y_3y_3,\ y_2y_1 \to y_1y_2,\ y_2y_3 \to y_3y_2,\ y_4y_3 \to y_3y_4. \end{aligned} \tag{2.6}$$

The smallest fiber with homology in two different dimensions appears for $\lambda = (8,9)$. Here $\Delta(\lambda)$ has 89 facets of dimensions ranging from 6 to 3. Starting with the unique standard monomial, the 89 facets are ordered by the term order $\lessdot$ as follows:

$$\begin{aligned} &y_3y_2y_2y_2y_2y_2y_2,\ y_2y_3y_2y_2y_2y_2y_2,\ y_2y_2y_3y_2y_2y_2y_2,\ \ldots, y_2y_2y_2y_2y_2y_2y_3, \\ &y_1y_3y_3y_2y_2y_2,\ y_1y_3y_2y_3y_2y_2,\ y_1y_3y_2y_2y_3y_2,\ \ldots, y_2y_2y_2y_3y_3y_1, \\ &y_1y_1y_3y_3y_3,\ y_1y_3y_1y_3y_3,\ \ldots, \underline{y_3y_1y_3y_1y_3},\ \ldots, y_3y_3y_3y_1y_1, \\ &y_1y_1y_2y_4,\ y_1y_2y_1y_4,\ \ldots, \underline{y_1y_4y_2y_1},\ \ldots, \underline{y_2y_1y_4y_1},\ \ldots, y_4y_1y_2y_1,\ y_4y_2y_1y_1. \end{aligned}$$

Every monomial in this list (except the first one) can be reduced to an earlier monomial via (2.6) by replacing a quadratic factor. Such an ordering is a shelling. Precisely three monomials in this list have the property that all their quadratic factors lie in $in_\lessdot(J_\Lambda)$. They are the three underlined monomials. Corollary 3.7 implies that $\Delta(\lambda)$ is homotopy equivalent to the wedge of spheres $\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2$.

Theorem 3.8 gives the following formula for the Poincaré series of Λ:

$$P_k^{k[\Lambda]}(t,\, z_1, z_2) \quad = \quad \frac{(1+tz_1)(1+tz_1z_2)(1+tz_1^2z_2^3)(1+tz_1^5z_2^8)}{1 - t^2z_1^3z_2^3 - t^2z_1^6z_2^8 - t^2z_1^6z_2^9 - t^3z_1^8z_2^{11} - t^3z_1^7z_2^9}$$

The coefficient of $z_1^{\lambda_1}z_2^{\lambda_2}$ in this series equals t^2 times the Poincaré polynomial of the simplicial complex $\Delta(\lambda)$. In particular, the coefficient of $z_1^8z_2^9$ is $2t^4 + t^5$.

III. Non-commutative Gröbner bases and non-pure shellings.

A monomial ideal $M \subset k[x_1, \ldots, x_n]$ is said to be *quasi-poset* if M is generated by quadrics and the following condition holds: If $x_ix_j \in M$ and $i < l < j$, then $x_lx_i \in M$ or $x_lx_j \in M$. This choice of nomenclature is natural:

Lemma 3.1. *A simplicial complex is the chain complex of a poset if and only if its Stanley-Reisner ideal is quasi-poset, after relabeling the variables if necessary.*

Proof. For any square-free quadratic monomial ideal M we can define a binary relation P on $\{x_1, \dots, x_n\}$ by setting $x_i <_P x_j$ whenever $i < j$ and $x_i x_j \notin M$. The relation P is transitive if and only if M is quasi-poset. Moreover, if M is the Stanley-Reisner ideal of a poset Q and $x_1 < x_2 < \dots < x_n$ is a linear extension of Q then $P = Q$. $\square$

We conclude that a quasi-poset monomial ideal is poset if and only if it is square-free. Naturally, there are many quasi-poset ideals which are not square-free. For instance, every quadratically generated Borel-fixed ideal is quasi-poset.

Let Λ be a submonoid of $\mathbb{N}^d$ with n minimal generators. We say that Λ is *quasi-poset* if there exists a term order $\prec$ on $k[x_1, \dots, x_n]$ such that the initial ideal $in_\prec(I_\Lambda)$ is quasi-poset. Assuming that this holds, we extend the term order $\prec$ to a term order $\lessdot$ on the non-commutative polynomial ring $k\langle y_1, \dots, y_n\rangle$ as follows:

$$\begin{aligned} &y_{i_1} y_{i_2} \cdots y_{i_r} \lessdot y_{j_1} y_{j_2} \cdots y_{j_s} \quad :\Longleftrightarrow \quad x_{i_1} x_{i_2} \cdots x_{i_r} \prec x_{j_1} x_{j_2} \cdots x_{j_s} \\ &\text{or} \quad (x_{i_1} \cdots x_{i_r} = x_{j_1} \cdots x_{j_s} \text{ and } y_{i_1} \cdots y_{i_r} \text{ is before } y_{j_1} \cdots y_{j_r} \text{ lexicographically}). \end{aligned}$$

(For an introduction to non-commutative term orders and Gröbner bases see [Mo].) We shall prove that the non-commutative ideal J_Λ has a quadratic Gröbner basis.

Theorem 3.2. *Let Λ be a quasi-poset monoid and $\prec$ a term order such that $in_\prec(I_\Lambda)$ is quasi-poset. If $\lessdot$ is the induced non-commutative term order, then*

$$in_\lessdot(J_\Lambda) \quad = \quad \langle \{ y_i y_j \mid j < i \} \cup \{ y_i y_j \mid i \le j \text{ and } x_i x_j \in in_\prec(I_\Lambda) \} \rangle.$$

Proof. The isomorphism $k\langle y_1, \dots, y_n\rangle / J_\Lambda \cong k[x_1, \dots, x_n]/I_\Lambda$ and our choice of term order $\lessdot$ imply that a monomial $y_{i_1} y_{i_2} \cdots y_{i_r}$ is not in $in_\lessdot(J_\Lambda)$ if and only if

$$i_1 \le i_2 \le \cdots \le i_r \quad \text{and} \quad x_{i_1} x_{i_2} \cdots x_{i_r} \notin in_\prec(I_\Lambda). \tag{3.1}$$

