Lifting Standard Bases in Filtered Structures

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Abstract

In this paper we further study the theory of standard bases in filtered structures as introduced by Robbiano. In particular, we study the situation in which a commutative ring A is filtered by commutative ordered monoids Δ and Γ in such a way that the Γ filtration refines the Δ filtration. We use these results to determine conditions under which a Δ standard basis can be lifted to a Γ standard basis. Finally, we interpret these results in three situations where the refinements are given by changing the status of the variables in a polynomial ring, refining a partial order on the monomials in a polynomial ring, and refining a filtration by an I-adic filtration for an ideal I in the ring A.

1 Introduction

The concept of a *Gröbner basis* in polynomial rings as defined originally by Buchberger (see [5] and [6]) has been extended to more general settings by many authors (see [4], [8], and [15]) where such bases are called *standard bases*. The notions of Gröbner and standard bases are dual to each other. Indeed, while Gröbner bases are defined with respect to orders that are well-orderings, standard bases are defined for orders that are negative well-orderings (see Examples 2.2 (3)). Standard bases play a role in the study of local algebra similar to the role of Gröbner bases in polynomial rings (see for example [11]).

Much work has been done for the study and computation of Gröbner bases. However, standard bases have received less attention because of the inherent difficulty in computing such bases (this is due to the existence of infinite reduction that might make Buchberger's algorithm fail to terminate). Recently, with the introduction of the Tangent Cone Algorithm (see [10], [11]), there has been a renewed interest in the study of standard bases. Even though there is no general algorithm for computing standard bases, the Tangent Cone Algorithm can be applied in certain instances to compute standard bases (e.g. in the ring of algebraic power series and in the localization at a prime ideal of a coordinate ring).

The most general setting for the theory of standard bases is the one of *filtered structures* (see Definition 2.1) as originally defined by Robbiano in [15]. The filtration is indexed by a monoid Γ and induces a topology on the ring A. To guarantee that A is a topological ring, and, more generally, to guarantee that the theory works and is broad enough to contain the examples we

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present in Section 5, we need to impose a condition on the monoid Γ (Definitions 2.3 and 2.4). It is a generalization of Mora's inf-limited condition [12]. A standard basis is then defined in terms of the completion of the ring A with respect to the topology (Definition 2.6) or, equivalently, in terms of the associated graded ring (Theorem 2.8). The basic examples (see Examples 2.2, 2.7 and Section 5) of filtered structures that are of interest in this paper are polynomial rings with a total or a partial order on the monomials, power series rings, and rings filtered by an ideal I, where I satisfies $\bigcap_{n=1}^{\infty} I^n = 0$.

In Section 3, we analyze the situation in which a filtration given by a monoid Γ refines a filtration given by a monoid Δ on the same ring. The notion of refinement was originally introduced by Robbiano [15] when he defined double structures (Definition 3.1). A filtration given by Γ refines one given by Δ if there is an ordered monoid homomorphism $\alpha : \Gamma \longrightarrow \Delta$ which induces a refinement of the Δ topology by the Γ topology (Corollary 3.4). We concentrate our attention on inf-limited double structures; i.e., the filtered structures induced by Γ and Δ are both inf-limited. In Lemma 3.3 we give criteria for determining whether a given double structure is inf-limited.

We next consider the special case when the homomorphism given by $\alpha : \Gamma \longrightarrow \Delta$ is a split map. The double structures with such a property are called *split double structures*. We prove in Proposition 3.9 that if the associated graded ring $gr_\Gamma(A)$ of a double structure is finitely generated as an algebra over its homogeneous direct summand G_0, and if Γ is a group then the double structure is in fact a split double structure. If Γ is not a group, we show that the double structure is contained in a split double structure. We also show how to construct a split double structure from two filtrations on the same ring (under certain conditions). This construction can then be iterated. A special case of this construction is the filtration given by linear term orders defined by vectors $u_1, \ldots, u_n$ as described in [13, 15].

In Section 4 we study the following question: given that a ring A is filtered by a monoid Δ, and that the monoid Γ gives a refined filtration on A, under what conditions can we lift a Δ standard basis to a Γ standard basis? Robbiano [15] proved that if $\{a_1, \ldots, a_r\}$ is a standard basis with respect to a filtration, it is also a standard basis with respect to a coarser filtration (see Theorem 4.2). In the present paper we extend this idea and give equivalent conditions that relate the two standard bases (Theorem 4.4). We then give an algorithm that inputs a standard basis for an ideal K with respect to a filtration, and outputs a standard basis for the same ideal with respect to a finer filtration. This process is based on the fact that the associated graded ring with respect to the coarser filtration can be filtered by the finer filtration.

In the last section we interpret the results of Sections 3 and 4 in the case of our basic examples:

1. lifting Gröbner bases from $(k[y_1, \ldots, y_m])[x_1, \ldots, x_n]$ (polynomials are viewed in the x variables with coefficients polynomials in the y variables) to $k[y_1, \ldots, y_m, x_1, \ldots, x_n]$;
2. lifting standard bases with respect to partial orderings on the monomials to standard bases with respect to finer partial orderings in $k[x_1, \ldots, x_n]$ (this process can be iterated to obtain Gröbner bases);
3. lifting standard bases in the I-adic case when we refine the I-adic filtration by another ideal (this procedure can also be iterated).

Special instances of these results appear, for example, in [1], [2], and [10].

Finally, we note that even though we present an algorithm for the computation of standard bases, the primary purpose of this paper is to further develop the general theory of standard bases in filtered structures. The computational efficacy of the algorithm presented is uncertain.

2 Background

We first define a *filtered structure*. This definition is the same as the one given in [15] except, following [12], we use a monoid instead of a group. Throughout this paper we say that Γ is an *ordered monoid* provided that Γ is an additively written commutative monoid with a total order compatible with addition. The latter means that if $a, b, c \in \Gamma$ and $a < b$, then $a + c < b + c$. We note that these conditions imply that Γ is a cancellative monoid.

Definition 2.1 *Let A be a commutative ring with unity, and let Γ be an ordered monoid. Let $\mathcal{F}_\Gamma = \{F_\gamma A\}_{\gamma \in \Gamma}$ be a set of additive subgroups of A with the following properties:*

1. *for all $\gamma, \gamma' \in \Gamma$ with $\gamma < \gamma'$, we have $F_\gamma A \subseteq F_{\gamma'} A$;*
2. *for all $\gamma, \gamma' \in \Gamma$, $F_\gamma A \cdot F_{\gamma'} A \subseteq F_{\gamma+\gamma'} A$;*
3. *for every $a \in A, a \neq 0$, there exists a minimum $\gamma \in \Gamma$ such that $a \in F_\gamma A$. (We assume that the minimum γ for 1 is $\gamma = 0$.)*

The triple $\mathcal{A} = (A, \Gamma, \mathcal{F}_\Gamma)$ is called a filtered structure.

We associate with any filtered structure a graded ring in the following way. Set

$$gr_\Gamma(A) = \bigoplus_{\gamma \in \Gamma} F_\gamma A / F_\gamma^\circ A, \text{ where}$$

$$F_\gamma^\circ A = \bigcup_{\gamma' < \gamma} F_{\gamma'} A.$$

(If no such γ' exists, we set $F_\gamma^\circ A = \{0\}$.) The multiplication in $gr_\Gamma(A)$ is defined by the following multiplication on homogeneous elements:

$$(a + F_\gamma^\circ A)(a' + F_{\gamma'}^\circ A) = aa' + F_{\gamma+\gamma'}^\circ A.$$

It is then easily seen that $gr_\Gamma(A)$ is a graded ring.

It is convenient to consider an element $-\infty$, which we assume is not in Γ, and to define an order on $\Gamma \cup \{-\infty\}$ by $-\infty < \gamma$, for all $\gamma \in \Gamma$. Moreover we formally define $-\infty + \gamma = -\infty$ for all $\gamma \in \Gamma \cup \{-\infty\}$. We now define two functions on A. First define

$$v_\Gamma : A \longrightarrow \Gamma \cup \{-\infty\}$$

as follows: $v_\Gamma(0) = -\infty$ and if $a \neq 0$, $v_\Gamma(a)$ is the minimal $\gamma \in \Gamma$ such that $a \in F_\gamma A$ (this is thought of as the value of a). Second,

$$in_\Gamma : A \longrightarrow gr_\Gamma(A)$$

is defined by $in_\Gamma(0) = 0$ and if $a \neq 0$, $in_\Gamma(a) = a + F^\circ_\gamma A$, where $v_\Gamma(a) = \gamma$ (this is thought of as the initial term of a). We abbreviate

$$G_\gamma = F_\gamma A / F^\circ_\gamma A.$$

Then we have the following easily verifiable properties (see [15]).

A1 $in_\Gamma(a) \in G_{v_\Gamma(a)}$ for every $0 \neq a \in A$;

A2 $in_\Gamma(a) = 0$ if and only if $a = 0$;

A3 $Image(in_\Gamma) = \bigcup_{\gamma \in \Gamma} G_\gamma$;

For all $a, b \in A$,

A4 $v_\Gamma(ab) \leq v_\Gamma(a) + v_\Gamma(b)$;

A5 $v_\Gamma(ab) = v_\Gamma(a) + v_\Gamma(b) \Longrightarrow in_\Gamma(ab) = in_\Gamma(a) in_\Gamma(b)$;

A6 $v_\Gamma(ab) < v_\Gamma(a) + v_\Gamma(b) \Longrightarrow in_\Gamma(a) in_\Gamma(b) = 0$;

A7 $v_\Gamma(a+b) \leq max(v_\Gamma(a), v_\Gamma(b))$;

A8 $v_\Gamma(a+b) = v_\Gamma(a) = v_\Gamma(b) \Longrightarrow in_\Gamma(a+b) = in_\Gamma(a) + in_\Gamma(b)$;

A9 $v_\Gamma(a+b) < v_\Gamma(a) = v_\Gamma(b) \Longrightarrow in_\Gamma(a) + in_\Gamma(b) = 0$.

We now give four examples of filtered structures. They are well-known "generic" examples that have appeared in many places in the literature. They will be used throughout the paper to illustrate our theory.

Example 2.2

1. Let R be a commutative ring, and $A = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the ring R. We choose a term order, denoted $<_x$ on the monomials in $x_1, \ldots, x_n$. With this order we have the usual concepts of leading term ($lt_x()$), leading coefficient ($lc_x()$), leading monomial ($lm_x()$), and leading term ideal ($Lt_x()$). The order $<_x$ determines an order on the monoid $\Gamma = \mathbf{N}^n$. This order gives a natural Γ-filtration on A as follows: for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{N}^n$, we define $log(x_1^{\gamma_1} \cdots x_n^{\gamma_n}) = \gamma$ and for $\gamma \in \Gamma$,we set
$$F_\gamma A = \{f \in A \mid log(lm_x(f)) \leq \gamma\}.$$
It is easy to see that $\mathcal{A} = (A, \Gamma, \{F_\gamma A\}_{\gamma \in \Gamma})$ is a filtered structure. It is also clear that the associated graded ring $gr_\Gamma(A)$ is isomorphic to A.

2. Let $A = R[x_1, \ldots, x_n]$ be as in Example 1. As in [13] let $<$ be a partial order on $\mathbf{N}^n$ of linear type given by vectors $u_1, \ldots, u_s$ in $\mathbf{R}^n$. That is, an order on $\mathbf{N}^n$ defined as follows: for $\eta, \eta' \in \mathbf{N}^n$,
$$\eta < \eta' \Longleftrightarrow (\eta \cdot u_1, \ldots, \eta \cdot u_s) <_{lex} (\eta' \cdot u_1, \ldots, \eta' \cdot u_s),$$

where $<_{lex}$ is the usual lexicographic order (with larger entries on the left). These vectors generate a partial order, denoted $<_u$, on the monomials of $R[x_1, \ldots, x_n]$. Therefore we have the concepts of leading term ($lt_u()$) and leading term ideal ($Lt_u()$). Note that, since the order may not be a total order, we may not have a leading monomial or a leading coefficient, i.e. there may be more than one monomial in $lt_u()$. This partial order gives a Γ-filtration on A, where Γ is the submonoid of $\mathbf{R}^s$ (ordered by $<_{lex}$) consisting of all vectors $(\eta \cdot u_1, \ldots, \eta \cdot u_s)$ as η ranges through $\mathbf{N}^n$. This filtration is defined as follows: for $\gamma \in \Gamma$,

$$F_\gamma A = \{f \in A \mid (log(\mathbf{X}) \cdot u_1, \ldots, log(\mathbf{X}) \cdot u_s) \leq_{lex} \gamma,$$
$$\text{where } \mathbf{X} \text{ is any monomial that appears in } lt_u(f)\}.$$

It is easy to see that $\mathcal{A} = (A, \Gamma, \{F_\gamma A\}_{\gamma \in \Gamma})$ is a filtered structure. Of course, A is, in fact, graded by Γ, i.e. $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ where

$$A_\gamma = \{f \in A \mid (log(\mathbf{X}) \cdot u_1, \ldots, log(\mathbf{X}) \cdot u_s) = \gamma,$$
$$\text{for every monomial } \mathbf{X} \text{ that appears in } f\} \cup \{0\}.$$

Clearly, $F_\gamma A = \bigoplus\limits_{\gamma' \leq_{lex} \gamma} A_{\gamma'}$, $F_\gamma^\circ A = \bigoplus\limits_{\gamma' <_{lex} \gamma} A_{\gamma'}$ and $G_\gamma \cong A_\gamma$, so $gr_\Gamma(A) \cong A$. It was proved in [14] that Example 1 is a special case of this example.