Since $in_\prec(I_\Lambda)$ is quasi-poset, the condition (3.1) is equivalent to

$$i_1 \le i_2 \le \cdots \le i_r \quad \text{and} \quad \{x_{i_1} x_{i_2}, x_{i_2} x_{i_3}, \dots, x_{i_{r-1}} x_{i_r}\} \cap in_\prec(I_\Lambda) \; = \; \emptyset. \tag{3.2}$$

The condition (3.2) means that $y_{i_1} y_{i_2} \cdots y_{i_r}$ is not in the ideal on the right hand side of the equation asserted in Theorem 3.2. $\square$

Remark 3.3. Gröbner bases in non-commutative rings are usually infinite sets, and Theorem 3.2 is more delicate than it may appear. Consider the monoid generated by $\alpha_1 = (2,0), \alpha_2 = (1,1), \alpha_3 = (0,2) \in \mathbb{N}^2$. Its toric ideal is $I_\Lambda = \langle x_1 x_3 - x_2^2 \rangle$. Let $\prec$ be any term order with $in_\prec(I_\Lambda) = \langle x_1 x_3 \rangle$, and let $\lessdot$ be the induced term order on $k\langle y_1, y_2, y_3\rangle$. Then $in_\lessdot(J_\Lambda)$ is minimally generated by the infinite set $\{y_3 y_1, y_2 y_1, y_3 y_2\} \cup \{y_1 y_2^m y_3 \,:\, m \ge 0\}$. Thus, even in this trivial example we get an infinite Gröbner basis for J_Λ unless we order the variables in a special way. $\square$

Our main goal in this section is to produce a shelling for the finite intervals $\Delta(\lambda)$ of a quasi-poset monoid Λ. We recall the definition of *non-pure shelling* due to Björner and Wachs [BW]: A linear ordering $<$ on the facets of any simplicial complex Δ is a *shelling order* if the following property holds:

> For any two facets $F' < F$ there exists a third facet $G < F$ such that $F' \cap F \subseteq G \cap F$ and $G \cap F$ has codimension 1 in F.

The topological significance of this condition is that Δ can be "built up" from its facets in the order $<$ while maintaining tight control of the homotopy type at each stage. A facet F of Δ is called *fully attached* if every boundary face of F (or equivalently every codimension 1 boundary face) is contained in some earlier facet.

Theorem 3.4 [BW, Theorems 3.4 and 4.1]**.** *Let $<$ be a shelling order on the facets of a simplicial complex Δ. Then Δ is homotopy equivalent to a wedge of spheres*

$$\bigvee_F \mathbb{S}^{\dim(F)}$$

where F runs over all fully attached facets of Δ.

We now return to our algebraic discussion regarding integer monoids Λ.

Theorem 3.5. *Let Λ be a submonoid of $\mathbb{N}^d$ with n generators and let $\lessdot$ be any term order on the free associative algebra $k\langle y_1, \ldots, y_n\rangle$. The following are equivalent:*

(1) *The initial ideal $in_{\lessdot}(J_\Lambda)$ is generated by quadratic monomials.*
(2) *For every $\lambda \in \Lambda$ the order $\lessdot$ on the non-commutative fiber of λ (starting with the standard monomial) gives a shelling order on the facets of $\Delta(\lambda)$.*

This construction gives a uniform rule for simultaneously shelling all finite intervals of Λ. The underlying term order $\lessdot$ ensures the following natural condition:

> If F' is a facet preceding F in the shelling of $\Delta(\lambda)$ then y_iF' comes before y_iF and $F'y_i$ comes before Fy_i in the shelling of $\Delta(\lambda+\alpha_i)$.

Proof. We identify the facets of $\Delta(\lambda)$ with the monomials in the non-commutative fiber of $\lambda \in \Lambda$. Two facets F and G intersect in a codimension 1 subface of F if and only if there exist two monomials E, H and a binomial $y_iy_j - y_{s_1}\cdots y_{s_r} \in J_\Lambda$ such that $F = Ey_iy_jH$ and $G = Ey_{s_1}\cdots y_{s_r}H$. The facet G comes before F in the proposed shelling order if and only if y_iy_j is the $\lessdot$-initial term of $y_iy_j - y_{s_1}\cdots y_{s_r}$.

(2) implies (1): Let F be a non-standard monomial. Pick any earlier monomial $F' \lessdot F$ in the same fiber. In the given shelling there exists a monomial $G \lessdot F$ such that $G \cap F$ is a codimension 1 subface of F. As discussed above, this means that F is divided by some quadratic monomial y_iy_j in $in_{\lessdot}(J_\Lambda)$.

(1) implies (2): Let $F' \lessdot F$ be any two facets of $\Delta(\lambda)$. Factor these monomials

$$F' \;=\; F'_1F'_2\cdots F'_l \qquad \text{and} \qquad F \;=\; F_1F_2\cdots F_l$$

where $F_i = F'_i$ modulo J_Λ. Since $F \lessdot F'$, there must be some i for which $F_i \lessdot F'_i$. Hence it suffices to assume $F = F_i, F' = F'_i$ and to prove the following:

> If F', F are two facets of $\Delta(\lambda)$ *with no partial products equal* and $F' \lessdot F$, then there exists $G \lessdot F$ with $G \cap F$ a codimension 1 subface of F.

If $F' \lessdot F$ then $F \in in_{\lessdot}(J_\Lambda)$. Since $in_{\lessdot}(J_\Lambda)$ is quadratic, some generator y_iy_j of $in_{\lessdot}(J_\Lambda)$ divides F. Write $F = Ey_iy_jH$ and choose a binomial $y_iy_j - y_{s_1}\cdots y_{s_r} \in J_\Lambda$ with initial term y_iy_j. The facet $G = Ey_{s_1}\cdots y_{s_r}H$ satisfies the requirement. $\square$

Combining Theorems 3.2, 3.4 and 3.5 we get the following result.