3. Let R be a commutative ring, and let $A = R[[x_1, \ldots, x_n]]$ be the power series ring in n variables over the ring R. If $<$ is any partial order of linear type on monomials as in Example 2 such that Γ is well-ordered, we can define an order $\prec$ on the monomials of A by "reversing" the order $<$; i.e., we define $\mathbf{X}_1 \prec \mathbf{X}_2$ to mean $\mathbf{X}_1 > \mathbf{X}_2$. We will call the order $\prec$ the negative of $<$. For example, negative degree lexicographic on two variables gives $1 > x > y > x^2 > xy > y^2 > \cdots$. Then we have the usual concept of leading term ($lt_\prec()$), and leading term ideal ($Lt_\prec()$). This of course defines a filtration on A as follows: for $\gamma \in \Gamma$,

$$F_\gamma A = \{f \in A \mid (log(\mathbf{X}) \cdot u_1, \ldots, log(\mathbf{X}) \cdot u_s) \geq_{lex} \gamma,$$
$$\text{where } \mathbf{X} \text{ is any monomial that appears in } lt_\prec(f)\}.$$

It is easy to see that $\mathcal{A} = (A, \Gamma^\circ, \{F_\gamma A\}_{\gamma \in \Gamma^\circ})$ is a filtered structure, where Γ° is Γ with its order reversed. It is not generally true that $gr_\Gamma(A)$ is isomorphic to A. As an example, if the order is given by total degree (i.e., $s = 1$ and $u_1 = (1, \ldots, 1)$), then $gr_\Gamma(A)$ is isomorphic to the polynomial ring $R[x_1, \ldots, x_n]$.

4. Let A be a commutative ring, and let I be an ideal of A such that $\cap_{n=1}^\infty I^n = \{0\}$. Let $\Gamma = -\mathbf{N}$. Then I defines a Γ-filtration on A as follows: for $n \in -\mathbf{N}$,

$$F_n A = I^{-n},$$

where we define $I^0 = A$. Again, it is easy to see that $\mathcal{A} = (A, \Gamma, \{F_\gamma A\}_{\gamma \in \Gamma})$ is a filtered structure. The associated graded ring is

$$gr_\Gamma(A) = \bigoplus_{n \leq 0} I^{-n}/I^{-n+1}.$$

In this context we will always use the notation gr_I, in_I, and v_I instead of the corresponding notations with the subscript Γ. As an example, consider the case of $A = k[x_1, \ldots, x_n]$, where k is a field. Let $I = \langle x_1, \ldots, x_n \rangle$. Also let $u_1 = (-1, \ldots, -1)$ as in Example 2 with $s = 1$. Then the I-adic filtration on A is the same as the filtration given by u_1.

We now return to the general theory. We need one further condition in order to guarantee that the theory works and is sufficiently broad to contain the examples we present later on this paper. Generalizing the definition of Mora [12], we give

Definition 2.3 *For a monoid Γ and a subset $\Gamma_0 \subseteq \Gamma$, the pair (Γ, Γ_0) is said to be inf-limited provided that given any descending sequence $\gamma_1 > \gamma_2 > \cdots$ of elements of Γ_0, and $\gamma \in \Gamma_0, \gamma' \in \Gamma$ there is an n such that $\gamma' > \gamma + \gamma_n$.*

Definition 2.4 *Let $\mathcal{A} = (A, \Gamma, \{F_\gamma A\}_{\gamma\in\Gamma})$ be a filtered structure. We define*

$$\Gamma_0 = \{\gamma \in \Gamma \mid F_\gamma^\circ A \subsetneqq F_\gamma A\}.$$

We say that $\mathcal{A}_\Gamma$ is an inf-limited filtered structure provided that (Γ, Γ_0) is inf-limited.

We note that, since we have assumed that $v_\Gamma(1) = 0$, we have $0 \in \Gamma_0$. Thus in Definition 2.3 we may use $\gamma = 0$. Also, in case $\Gamma = \Gamma_0$, then Definition 2.3 is equivalent to Mora's definition of inf-limited (see [12]).

Now consider an inf-limited filtered structure $\mathcal{A} = (A, \Gamma, \{F_\gamma A\}_{\gamma\in\Gamma})$. We define a topology on A by using the collection $\{F_\gamma^\circ A\}_{\gamma\in\Gamma}$ as a basis of open sets for the element 0. Since for all $a \in A$, $a \neq 0$, $a \notin F_{v_\Gamma(a)}^\circ A$, we see that the topology is *separated*, i.e.

$$\bigcap_{\gamma\in\Gamma} F_\gamma^\circ A = \{0\}.$$

Moreover, if Γ_0 contains a least element $\gamma_0 \in \Gamma_0$, we see that $F_{\gamma_0}^\circ A = \{0\}$ (since every $a \neq 0$ in $F_{\gamma_0}^\circ A$ would satisfy $v_\Gamma(a) < \gamma_0$), and so the topology is discrete.

More generally, we have

Lemma 2.5 *The collection of open sets $\{F_\gamma^\circ A\}_{\gamma\in\Gamma_0}$ defines the same topology on A as $\{F_\gamma^\circ A\}_{\gamma\in\Gamma}$.*

Proof. First observe that for all $\gamma \in \Gamma$ we have

$$F_\gamma^\circ A = \bigcup_{\gamma'<\gamma} F_{\gamma'} A = \bigcup_{\substack{\gamma' < \gamma \\ \gamma' \in \Gamma_0}} F_{\gamma'} A.$$

(If no such γ' exists, then $F_\gamma^\circ A = \{0\}$.) The second equality follows from the fact that for every $0 \neq a \in F_\gamma^\circ A$, $v_\Gamma(a) < \gamma$ and $v_\Gamma(a) \in \Gamma_0$. It now suffices to show that each $F_{\gamma'} A$ in the second union above is a finite intersection of open sets $F_{\gamma''}^\circ A$ with $\gamma'' \in \Gamma_0$. We see this as follows. First note that

$$F_{\gamma'} A = \bigcap_{\substack{\gamma'' > \gamma' \\ \gamma'' \in \Gamma_0}} F_{\gamma''}^\circ A.$$

(If no such γ'' exists then $F_{\gamma'}A = A$). This is equal to a finite intersection because for any fixed $\gamma_1'' \in \Gamma_0$ with $\gamma_1'' > \gamma'$ there are only finitely many $\gamma'' \in \Gamma_0$ such that $\gamma_1'' > \gamma'' > \gamma'$, since (Γ, Γ_0) is inf-limited. $\square$

We further note that $\mathcal{A}$ inf-limited implies that A is a topological ring. To see this first, note that if Γ_0 contains a least element we are done. Otherwise the only issue is the continuity of multiplication, and for this it suffices to show that for all non zero $a \in A$ and $\gamma_0 \in \Gamma$, there is a $\gamma \in \Gamma$ such that $aF_\gamma^\circ A \subseteq F_{\gamma_0}^\circ A$, or that $v_\Gamma(a) + \gamma \leq \gamma_0$. To obtain γ, we let $\gamma_1 > \gamma_2 > \cdots$ be a descending sequence of elements in Γ_0, then there is an n such that $\gamma_0 > v_\Gamma(a) + \gamma_n$, since (Γ, Γ_0) is inf-limited, and we are done.

Since we are dealing with filtered structures in which there may be infinite descending chains $F_{\gamma_1}^\circ A \supset F_{\gamma_2}^\circ A \supset \cdots \supset F_{\gamma_n}^\circ A \supset \cdots$, the analog of the usual concepts of reduction may require "infinitely" many steps, and thus we need to define standard bases in terms of the completion $\hat{A}$ of A (see [12]). This is the usual completion of the topological space A with respect to the topology defined by $\{F_\gamma^\circ A\}_{\gamma\in\Gamma}$, which can be defined since $\mathcal{A}$ is assumed to be inf-limited. In this case it can be shown [12] that the functions v_Γ and in_Γ can be extended naturally to $v_\Gamma : \hat{A} \longrightarrow \Gamma \cup \{-\infty\}$ and $in_\Gamma : \hat{A} \longrightarrow gr_\Gamma(\hat{A})$. This is done in such a way that, if $0 \neq \hat{a} \in \hat{A}$ and $\{a_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence of elements of A converging to $\hat{a}$, then $v_\Gamma(\hat{a}) = v_\Gamma(a_n)$ and $in_\Gamma(\hat{a}) = in_\Gamma(a_n)$ for all n large enough. Thus the ring $\hat{A}$ can be filtered by Γ by defining $F_\gamma\hat{A} = \{\hat{a} \in \hat{A} \mid v_\Gamma(\hat{a}) \leq \gamma\}$ for every $\gamma \in \Gamma$. Thus $(\hat{A}, \Gamma, \{F_\gamma\hat{A}\}_{\gamma\in\Gamma})$ is an inf-limited filtered structure whose associated graded ring is $gr_\Gamma(\hat{A}) \cong gr_\Gamma(A)$. Also, if the topology is discrete, then $\hat{A} = A$, so in particular $\hat{A} = A$ whenever Γ_0 contains a least element. A more detailed description of these results can be found in [12].

In this setting, we can define *standard bases.*

Definition 2.6 *Let $\mathcal{A} = (A, \Gamma, \mathcal{F}_\Gamma)$ be an inf-limited filtered structure, and let K be an ideal of A. Let $a_1, \ldots, a_r$ be non-zero elements of K. We say that the set $\{a_1, \ldots, a_r\}$ is a standard basis (denoted std-basis) of K if for every non-zero element a of K there exist $\hat{c}_1, \ldots, \hat{c}_r \in \hat{A}$ such that*

$$a = \sum_{i=1}^{r} \hat{c}_i a_i, \text{ where}$$

$$v_\Gamma(a) = \max_{\hat{c}_i \neq 0}(v_\Gamma(a_i) + v_\Gamma(\hat{c}_i)).$$

To illustrate the above definition, we now go back to the four examples presented earlier.

Example 2.7

1. We first consider the polynomial ring $A = R[x_1, \ldots, x_n]$ as in Example 2.2 (1). Note that the topology induced by the filtration is discrete, since 0 is least in $\Gamma = \mathbf{N}^n$, so that $\hat{A} = A$. The concept of standard bases in this context is then the usual concept of Gröbner bases as introduced by Buchberger in [5] and [6] when R is a field, and by Moller in [9] for the more general situation of a commutative ring.

2. Now consider the filtration of $A = R[x_1, \ldots, x_n]$ given in Example 2.2 (2). It is not automatic that $\mathcal{A}$ is inf-limited (e.g. in the case of two variables, take $u_1 = (-1, 0), u_2 = (0, -1)$; then

$\Gamma = \Gamma_0 = -\mathbf{N} \times -\mathbf{N}$ ordered by lex) and so this must be assumed in order to apply the results of this paper. The topology need not be discrete, for example the I-adic topology in Example 2.2 (4) is usually not discrete. But, if the order is given by total degree, or more generally, if $x_i > 1$ for all i, then the topology is discrete, since this just says that $0 \in \Gamma_0$ is least. A std-basis for the total degree order is a so-called Macaulay basis.

3. We now consider the power series ring $A = R[[x_1, \ldots, x_n]]$ of Example 2.2 (3). If the order is a negative term order, then the concept of standard bases is the concept of Hironaka bases as introduced in [7]. For the more general case of partial orders, we refer the reader to [12].

4. To illustrate Example 2.2 (4), consider the special case where $A = \mathbf{Z}$, and $I = \langle m \rangle$, for some integer m. Then the completion of A is the usual m-adic integers which we denote $\hat{\mathbf{Z}}_m$. We first note that it is not generally true that the integer n is a std-basis for the ideal $K = n\mathbf{Z}$. For example, if $I = 6\mathbf{Z}$ and $K = 3\mathbf{Z}$, then $\{3\}$ is not a std-basis of K. In fact, for any integer $a \in K$, $\{a\}$ cannot be a std-basis for K with the I-adic filtration. This follows since if $\{a\}$ is a std-basis then, of course, $3|a$. Also, $3 \in K$ implies $3 = ca$ for some $c \in \hat{\mathbf{Z}}_6$ such that $0 = v_I(3) = v_I(c) + v_I(a)$, so that $v_I(a) = 0$. Moreover, $6 \in K$ and so $6 = da$ for some $d \in \hat{\mathbf{Z}}_6$ such that $-1 = v_I(6) = v_I(d) + v_I(a) = v_I(d)$. Since, $\hat{\mathbf{Z}}_6 = \hat{\mathbf{Z}}_2 \times \hat{\mathbf{Z}}_3$ with $\mathbf{Z}$ embedded on the diagonal (by the Chinese Remainder Theorem), we can write $d = (d_2, d_3)$ with $d_2 \in \hat{\mathbf{Z}}_2$ and $d_3 \in \hat{\mathbf{Z}}_3$. Then we have $6 = d_3 a$ and moreover $v_I(d) = -1$ implies that $3|d_3$ and so $3 \not| \, a$, a contradiction. We now observe that $\{3, 6\}$ is a std-basis for K with respect to the I-adic filtration, since for $k = 3b \in K$ with $2 \not| \, b$ then $k = 3b$ gives the desired expression, and if $2|b$ then $k = 6\frac{b}{2}$ gives the desired expression. Note that $\{3, 42\}$ is also a std-basis of K, since $6 = 42 \cdot \frac{1}{7}$ and $\frac{1}{7} \in \hat{\mathbf{Z}}_6$.

We now return to the general theory.