Corollary 3.6. *Let Λ be a quasi-poset monoid. Then, for all $\lambda \in \Lambda$, the simplicial complex $\Delta(\lambda)$ is shellable. In particular, the Betti numbers of k over the monoid algebra $k[\Lambda]$ do not depend on the characteristic of k.*

We next compute the Poincaré series of $k[\Lambda]$. Consider a non-commutative monomial $m = y_{i_1} y_{i_2} \cdots y_{i_r}$ of *length* r and *degree* $\lambda = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}$. It corresponds to an $(r-2)$-dimensional facet in $\Delta(\lambda)$. This facet is fully attached in the shelling order specified in Theorem 3.5 if and only if $y_{i_j} y_{i_{j+1}} \in in_{\lessdot}(I_\Lambda)$ for $j = 1, 2 \ldots, r-1$. We therefore call a monomial m in $k\langle y_1, \ldots, y_n \rangle$ *fully attached* if each quadratic factor of m lies in $in_{\lessdot}(J_\Lambda)$. Theorems 3.4 and 3.5 imply the following result.

Corollary 3.7. *Let Λ be a quasi-poset monoid and $\lambda \in \Lambda$. Then $\Delta(\lambda)$ is homotopy equivalent to the wedge of spheres*

$$\bigvee_m \mathbb{S}^{\mathrm{length}(m)-2}$$

where m runs over all fully attached non-commutative monomials of degree λ.

Theorem 3.8. *The Λ-graded Poincaré series (1.1) of k over a quasi-poset monoid algebra $k[\Lambda] \cong k[x_1, \ldots, x_n]/I_\Lambda$ coincides with the Λ-graded Poincaré series of k over $k[x_1, \ldots, x_n]/in_{\prec}(I_\Lambda)$, and equals the inverted Hilbert series*

$$\frac{1}{[\mathrm{Hilb}(k[x_1, \ldots, x_n]/in_{\prec}(I_\Lambda); \mathbf{x})]_{x_i \mapsto -t\mathbf{z}^{\alpha_i}}}. \tag{3.3}$$

Proof. The algebra $R = k[x_1, \ldots, x_n]/in_{\prec}(I_\Lambda)$ is a Koszul algebra because $in_{\prec}(I_\Lambda)$ is generated by quadratic monomials. Using Corollary 2.2 and Remark 3.9 below, this implies that the Poincaré series of R equals the inverted Hilbert series (3.3).

On the other hand, Fröberg [Fr] has shown that the non-commutative algebra

$$R\langle y_1, \ldots, y_n \rangle / \langle y_l^2, y_i y_j + y_j y_i \rangle_{x_l^2, x_i x_j \notin in_{\prec}(I_\Lambda)} \tag{3.4}$$

carries the structure of a multigraded minimal free resolution of k over R, where a monomial $m = y_{i_1} \cdots y_{i_r}$ has homological degree r. The quadratic generators of the presentation ideal in (3.4) form a Gröbner basis with respect to the term order $\lessdot$. To see this, form critical pairs and note that they are trivial by the quasi-poset hypothesis. We conclude that (3.4) is isomorphic as a multigraded R-module to

$$R\langle y_1, \ldots, y_n \rangle / \langle y_l^2, y_i y_j \rangle_{y_l^2, y_i y_j \notin in_{\lessdot}(J_\Lambda)} \tag{3.5}$$

The set of fully attached monomials is a free basis for (3.5) as a multigraded R-module, and therefore the Poincaré series of k over R equals

$$\sum_{m \text{ fully attached}} \mathbf{z}^{\deg(m)} \cdot t^{\mathrm{length}(m)}.$$

This series equals the Poincaré series of k over $k[\Lambda]$ by Corollary 3.7 and (1.2). $\square$

Theorem 3.8 gives a rational formula for the Poincaré series (1.1) of a quasi-poset monoid Λ. It remains an open problem whether (1.1) is rational for all monoids Λ. The following weaker result shows that the difficulty lies in controlling cancellations.

Remark 3.9. *For any submonoid Λ of $\mathbb{N}^d$ we have*

$$\sum_{\lambda\in\Lambda} \tilde{\chi}(\mathrm{Tor}_*^{k[\Lambda]}(k,k)_\lambda)\ \mathbf{z}^\lambda \;=\; \frac{1}{\mathrm{Hilb}(k[\Lambda];\mathbf{z})} \;=\; \frac{1}{\sum_{\lambda\in\Lambda}\mathbf{z}^\lambda} \tag{3.6}$$

where $\tilde{\chi}(\mathrm{Tor}_*^{k[\Lambda]}(k,k)_\lambda)$ is the (reduced) Euler characteristic

$$\sum_{i\geq -1} (-1)^i \dim_k \left(\mathrm{Tor}_i^{k[\Lambda]}(k,k)_\lambda\right). \tag{3.7}$$

Proof. The coefficient of $\mathbf{z}^\lambda$ in the right hand side of (3.6) equals the alternating sum of the face numbers of $\Delta(\lambda)$. This number coincides with the Euler characteristic of $\Delta(\lambda)$, and, by (1.2), it is equal to the alternating sum (3.7). □

IV. The ubiquity of poset monoids.

In this section we study graded monoids which possess a poset initial ideal.

Example 4.1. Suppose that the monoid algebra $k[\Lambda]$ is an *algebra with straightening law* (abbreviated *ASL*) over a poset P. The axioms of an ASL (see e.g. Section 7.1 in [BH]) stipulate that the toric ideal I_Λ is generated by *straightening relations*

$$x_i x_j - (\text{terms, each of which is divisible by a variable } <_P \text{ than } x_i \text{ and } x_j),$$

where $\{x_i, x_j\}$ runs over incomparable pairs in P. If $\succ$ is the reverse lexicographic term order induced by any linear extension of $<_P$ then $in_\prec(I_\Lambda)$ equals the Stanley-Reisner ideal of P. The prototype of a toric ASL arises from the following construction (see [Hi]): Let $P = J(R)$ be any distributive lattice, consisting of the order ideals of a poset R, and let Λ be the monoid of order preserving maps from R into the non-negative integers. Then I_Λ is generated by the relations $x_i \cdot x_j - (x_i \vee x_j)\cdot(x_i \wedge x_j)$, where $\wedge, \vee$ are the lattice operations. These ASL's include as special cases the coordinate rings of 2-by-2 determinantal varieties, and the toric deformations of flag varieties G/P and their Schubert subvarieties [Lak]. □

Throughout this section $\mathcal{A}$ denotes a configuration in $\mathbb{N}^d$ which is graded in the sense of Remark 2.1. Let Λ be the monoid spanned by $\mathcal{A}$ and write $I_\mathcal{A} := I_\Lambda$ and $k[\mathcal{A}] := k[\Lambda]$. We say that $\mathcal{A}$ *supports a poset* if $I_\mathcal{A}$ has an initial ideal which is the Stanley-Reisner ideal of a poset. If $\mathcal{A}_1 \subset \mathbb{N}^d$ and $\mathcal{A}_2 \subset \mathbb{N}^e$ then their *direct sum* is the configuration $\mathcal{A}_1 \oplus \mathcal{A}_2 := \{(a,b) \in \mathbb{N}^{d+e} : a \in \mathcal{A}_1, b \in \mathcal{A}_2\}$. The monoid algebra $k[\mathcal{A}_1 \oplus \mathcal{A}_2]$ is the *Segre product* of $k[\mathcal{A}_1]$ and $k[\mathcal{A}_2]$.

Theorem 4.2. *Let $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$ be graded configurations.*

1. *If $\mathcal{A}$ supports a poset and d is a positive integer, then $d\mathcal{A}$ supports a poset.*
2. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ support posets then so does their direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$.*

Proof. Let $(P, \leq)$ be any poset. We define its *d-th symmetric power* $(P^{(d)}, \preceq)$ as follows. The elements of $P^{(d)}$ are strings $x_1 x_2 x_3 \cdots x_d$ of d elements in P which form a chain $x_1 \leq x_2 \leq \cdots \leq x_d$. The partial order $\preceq$ on $P^{(d)}$ is defined by setting

$x_1x_2x_3\cdots x_d \le y_1y_2y_3\cdots y_d$ if $x_i \le_P y_i$ for all odd i and $y_i \le_P x_i$ for all even i. This relation is indeed transitive. A sequence of elements in $P^{(d)}$ is a chain

$$x_1x_2x_3\cdots x_d \preceq y_1y_2y_3\cdots y_d \preceq \cdots \preceq z_1z_2z_3\cdots z_d \tag{4.1}$$

if and only if the entries are sorted in the following "snake-like" pattern:

$$x_1 \le y_1 \le \cdots \le z_1 \le z_2 \le \cdots \le y_2 \le x_2 \le x_3 \le y_3 \le \cdots \le z_3 \le z_4 \le \cdots\cdots \tag{4.2}$$

Suppose that the Stanley-Reisner ideal of P is an initial ideal of $I_{\mathcal{A}}$. We shall prove that the Stanley-Reisner ideal of $P^{(d)}$ is an initial ideal of $I_{d\mathcal{A}}$. We identify the elements of P with the elements in $\mathcal{A}$. This induces a bijection between the elements $x_1x_2x_3\cdots x_d$ of $P^{(d)}$ and the elements $x_1+x_2+x_3+\cdots+x_d$ of $d\mathcal{A}$. We introduce a variable $T_{x_1x_2\cdots x_d}$ for each such element and we regard $I_{d\mathcal{A}}$ as an ideal in the polynomial ring in these variables. We claim that any monomial

$$T_{x_1x_2x_3\cdots x_d}T_{y_1y_2y_3\cdots y_d}\cdots T_{z_1z_2z_3\cdots z_d}, \tag{4.3}$$

can be rewritten uniquely modulo $I_{d\mathcal{A}}$ as a monomial satisfying (4.2). This is accomplished by the following algorithm: Suppose the monomial (4.3) does not satisfy (4.2). Along the inequality chain (4.2) there exists a pair which is either in the wrong order or incomparable. In the first case we switch the order and in the second case we replace the incomparable pair by the corresponding standard monomial modulo $I_{\mathcal{A}}$. That standard monomial is again quadratic (since $\mathcal{A}$ is graded), hence the resulting string is a new monomial of the form (4.3). That process will eventually terminate because in the first case the number of inversions along (4.2) decreases, and in the second case the monomial $x_1x_2\cdots x_dy_1y_2\cdots y_dz_1z_2\cdots z_d$ decreases in the given term order for $I_{\mathcal{A}}$.

For any incomparable pair $x_1x_2x_3\cdots x_d$, $y_1y_2y_3\cdots y_d$ of elements in $P^{(d)}$ the above algorithm returns two comparable elements $u_1u_2u_3\cdots u_d$ and $v_1v_2v_3\cdots v_d$. The corresponding quadratic binomial

$$T_{x_1x_2x_3\cdots x_d}T_{y_1y_2y_3\cdots y_d} - T_{u_1u_2u_3\cdots u_d}T_{v_1v_2v_3\cdots v_d} \tag{4.4}$$

is marked to have the first term as the "leading term". These marked quadratic binomials generate the ideal $I_{d\mathcal{A}}$, and the reduction relation modulo these marked binomials is Noetherian. By the result of [RS] there exists a term order $\prec$ on the polynomial ring in the variables $T_{x_1x_2\cdots x_d}$ which induces this marking, i.e., the left term of (4.4) is $\prec$-leading for any two incomparable pairs in $P^{(d)}$. Hence $in_{\prec}(I_{d\mathcal{A}})$ equals the Stanley-Reisner ideal of the d-th symmetric power poset $P^{(d)}$.