For a subset S of A, we define $gr_\Gamma(S)$ to be the ideal of $gr_\Gamma(A)$ generated by all $in_\Gamma(s), s \in S$.

The following theorem is essentially due to Mora ([12]). (For the sake of completeness, and since the proof is short, we will include it here.)

Theorem 2.8 *Let $\mathcal{A} = (A, \Gamma, \mathcal{F}_\Gamma)$ be an inf-limited filtered structure. Then the following statements are equivalent for an ideal K of A.*

1. *$a_1, \ldots, a_r$ is a std-basis of K;*
2. *$gr_\Gamma(\{a_1, \ldots, a_r\}) = gr_\Gamma(K)$.*

Proof. (1) $\Longrightarrow$ (2). Let $0 \neq a \in K$, then by assumption we may write

$$a = \sum_{i=1}^{r} \hat{c}_i a_i, \text{ where}$$

$$v_\Gamma(a) = \max_{\hat{c}_i \neq 0}(v_\Gamma(a_i) + v_\Gamma(\hat{c}_i)).$$

Then

$$in_\Gamma(a) = a + F^\circ_{v_\Gamma(a)}\hat{A} = \sum_{i=1}^{r}(\hat{c}_i a_i + F^\circ_{v_\Gamma(a)}\hat{A}) = \sum in_\Gamma(\hat{c}_i) in_\Gamma(a_i),$$

where the last sum is taken over all i for which $v_\Gamma(a) = v_\Gamma(a_i) + v_\Gamma(\hat{c}_i) = v_\Gamma(a_i\hat{c}_i)$. The last equality follows since, if $v_\Gamma(a) > v_\Gamma(a_i\hat{c}_i)$ then, $a_i\hat{c}_i + F^\circ_{v_\Gamma(a)}\hat{A} = 0$, and if $v_\Gamma(a) = v_\Gamma(a_i\hat{c}_i)$, then $a_i\hat{c}_i + F^\circ_{v_\Gamma(a)}\hat{A} = in_\Gamma(a_i)in_\Gamma(\hat{c}_i)$.

(2) $\Longrightarrow$ (1). Let $0 \neq a \in K$. Then by hypothesis, we may write $in_\Gamma(a) = \sum_{i=1}^r \bar{b}_{i,0}in_\Gamma(a_i)$, where $\bar{b}_{i,0} \in gr_\Gamma(A)$. Since $in_\Gamma(a)$ is homogeneous, we may assume that $\bar{b}_{i,0} = b_{i,0} + F^\circ_{\gamma_i}A$, where $b_{i,0} \in A$, $\gamma_i = v_\Gamma(b_{i,0})$, and $v_\Gamma(a) = v_\Gamma(b_{i,0}) + v_\Gamma(a_i)$ for all i such that $\bar{b}_{i,0} \neq 0$. Let $a^{(1)} = a - \sum b_{i,0}a_i$. Then $a^{(1)} \in F^\circ_{v_\Gamma(a)}A$, that is $v_\Gamma(a^{(1)}) < v_\Gamma(a)$. We now apply this construction to $a^{(1)}$ and continue inductively. Thus we have $a^{(n+1)} = a^{(n)} - \sum_{i=1}^r b_{i,n}a_i$, where $b_{i,n} \in A$, $v_\Gamma(a^{(n)}) = v_\Gamma(b_{i,n}) + v_\Gamma(a_i)$, and $v_\Gamma(a^{(n+1)}) < v_\Gamma(a^{(n)})$. Therefore $a = a^{(n+1)} + \sum_{i=1}^r (b_{i,0} + \cdots + b_{i,n})a_i$. Since $\mathcal{A}$ is inf-limited, we easily see that $a^{(n)}$ converges to 0. Also the sequence $(b_{i,0} + \cdots + b_{i,n})$ is a Cauchy sequence, since $b_{i,n}$ converges to 0, again by the inf-limited hypothesis, noting that $v_\Gamma(a^{(n)}) = v_\Gamma(b_{i,n}) + v_\Gamma(a_i)$. If we define $\hat{c}_i$ to be the limit of the sequence $(b_{i,0} + \cdots + b_{i,n})$, we get the desired result. □

To illustrate this result we return to Example 2.7 (4). We noted before that $\{3\}$ is not a std-basis of the ideal $K = 3\mathbf{Z}$ with respect to the filtration defined by the ideal $I = 6\mathbf{Z}$, whereas $\{3, 6\}$ and $\{3, 42\}$ are. It is instructive to look at the graded ring $gr_I(\mathbf{Z})$ to obtain this result. First note that $gr_I(\mathbf{Z}) = (\mathbf{Z}/6\mathbf{Z})[t]$ for some variable t, where the identification is given by $a + 6^{-v_I(a)+1}\mathbf{Z}$ corresponds to $(\frac{a}{6^{-v_I(a)}} + 6\mathbf{Z})t^{-v_I(a)}$. Then $gr_I(K) = \langle 3 + 6\mathbf{Z}, t\rangle = \langle 3 + 6\mathbf{Z}, (7 + 6\mathbf{Z})t\rangle$ and these two bases correspond to the standard bases given before.

Corollary 2.9 *Let $\mathcal{A} = (A, \Gamma, \mathcal{F}_\Gamma)$ be an inf-limited filtered structure. Let K be an ideal of A. If $\{a_1, \ldots, a_r\}$ is a std-basis of K, then $\{a_1, \ldots, a_r\}$ is also a std-basis of the completion $\hat{K}$ of K in $\hat{A}$. Moreover $\hat{K} = \langle a_1, \ldots, a_r\rangle \subseteq \hat{A}$.*

Proof. Since $gr_\Gamma(\hat{K}) = gr_\Gamma(K)$, the result follows from Theorem 2.8.□

Note. In general a std-basis of an ideal K need not be a basis of K (see [12]).

As defined in [15], a filtered structure $\mathcal{A} = (A, \Gamma, \mathcal{F}_\Gamma)$ is called noetherian if both the rings A and $gr_\Gamma(A)$ are noetherian. We note that if $\mathcal{A}$ is a noetherian filtered structure, then every ideal has a std-basis. This follows easily from Theorem 2.8.

3 Double Structures

The question we will investigate in Section 4 is that if A is filtered by a monoid Δ, and if the monoid Γ gives a refined filtration on A, can we lift a Δ-std-basis to a Γ-std-basis. In this section we will define formally what we mean by a refinement. That is, we take from Robbiano, [15], the following

Definition 3.1 *Let*

$$\mathcal{A}_\Gamma = (A, \Gamma, \mathcal{F}_\Gamma) \text{ and } \mathcal{A}_\Delta = (A, \Delta, \mathcal{F}_\Delta)$$

be two filtered structures over the ring A, where

$$\mathcal{F}_\Gamma = \{F_\gamma A\}_{\gamma\in\Gamma}, \text{ and } \mathcal{F}_\Delta = \{F_\delta A\}_{\delta\in\Delta}.$$

Let $\alpha:\Gamma \longrightarrow \Delta$ be an order preserving surjective monoid homomorphism such that

$$\alpha(v_\Gamma(a)) = v_\Delta(a) \text{ for every } 0 \neq a \in A.$$

We assume that $\Delta_0 \neq \{0\}$. The triple $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is called a double structure on A. If $\mathcal{A}_\Gamma$ and $\mathcal{A}_\Delta$ are both inf-limited filtered structures, then $\mathcal{U}_\alpha$ is called an inf-limited double structure.

We set

$$gr_\Gamma(A) = \bigoplus_{\gamma\in\Gamma} G_\gamma, \text{ where } G_\gamma = F_\gamma A/F^\circ_\gamma A,$$

$$gr_\Delta(A) = \bigoplus_{\delta\in\Delta} G_\delta, \text{ where } G_\delta = F_\delta A/F^\circ_\delta A,$$

and define v_Γ, in_Γ, and v_Δ, in_Δ for $\mathcal{A}_\Gamma$ and $\mathcal{A}_\Delta$ respectively.

Examples of such refinements and double structures are given in Section 5.

It may seem unnecessarily restrictive to assume that α is surjective in the definition of a double structure. But, we note that the "interesting" part of Γ is Γ_0. Since $v_\Gamma(a) = \gamma$ implies $v_\Delta(a) = \alpha(\gamma)$, we see that $\alpha(\Gamma_0) = \Delta_0$ even without the assumption that α is surjective. So making the assumption that Γ and Δ contain no superfluous elements (i.e., Γ_0 and Δ_0 generate Γ and Δ respectively) it would follow that α is surjective.

Lemma 3.2 *Let $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ be a double structure. Then for all $\delta \in \Delta$,*

$$F_\delta A = \bigcup_{\alpha(\gamma)=\delta} F_\gamma A,$$

and

$$F^\circ_\delta A = \bigcap_{\alpha(\gamma)=\delta} F^\circ_\gamma A.$$

Proof. Let $0 \neq a \in F_\delta A$. Suppose $v_\Delta(a) < \delta$. If $\gamma \in \Gamma$ is such that $\alpha(\gamma) = \delta$ and $v_\Gamma(a) \geq \gamma$ then applying α we have a contradiction. Thus $a \in \bigcap_{\alpha(\gamma)=\delta} F^\circ_\gamma A$. Suppose $v_\Delta(a) = \delta$. Then $\alpha(v_\Gamma(a)) = \delta$ and so $a \in \bigcup_{\alpha(\gamma)=\delta} F_\gamma A$. Conversely, suppose that $a \in F_\gamma A$, where $\alpha(\gamma) = \delta$. Then, $v_\Gamma(a) \leq \gamma$ and so $v_\Delta(a) \leq \delta$, thus $a \in F_\delta A$. Finally, if $a \in F^\circ_\gamma A$ for all γ such that $\alpha(\gamma) = \delta$, then $v_\Gamma(a) < \gamma$ for all γ such that $\alpha(\gamma) = \delta$. We see that $v_\Delta(a) \leq \delta$ and if $v_\Delta(a) = \delta$ then we contradict the previous statement with $\gamma = v_\Gamma(a)$. Hence, $a \in F^\circ_\delta A$. □

We now give criteria for determining whether a given double structure is an inf-limited double structure.

Lemma 3.3 *Let $\alpha : \Gamma \longrightarrow \Delta$ be an order preserving surjective ordered monoid homomorphism. Consider subsets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $0 \in \Gamma_0$, $\Delta_0 \neq \{0\}$ and $\alpha(\Gamma_0) = \Delta_0$. Then the following statements are equivalent.*

1. (Γ, Γ_0) *is inf-limited;*

2. (a) (Δ, Δ_0) *is inf-limited;*
   (b) *for all* $\delta \in \Delta_0$, $\alpha^{-1}(\delta) \cap \Gamma_0$ *is well-ordered.*

Proof. (1) $\Longrightarrow$ (2). We first prove that (Δ, Δ_0) is inf-limited. Let $\delta_1 > \delta_2 > \cdots$ be a decreasing sequence of elements of Δ_0, and let $\delta \in \Delta_0$, $\delta' \in \Delta$. By assumption, there exist $\gamma_i \in \Gamma_0$ such that $\alpha(\gamma_i) = \delta_i, i = 1, 2, \ldots$, and $\gamma \in \Gamma_0$, $\gamma' \in \Gamma$ such that $\alpha(\gamma) = \delta$ and $\alpha(\gamma') = \delta'$. Then we have a decreasing sequence $\gamma_1 > \gamma_2 > \cdots$ of elements of Γ_0, so there exists n such that $\gamma' \geq \gamma + \gamma_n$. Thus $\delta' = \alpha(\gamma') \geq \alpha(\gamma + \gamma_n) = \delta + \delta_n$ as desired. We next prove that $\alpha^{-1}(\delta) \cap \Gamma_0$ is well-ordered for every $\delta \in \Delta_0$. On the contrary, suppose that $\gamma_1 > \gamma_2 > \cdots$ is a decreasing sequence in Γ_0 such that $\alpha(\gamma_i) = \delta$ for $i = 1, 2, \ldots$. If δ is not least in Δ_0, choose $\delta' \in \Delta_0$ such that $\delta' < \delta$ and choose $\gamma' \in \Gamma_0$ such that $\alpha(\gamma') = \delta'$. Since (Γ, Γ_0) is inf-limited, there is an n such that $\gamma' > \gamma_n$ (recall $0 \in \Gamma_0$). Applying α we get $\delta' = \alpha(\gamma') \geq \alpha(\gamma_n) = \delta$ which is a contradiction. Now, consider the case where δ is least in Δ_0. Let $\gamma \in \Gamma_0$ be arbitrary. Since (Γ, Γ_0) is inf-limited, there is an n such that $2\gamma_1 > \gamma + \gamma_n$ ($2\gamma_1 \in \Gamma$). Applying α we see that $2\delta \geq \alpha(\gamma) + \delta$, so that $\delta \geq \alpha(\gamma)$, and thus, since δ is least in Δ_0, $\delta = \alpha(\gamma)$. Since $\gamma \in \Gamma_0$ is arbitrary and α maps Γ_0 onto Δ_0, we conclude that Δ_0 is a singleton and therefore $\Delta_0 = \{0\}$, which is again a contradiction.