We now prove part (2) of the theorem. Let P_1, P_2 be posets and $\prec_1, \prec_2$ be term orders such that $in_{\prec_i}(I_{\mathcal{A}_i})$ is the Stanley-Reisner ideal of P_i for $i = 1, 2$. Consider the direct product of posets $P = P_1 \times P_2$. The elements (a, b) of P are identified with variables y_{ab} for the Segre ideal $I_{\mathcal{A}_1\oplus\mathcal{A}_2}$. Any monomial $y_{a_1b_1}y_{a_2b_2}\cdots y_{a_nb_n}$ can be rewritten uniquely modulo $I_{\mathcal{A}_1\oplus\mathcal{A}_2}$ as $y_{a'_1b'_1}y_{a'_2b'_2}\cdots y_{a'_nb'_n}$ where $a'_1 \le a'_2 \le \cdots \le a'_n$ in P_1 and $b'_1 \le b'_2 \le \cdots \le b'_n$ in P_2. Here $x_{a'_1}x_{a'_2}\cdots x_{a'_n}$ is the $\prec_1$-normal form of $x_{a_1}x_{a_2}\cdots x_{a_n}$, and $x_{b'_1}x_{b'_2}\cdots x_{b'_n}$ is the $\prec_2$-normal form of $x_{b_1}x_{b_2}\cdots x_{b_n}$. Moreover, this rewriting can be done by a sequence of quadratic moves ($n = 2$). By the same argument as above, there exists a term order $\prec$ such that $in_{\prec}(I_{\mathcal{A}_1\oplus\mathcal{A}_2})$ equals the Stanley Reisner ideal of $P_1 \times P_2$. $\square$

Corollary 4.3. *All Veronese varieties and all Segre varieties support posets.*

Proof:. The vertex set of a regular r-simplex supports the $(r+1)$-chain. The configurations obtained from these trivial examples by iterating the constructions of Theorem 4.2 are precisely the Veronese varieties and the Segre varieties. □

There is a big difference between parts (1) and (2) of Theorem 4.2 as far as lifting of term orders is concerned. In part (2) one simply takes $\prec$ as a lexicographic product of $\prec_1$ and $\prec_2$. This generalizes the familiar "stair case Gröbner basis" for the ideal of 2×2-minors of a matrix of indeterminates. On the other hand, in part (1) there seems to be no explicit construction of the required term order for $d\mathcal{A}$ from the given term order for $\mathcal{A}$. Even in the Veronese case (where $I_{\mathcal{A}}$ is the zero ideal) the Gröbner basis resulting from Theorem 4.2 (1) is generally not lexicographic, not even in the generalized sense of [St1]. The smallest example where the lexicographic property fails is the cubic Veronese embedding of $\mathbb{P}^4$ $(d=3, r=4,\ P = \text{a 5-chain})$.

The construction of the d-th symmetric power $P^{(d)}$ of a poset P specializes to the well-known construction of the interval poset $Int(P)$ when $d=2$. The *interval poset* $Int(P)$ is the set of all intervals $[x,y]$ in P ordered by inclusion. Hence $P^{(2)}$ is the order dual of $Int(P)$. It is well-known that the order complex of $Int(P)$ is homeomorphic to that of P; see [Wal]. The proof of Theorem 4.2 can be extended to give a similar proof that the order complex of $P^{(d)}$ is homeomorphic to that of P, for any poset P and any positive integer d.

Remark 4.4. Theorem 4.2 is false for non-graded configurations. Consider the configuration $\mathcal{A} = \{(5,1),(2,1),(1,2),(1,5)\}$ with toric ideal $I_{\mathcal{A}} = \langle \underline{x_1x_3} - x_2^3,\ \underline{x_1x_4} - x_2^2x_3^2,\ \underline{x_2x_4} - x_3^3\rangle$. It supports a poset via the underlined initial terms. But neither $2\mathcal{A}$ nor $\mathcal{A}\oplus\mathcal{A}$ support a poset because their ideals $I_{2\mathcal{A}}$ and $I_{\mathcal{A}\oplus\mathcal{A}}$ have no quadratic initial ideals at all. Similarly, $I_{2\mathcal{A}}$ has a minimal generator in degree $(10,11)$. There are three monomials of that degree: $y_{13}y_{33}^2, y_{22}^2y_{34}, y_{22}y_{23}x_{34}$. Each initial ideal of $I_{2\mathcal{A}}$ has one of these three cubic monomials as a minimal generator. For instance, $I_{\mathcal{A}\oplus\mathcal{A}}$ has a minimal generator of degree $(5,19,9,9)$. There are four monomials of that degree: $y_{42}^2y_{44}y_{32}^2$, $y_{42}^3y_{34}y_{32}$, $y_{43}^3y_{33}y_{31}$, $y_{41}y_{43}^2y_{33}^2$. Each initial ideal of $I_{\mathcal{A}\oplus\mathcal{A}}$ has one of these quintic monomials as a minimal generator. □

If $\mathcal{A}$ spans a graded monoid of rank d then $Q = conv(\mathcal{A})$ is a $(d-1)$-dimensional polytope. The results in Chapter of [St2] give the following reformulation.

Remark 4.5. *A graded configuration $\mathcal{A} \subset \mathbb{N}^d$ supports a poset if and only if the polytope $Q = conv(\mathcal{A})$ has a unimodular regular triangulation Δ with vertices in $\mathcal{A}$ such that Δ is the chain complex of a poset. In this case $\mathcal{A} = Q \cap \mathbb{N}^d$.*

In the remainder of this section we study the special case $d=3$. We thus assume that Q is a planar lattice polygon and $\mathcal{A}$ the set of all lattice points in Q.

Proposition 4.6. *A lattice polygon Q supports a poset if and only if there exists a triangulation Δ of Q having the following properties:*

(1) *Δ is a regular triangulation which uses all lattice points in Q.*
(2) *The vertices of Δ can be properly 3-colored, i.e. so that no two vertices connected by an edge have the same color.*

(3) *In the proper 3-coloring of Δ, one of the colors only appears on internal vertices having degree 4 and on boundary vertices of degree 2 or 3.*

Proof. Suppose that $in_{\prec}(I_{\mathcal{A}})$ is the Stanley-Reisner ideal of a poset P. Then the term order $\prec$ gives rise to a regular triangulation $\Delta = \Delta_{\prec}(I_{\mathcal{A}})$ of $\mathcal{A}$, and the square-freeness of $in_{\prec}(I_{\mathcal{A}})$ implies that Δ is unimodular (Corollary 8.9 in [St2]). Since Δ is pure 2-dimensional, the poset P is graded of rank 2, and the labeling of vertices by their rank in P gives a proper 3-coloring of Δ. This explains (1) and (2). To see (3) use the transitivity of the partial order P: the vertices in the middle rank of P cannot have edges to two elements of different rank, unless those other two elements also share an edge. This means that a vertex in the middle rank of P cannot be an internal vertex of degree 5 or more (think about the coloring of its neighbors) nor can it be a boundary vertex with degree 4 or more. Since an internal vertex of degree 2 or smaller is impossible in a triangulation, and also degree 3 is impossible because of 3-colorability, we deduce (3).