(2) $\Longrightarrow$ (1). Assume that $\gamma_1 > \gamma_2 > \cdots$ and γ are all in Γ_0 and $\gamma' \in \Gamma$. We then have a sequence $\alpha(\gamma_1) \geq \alpha(\gamma_2) \geq \cdots$ in Δ_0. There is a subsequence $\alpha(\gamma_{i_1}) > \alpha(\gamma_{i_2}) > \cdots$, since otherwise there would be an N such that $\delta = \alpha(\gamma_N) = \alpha(\gamma_{N+1}) = \cdots$ which violates the hypothesis that $\alpha^{-1}(\delta) \cap \Gamma_0$ is well-ordered. Since (Δ, Δ_0) is inf-limited, there is an n such that $\alpha(\gamma') > \alpha(\gamma) + \alpha(\gamma_{i_n})$, and so $\gamma' > \gamma + \gamma_{i_n}$ as desired. $\square$

Corollary 3.4 *Let* $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ *be an inf-limited double structure. Then the* Γ *topology on* A *is a refinement of the* Δ *topology on* A.

Proof. We have observed in Lemma 2.5 that $\{F^\circ_\gamma A\}_{\gamma \in \Gamma_0}$ and $\{F^\circ_\delta A\}_{\delta \in \Delta_0}$ form bases of the Γ and Δ topologies respectively. Therefore it suffices to show that for each $\delta \in \Delta_0$ there is a $\gamma \in \Gamma_0$ such that $F^\circ_\delta A = F^\circ_\gamma A$. Let $\delta \in \Delta_0$, then by Lemma 3.2,

$$F^\circ_\delta A = \bigcap_{\alpha(\gamma) = \delta} F^\circ_\gamma A = \bigcap_{\substack{\alpha(\gamma) = \delta \\ \gamma \in \Gamma_0}} F^\circ_\gamma A. \tag{1}$$

For the second equality, let $0 \neq a \in \bigcap_{\substack{\alpha(\gamma) = \delta \\ \gamma \in \Gamma_0}} F^\circ_\gamma A$. Then $v_\Gamma(a) < \gamma$ for all $\gamma \in \Gamma_0$ with $\alpha(\gamma) = \delta$. Thus $v_\Delta(a) < \delta$, since $v_\Gamma(a) \in \Gamma_0$. Now let γ be any element of Γ such that $\alpha(\gamma) = \delta$. If $\gamma \leq v_\Gamma(a)$, then $\delta \leq v_\Delta(a)$, which is a contradiction. Thus $v_\Gamma(a) < \gamma$, and hence $a \in F^\circ_\gamma A$, so that $a \in \bigcap_{\alpha(\gamma) = \delta} F^\circ_\gamma A$.

Now the last intersection in Equation(1) is equal to $F^\circ_{\gamma_0} A$, where γ_0 is least in $\alpha^{-1}(\delta) \cap \Gamma_0$ (such a γ_0 exists by Lemma 3.3). $\square$

The next lemma shows how to construct inf-limited double structures.

Lemma 3.5 *Let $\mathcal{A}_\Gamma = (A, \Gamma, \mathcal{F}_\Gamma)$ be a filtered structure, let Δ be another ordered monoid, and let $\alpha : \Gamma \longrightarrow \Delta$ be a surjective order preserving monoid homomorphism such that $\alpha(\Gamma_0) \neq \{0\}$. For all $\delta \in \Delta$, define*

$$F_\delta A = \bigcup_{\alpha(\gamma)=\delta} F_\gamma A.$$

Then $\mathcal{A}_\Delta = (A, \Delta, \mathcal{F}_\Delta)$ is a filtered structure, and $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is a double structure. If $\mathcal{A}_\Gamma$ is inf-limited, then $\mathcal{U}_\alpha$ is an inf-limited double structure.

Proof. That $\mathcal{U}_\alpha$ is a double structure is easily verified. Then, that $\mathcal{U}_\alpha$ is an inf-limited double structure when $\mathcal{A}_\Gamma$ is inf-limited follows from Lemma 3.3. $\square$

In some of the examples of double structures we will consider the monoids Γ are of a special type.

Definition 3.6 *Let Δ and Λ be ordered monoids, and let $\Gamma = \Lambda \times \Delta$. An elimination order on Γ with Δ largest is defined as follows:*

$$(\lambda, \delta) > 0 \textit{ if and only if } \begin{cases} \delta > 0 \textit{ or} \\ \delta = 0 \textit{ and } \lambda > 0 \end{cases}$$

We recall that one says that a monoid homomorphism $\alpha : \Gamma \longrightarrow \Delta$ splits if and only if for some submonoid Λ of Γ, we have an isomorphism $\phi : \Gamma \longrightarrow \Lambda \times \Delta$ satisfying $\phi(\lambda) = (\lambda, 0)$ for all $\lambda \in \Lambda$ and such that

$$\begin{tikzcd} \Gamma \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & \Lambda \times \Delta \arrow[dl, "\pi"] \\ & \Delta & \end{tikzcd}$$

commutes, where π is the projection. In this case $\Lambda = ker(\alpha)$. We also have the monoid homomorphism $\beta : \Delta \longrightarrow \Gamma$ defined by $\beta(\delta) = \phi^{-1}(0, \delta)$ for all $\delta \in \Delta$ which satisfies $\alpha \circ \beta =$ identity on Δ. Further, $\phi^{-1}(\lambda, \delta) = \lambda + \beta(\delta)$. Of course, if Γ and Δ are groups, then the existence of β is enough to guarantee that α splits, but for monoids this is not the case.

Lemma 3.7 *In the above situation assume that Γ, Δ are ordered monoids and that α is order preserving. Then β is order preserving and the isomorphism ϕ is order preserving where the order on $\Lambda \times \Delta$ is the elimination order with Δ largest.*

Proof. We first prove that β is order preserving. Let $\delta_1 < \delta_2$ be in Δ. If $\beta(\delta_1) \geq \beta(\delta_2)$, then $\alpha(\beta(\delta_1)) \geq \alpha(\beta(\delta_2))$, that is $\delta_1 \geq \delta_2$, since α is order preserving, which is a contradiction. To prove the second statement it suffices to show that $(\lambda_1, \delta_1) > (\lambda_2, \delta_2)$ if and only if $\lambda_1 + \beta(\delta_1) > \lambda_2 + \beta(\delta_2)$, for all $\lambda_1, \lambda_2 \in \Lambda$, $\delta_1, \delta_2 \in \Delta$. We first note that if $\delta_1 = \delta_2$, both conditions state that $\lambda_1 > \lambda_2$. On the other hand, if $\delta_1 \neq \delta_2$, we have after applying α to $\lambda_1 + \beta(\delta_1)$ and $\lambda_2 + \beta(\delta_2)$, that both conditions are equivalent to $\delta_1 > \delta_2$. $\square$

Now keeping in mind Lemma 3.7 we make the following

Definition 3.8 *A double structure* $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ *is called a split double structure if the map* α *splits. If in addition* $\mathcal{U}_\alpha$ *is an inf-limited double structure, we call it an inf-limited split double structure.*

We next observe that split double structures are common, at least when Γ and Δ are groups. We say that a double structure $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is *contained* in the double structure $\mathcal{U}_{\alpha'} = (\mathcal{A}_{\Gamma'}, \mathcal{A}_{\Delta'}, \alpha')$ provided that the underlying rings A are the same, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ with compatible orders, $gr_\Gamma(A) = gr_{\Gamma'}(A)$, $gr_\Delta(A) = gr_{\Delta'}(A)$, and finally $\alpha'|\Gamma = \alpha$.