Conversely, suppose such a triangulation Δ satisfying (1),(2), and (3) exists. It corresponds to a term order $\prec$, and, since Δ is unimodular, $in_{\prec}(I_{\mathcal{A}})$ is generated by the monomials which do not lie on a face of Δ. We construct a poset P on $\{x_1, \dots, x_n\}$ as follows. Let the three colors in (2) be red, blue, and yellow, with blue the color specified in (3). These are the three ranks of P. We set $x_1 < x_2$ in P if x_1, x_2 are connected by an edge of Δ and either

- x_1 is red, x_2 is blue, or
- x_1 is red, x_2 is yellow, or
- x_1 is blue, x_2 is yellow.

Property (3) ensures that no other order relations will be implied by transitivity. □

Using the previous proposition, we can derive sufficient conditions for a lattice polygon to support a poset. Thinking of $\mathcal{A}$ as lying in the integer lattice $\mathbb{Z}^2$ of the x, y-plane, say that $\mathcal{A}$ is *integrally framed* if for every integer i, the horizontal line $y = i$ intersects the convex hull of A in a line segment having integral endpoints.

Proposition 4.7. *If $\mathcal{A}$ is integrally framed then it supports a poset.*

Proof. By Proposition 4.6, it suffices to construct a triangulation Δ having properties (1)-(3) when $\mathcal{A}$ is integrally framed. Suppose the orthogonal projection of $\mathcal{A}$ onto the y-axis has points $0, 1, 2, \dots, s$. The construction of Δ begins by drawing in all of the edges between adjacent vertices on the boundary of the convex hull of $\mathcal{A}$. Then add in all edges between vertices of $\mathcal{A}$ whose difference vector is horizontal. Lastly add in the "zig-zag" of edges of Δ which connect the following vertices in sequence: the rightmost point of $\mathcal{A}$ lying on the line $y = 0$, the leftmost lying on $y = 1$, the rightmost lying on $y = 2$, the leftmost lying on $y = 3$, etc...

There is a unique way to complete this to a unimodular triangulation of $\mathcal{A}$: within each triangle of the picture so far connect the apex to all points along the base. This triangulation is easily seen to be regular, and it is unimodular by construction. It is 3-colorable by the following scheme: Color the vertices on the zig-zag alternately yellow, red, yellow, red etc. Then color the rest of the vertices along any horizontal line $y = i$ alternating red and blue if i is even, or alternating yellow and blue if i is odd. This gives a proper 3-coloring. The blue vertices always have degree 4 when they are internal and degree at most 3 when they lie on the boundary. □

An example of a configuration $\mathcal{A}$ which is *not* integrally framed is

$$(4.5) \qquad \mathcal{A} \quad = \quad \begin{array}{cccc} & (1,3) & & \\ & (1,2) & (2,2) & (3,2) \\ (0,1) & (1,1) & (2,1) & \\ & & (2,0) & \end{array}$$

This configuration does not support a poset. Define the *width* of $P = conv(\mathcal{A})$ to be the minimum cardinality of $\phi(\mathcal{A})$ for any linear functional $\phi : \mathbb{Z}^2 \to \mathbb{Z}$. The configuration in (4.5) has width 4. This is smallest possible by the next proposition.

Proposition 4.8. *Let P be a lattice polygon with at least 4 points on the boundary of its convex hull and width at most 3. Then P supports a poset.*

We omit the proof of Proposition 4.8; it is an explicit elementary construction.

VI. Minimal free resolution.

The results in Section 3 raise the question how to construct a minimal free resolution of k over a quasi-poset monoid algebra $k[\Lambda]$. In this section we answer this question for the special case studied in [LS]. We shall construct an explicit minimal free resolution $\mathbb{F}$ of the residue field k over $k[\Lambda]$ for a normal 2-dimensional submonoid Λ of $\mathbb{N}^2$. Let $\alpha_1, \ldots, \alpha_n$ denote the minimal generators of Λ and assume (2.3).

The toric ideal I_Λ has a minimal Gröbner basis of this form:

$$(5.1) \qquad \underline{x_i x_{i+j}} - x_{i+1} x_{i+j-1} \cdot \prod_{p=i+1}^{i+j-1} x_p^{c_p}, \qquad 1 \leq i \leq n-2, \ 2 \leq j \leq n-i.$$

The underlined monomials span the initial ideal $in_\prec(I_\Lambda)$. A minimal free resolution of k over $k[x_1, \ldots, x_n]/in_\prec(I_\Lambda)$ is given in [Fr, Theorem in §3]. Our construction lifts this resolution to $k[\Lambda]$ using the specific relations (5.1). We first compute the total Betti numbers of $k[\Lambda]$. The following result improves Corollary 2.20 in [LS].

Theorem 5.1. *Let Λ be a normal 2-dimensional monoid with n generators. Then*

$$\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k,k) \quad = \quad (n-2)^{i-2} \cdot (n-1)^2 \qquad \textit{for} \ \ i \geq 2.$$

Proof. By Theorem 3.8, the (ungraded) Poincaré series of $k[\Lambda]$ equals

$$(5.2) \quad \sum_{i=0}^{\infty} \Big(\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k,k) \Big) \cdot t^i \quad = \quad \frac{1}{[\mathrm{Hilb}(k[x_1, \ldots, x_n]/in_\prec(I_\Lambda); \mathbf{x})]_{\, x_i \mapsto -t}}.$$

Using the fact that $in_\prec(I_\Lambda)$ is the Stanley-Reisner ideal of a shellable 1-dimensional ball with $n-1$ facets, we evaluate the right hand side of (5.2) as follows:

$$(5.3) \quad \frac{(1+t)^2}{1-(n-2)\cdot t} \quad = \quad 1 \ + \ n \cdot t \ + \ \sum_{i=2}^{\infty} (n-2)^{i-2} \cdot (n-1)^2 \cdot t^i. \qquad \square$$

We now construct the minimal free resolution $\mathbb{F}$ of k over $k[\Lambda]$. We define $\mathbb{F}$ to be the free $k[\Lambda]$-module $k[\Lambda] \otimes_k Q$, where Q is the non-commutative k-algebra

$$Q \quad := \quad k\langle y_1, \ldots, y_n\rangle / \big(\langle y_i y_{i+1} + y_{i+1} y_i \rangle_{1 \le i \le n-1} \; + \; \langle \, y_j^2 \, \rangle_{1 \le j \le n} \big).$$

It can be seen that Hilbert series of Q equals (5.3), for instance, by the argument given after equation (3.4). We say that a y-monomial $y_{i_1} y_{i_2} \ldots y_{i_p} \in Q$ is *standard* if y_{i_1} is the smallest variable that can be factored to the left (modulo the relations defining Q) and $y_{i_2} \ldots y_{i_p}$ is standard. There are precisely $(n-2)^{i-2}(n-1)^2$ standard y-monomials of degree i; they form a basis for the free $k[\Lambda]$-module $\mathbb{F}_i$. Throughout this section the letter m denotes a standard y-monomial.

We next define a differential d on $\mathbb{F}$. For y-monomials of degree ≤ 2 we set

$$\begin{aligned} d(y_s) \;&:=\; x_s, \\ d(y_i y_{i+1}) \;&:=\; -d(y_{i+1} y_i) \;:=\; x_i\, y_{i+1} - x_{i+1}\, y_i, \\ d(y_i y_{i+j}) \;&:=\; x_i\, y_{i+j} \,-\, c_{ij}\, y_{i+j-1}, \\ d(y_{i+j} y_i) \;&:=\; x_{i+j}\, y_i \,-\, c'_{ij}\, y_{i+1}, \end{aligned} \tag{5.4}$$

where $1 \le s \le n$, $1 \le i \le n-2$ and $2 \le j \le n-i+2$. The coefficients $c_{ij} = x_{i+1} \prod_{p=i+1}^{i+j-1} x_p^{c_p}$ and $c'_{ij} = x_{i+j-1} \prod_{p=i+1}^{i+j-1} x_p^{c_p}$ are uniquely determined by (5.1).

Let $y_i y_j y_l\, m$ be an arbitrary standard y-monomial of degree ≥ 3. We define

$$d(y_i y_j y_l\, m) \quad := \quad \begin{cases} d(y_i y_l)\, y_j\, m & \text{if } i > l = j+1 \\ d(y_i y_j)\, y_l\, m & \text{otherwise.} \end{cases} \tag{5.5}$$

Finally we extend the action of the differential d to all of $\mathbb{F}$ by $k[\Lambda]$-linearity. The following theorem is the main result in this section.

Theorem 5.2. *The complex* $(\mathbb{F}, d)$ *is a minimal free resolution of* k *over* $k[\Lambda]$.

We shall prove Theorem 5.2 in a sequence of three lemmas:

Lemma 5.3. $(\mathbb{F}, d)$ *is a complex.*

Proof. In order to simplify the notation we consider $(\bar{\mathbb{F}}, \bar{d}) = (\mathbb{F}, d)/\langle x_i - 1 \rangle_{1 \le i \le n}$. Since $(\mathbb{F}, d)$ is multigraded and $k[\Lambda]$ is one-dimensional (over k) in each multidegree, it suffices to show that $\bar{d}^2$ annihilates all standard y-monomials. The action of $\bar{d}$ on a standard y-monomial M depends only on the arrangement of the three leftmost variables in M. Our proof splits into seven possible cases for these arrangements.

Case 1. If M has degree ≤ 2 then $\bar{d}^2(M) = 0$ by (5.4) and the relations (5.1).

In the remaining cases $degree(M) \ge 3$ and i, j, q, p are suitable positive integers.

Case 2. $M = y_i y_{i+j} y_{i+j+q} m$. Then

$$\begin{aligned} \bar{d}^2(M) \quad &= \quad \bar{d}(y_{i+j} y_{i+j+q} m - y_{i+j-1} y_{i+j+q} m) \\ &= \quad y_{i+j+q} m - y_{i+j+q-1} m - y_{i+j+q} m + y_{i+j+q-1} m \quad = \quad 0 \end{aligned}$$

Case 3. $M = y_i y_{i+j} y_{i+j-2} m$. Note that y_{i+j-1} cannot be the first variable of m since it commutes with $y_{i+j} y_{i+j-2}$ and M is standard. Hence

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_{i+j} y_{i+j-2} m - y_{i+j-1} y_{i+j-2} m) \\ &= y_{i+j-2} m - y_{i+j-1} m + y_{i+j-1} m - y_{i+j-2} m \quad = \quad 0 \end{aligned}$$

Case 4. $M = y_i y_{i+j} y_{i+j-p} m$, where $p \geq 3$. Then $\bar{d}^2(M) = \bar{d}(y_{i+j} y_{i+j-p} m - y_{i+j-1} y_{i+j-p} m)$. This evaluates to $y_{i+j-p} m - y_{i+j-p+1} m - y_{i+j-p} m + y_{i+j-p+1} m = 0$ if m does not start with $y_{i+j-p+1}$. Otherwise we write $m = y_{i+j-p+1} \tilde{m}$ and

$$\bar{d}^2(M) \quad = \quad \bar{d}(y_{i+j} y_{i+j-p+1}) y_{i+j-p} \tilde{m} \; - \; \bar{d}(y_{i+j-1} y_{i+j-p+1}) y_{i+j-p} \tilde{m} \quad = \quad 0.$$

Case 5. $M = y_{i+j} y_i y_{i+q} m$, where $q \geq 2$. Then

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_i y_{i+q} m - y_{i+1} y_{i+q} m) \\ &= y_{i+q} m - y_{i+q-1} m - y_{i+q} m + y_{i+q-1} m \quad = \quad 0 \end{aligned}$$

Case 6. $M = y_{i+j} y_i y_{i+1} m$, where $j \geq 2$. Here m cannot start with y_{i+1}; therefore

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}\big(\bar{d}(y_{i+j} y_{i+1}) y_i m)\big) \\ &= \bar{d}\big(y_{i+1} y_i m - y_{i+2} y_i m\big) \\ &= y_i m - y_{i+1} m - y_i m + y_{i+1} m \quad = \quad 0 \end{aligned}$$

Case 7. $M = y_{i+j} y_i y_{i-p} m$, where $p \geq 2$. Note that if $p = 2$ then m cannot start with y_{i-1} because y_{i-1} commutes with $y_i y_{i-2}$ and M is standard. We have $d^2(M) = \bar{d}(y_i y_{i-p} m - y_{i+1} y_{i-p} m)$. This evaluates to $y_{i-p} m - y_{i-p+1} m - y_{i-p} m + y_{i-p+1} m = 0$ if m does not start with y_{i-p+1}. Otherwise write $m = y_{i-p+1} \tilde{m}$ and

$$\bar{d}^2(M) \quad = \quad d(y_i y_{i-p+1})\, y_{i-p} \tilde{m} - d(y_{i+1} y_{i-p+1})\, y_{i-p} \tilde{m} \quad = \quad 0. \qquad \square$$

The complex $\mathbb{F}$ is spanned as a k-vector space by monomials in x and y. The commutative x-part obeys the relations in $k[\Lambda]$ and the non-commutative y-part obeys the relations in Q. We call such a monomial *spanning* if it has the form $x_i^c x_{i+1}^e y_{i+1} y_j m$ where either $y_{i+1} y_j m$ is standard or $i = j$ and $y_i y_{i+1} m$ is standard.

Lemma 5.4. *The set of spanning monomials spans the k-vector space $\mathbb{F}/\mathrm{Im}(d)$.*

Proof. Every monomial in $\mathbb{F}_{\geq 2}$ can be written in the form $M = x_i^c x_{i+1}^e y_r y_s m$ where $y_r y_s m$ is standard. If $c = e = 0$ then M is spanning. We therefore assume $c > 0$. We set $X(M) := i$ and $Y(M) := r$. Thus M is spanning if and only if $Y(M) - X(M) = 1$. We will show that every non-spanning monomial M is congruent modulo $\mathrm{Im}(d)$ to another monomial M' which is closer to being spanning. Depending whether $X(M) - Y(M)$ is positive, zero or negative, we use one of the following reduction procedures:

Negative reduction: Suppose $M = x_i^w x_{i+1}^v y_{i+j} y_z m$ with $j \geq 2$. Set

$$M' \quad := \quad M - d(x_i^{w-1} x_{i+1}^v y_i y_{i+j} y_z m).$$

Rewriting M' in standard form, we see that $Y(M) > Y(M')$ and $X(M) \leq X(M')$. Thus M' is congruent to M modulo $\mathrm{Im}(d)$ and $Y(M) - X(M) > Y(M') - X(M')$.

Positive reduction 1: Suppose $M = x_{i+j}^{w} x_{i+j+1}^{v} y_i y_z m$, $z \neq i+1$ and $j > 0$. Set $M' := M - d(x_{i+j}^{w-1} x_{i+j+1}^{v} y_{i+j} y_i y_z m)$. Rewriting M' in standard form, we find $Y(M) < Y(M')$ and $X(M) \geq X(M')$, and hence $X(M) - Y(M) > X(M') - Y(M')$.

Positive reduction 2: Suppose $M = x_{i+j}^{w} x_{i+j+1}^{v} y_i y_{i+1} m$ with $j > 0$. Set $M' := -M - d(x_{i+j}^{w-1} x_{i+j+1}^{v} y_{i+j} y_{i+1} y_i m)$. It can be shown that $X(M) - Y(M) > X(M') - Y(M')$.

Zero reduction: Suppose $M = x_i^w x_{i+1}^v y_i y_z m$, where $v > 0$ and $z \neq i+1$. Then $M' := d(x_i^w x_{i+1}^{v-1} y_i y_{i+1} y_z m) + M = x_i^{w+1} x_{i+1}^{v-1} y_{i+1} y_z m$ is a spanning monomial.

Starting with any monomial M in $\mathbb{F}_{\geq 2}$, we can use the four types of reduction presented above to replace M by a spanning monomial modulo $\mathrm{Im}(d)$. $\square$

Lemma 5.5. *The complex $(\mathbb{F}, d)$ is exact.*

Proof. Consider any non-zero Λ-homogeneous k-linear combination of spanning monomials,

$$N \quad = \quad \sum_s \; c_s \cdot x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s .$$

To prove Lemma 5.5, we must show that $d(N) \neq 0$. We shall assume that each monomial in N is written so that the differential acts on the first two y-variables (to obtain this we switch the second and third y-variable if necessary). We call z_s the *order* of the term $x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s$. Setting $u = \min\{z_s\}$, we can write

$$N \quad = \quad c \cdot x_q^w x_{q+1}^v y_{q+1} y_u m \; + \; \text{other terms of order} \geq u.$$

Since $k[\Lambda]$ is one-dimensional in each multidegree, and N is multihomogeneous, and each term of N is spanning, we find that there is only one term in N which has order u and ends on m.

Suppose that $d(N) = 0$. The term $L := c x_q^w x_{q+1}^{v+1} y_u m$ appears in $d(N)$ and must cancel. Since N contains no spanning monomials of order $u-1$, a scalar multiple of L must appear in $d(x_h^f x_{h+1}^g y_{h+1} y_{u+1} m)$ for some f, g, h. This is only possible if $u > q \geq h$, and, in view of $y_{q+1} y_u \neq 0$, we conclude $u \geq q+2$. But then the spanning monomial $c x_q^w x_{q+1}^v y_{q+1} y_u m$ contributes a term $\mathbf{x}^{\mathbf{p}} y_{u-1} m$ to the expansion of $d(N) = 0$. This term $\mathbf{x}^{\mathbf{p}} y_{u-1} m$ appears in the differential of another spanning monomial from N. Such a spanning monomial must have order u and end on m. Hence N contains two or more distinct spanning monomials of order u and ending on m. This is a contradiction, and the proof is complete. $\square$

This completes the proof of Theorem 5.2 since $(\mathbb{F}, d)$ is minimal by construction.

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