Proposition 3.9 *Let* $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ *be a double structure such that the ring* $gr_\Gamma(A)$ *is finitely generated as a* G_0*-algebra and* Γ *is generated by* Γ_0*. If* Γ *is a group, then* $\mathcal{U}_\alpha$ *is a split double structure, and* Γ *and* Δ *are finitely generated free abelian groups. In any case,* $\mathcal{U}_\alpha$ *is contained in a split double structure.*

Proof. Let $gr_\Gamma(A) = G_0[g_1, \ldots, g_t]$. We can assume that the g_i's are Γ-homogeneous, say $g_i \in G_{\gamma_i}$. Using the contrapositive of **A6** in Section 2 we see that every element in Γ_0 is a sum of the elements $\gamma_1, \ldots, \gamma_t$, and so by hypothesis $\gamma_1, \ldots, \gamma_t$ generate Γ as a monoid. Thus if Γ is a group, we see that Γ is a finitely generated torsion free abelian group and hence it is free. Then Δ is also a finitely generated torsion free abelian group, so it is also free. Thus we conclude that α splits.

In general, Γ, Δ are cancellative monoids, and so there are unique minimal ordered groups Γ', and Δ' containing Γ and Δ respectively, and an order preserving epimorphism $\alpha' : \Gamma' \longrightarrow \Delta'$ such that $\alpha'|\Gamma = \alpha$. The Γ and Δ filtrations extend uniquely to Γ' and Δ' filtrations in such a way that $gr_\Gamma(A) = gr_{\Gamma'}(A)$, and $gr_\Delta(A) = gr_{\Delta'}(A)$. This gives a double structure $\mathcal{U}_{\alpha'}$ containing $\mathcal{U}_\alpha$. Moreover, Γ generates Γ' and so $\Gamma_0 = \Gamma'_0$ generates Γ'. So applying the first result yields the second statement.□

Note. It was already observed in [15] that such groups are isomorphic to $\mathbf{Z}^n$.

The following is a general method for constructing inf-limited split double structures which we will find useful in the applications.

Let $\mathcal{A}_\Delta = (A, \Delta, \mathcal{F}_\Delta)$ be an inf-limited filtered structure with $\Delta_0 \neq \{0\}$. Let Λ be another ordered monoid. Assume that the associated graded ring $gr_\Delta(A)$ has the following structure: for all $\delta \in \Delta$, and for all $\lambda \in \Lambda$, we have an additive subgroup $F_\lambda G_\delta$ of G_δ. We define

$$F^\circ_\lambda G_\delta = \begin{cases} \cup_{\lambda' < \lambda} F_{\lambda'} G_\delta \\ \{0\} \text{ if no such } \lambda' \text{ exists} \end{cases}$$

For $\delta \in \Delta$, we define

$$\Lambda_0^{(\delta)} = \{\lambda \in \Lambda \mid F^\circ_\lambda G_\delta \subsetneqq F_\lambda G_\delta\},$$

Moreover assume that we have the following properties:

P1 If $\lambda, \lambda' \in \Lambda$ such that $\lambda < \lambda'$, then $F_\lambda G_\delta \subseteq F_{\lambda'} G_\delta$;

P2 For all $\lambda, \lambda' \in \Lambda$, and for all $\delta, \delta' \in \Delta$, $(F_\lambda G_\delta) \cdot (F_{\lambda'} G_{\delta'}) \subseteq F_{\lambda + \lambda'} G_{\delta + \delta'}$;

P3 For all $\delta \in \Delta$ and for all $0 \neq x \in G_\delta$, there exists a least $\lambda \in \Lambda$ such that $x \in F_\lambda G_\delta$;

P4 For all $\delta \in \Delta_0$ the set $\Lambda_0^{(\delta)}$ is well-ordered.

We now make A into an inf-limited filtered structure with respect to $\Gamma = \Lambda \times \Delta$. We impose the elimination order on Γ with Δ largest. For each $\lambda \in \Lambda$ and $\delta \in \Delta$ we define $F_{(\lambda,\delta)}A$ by the equation

$$F_\lambda G_\delta = F_{(\lambda,\delta)}A / F_\delta^\circ A.$$

Proposition 3.10 *$\mathcal{A}_\Gamma = (A, \Gamma, \mathcal{F}_\Gamma)$, where $\mathcal{F}_\Gamma = \{F_{(\lambda,\delta)}A\}_{(\lambda,\delta)\in\Gamma}$, is an inf-limited filtered structure. Moreover, if $\alpha : \Gamma = \Lambda \times \Delta \longrightarrow \Delta$ is the projection map, then $(\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is an inf-limited split double structure.*

Proof. First, let $(\lambda, \delta) < (\lambda', \delta')$. If $\delta < \delta'$, then

$$F_{(\lambda,\delta)}A \subseteq F_\delta A \subseteq F_{\delta'}^\circ A \subseteq F_{(\lambda',\delta')}A.$$

If $\delta = \delta'$, then $\lambda < \lambda'$, and $F_{(\lambda,\delta)}A \subseteq F_{(\lambda',\delta)}A$ by **P1**.

Now let $(\lambda, \delta), (\lambda', \delta') \in \Gamma$, and let $a \in F_{(\lambda,\delta)}A, a' \in F_{(\lambda',\delta')}A$. Then $a + F_\delta^\circ A \in F_\lambda G_\delta$, and $a' + F_{\delta'}^\circ A \in F_{\lambda'} G_{\delta'}$. Therefore by **P2**, we have

$$aa' + F_{\delta+\delta'}^\circ A = (a + F_\delta^\circ A)(a' + F_{\delta'}^\circ A) \in (F_\lambda G_\delta)(F_{\lambda'} G_{\delta'})$$

$$\subseteq F_{\lambda+\lambda'} G_{\delta+\delta'} = F_{(\lambda+\lambda',\delta+\delta')}A / F_{\delta+\delta'}^\circ A.$$

Therefore $aa' \in F_{(\lambda+\lambda',\delta+\delta')}A$.

Now let $0 \neq a \in A$. Let $\delta = v_\Delta(a)$, so $a + F_\delta^\circ A \neq 0$ and $a + F_\delta^\circ A \in G_\delta$. Then by **P3** there is a $\lambda \in \Lambda$ least such that $a + F_\delta^\circ A \in F_\lambda G_\delta = F_{(\lambda,\delta)}A / F_\delta^\circ A$. Then $a \in F_{(\lambda,\delta)}A$. Now let $(\lambda', \delta') < (\lambda, \delta)$. If $\delta' < \delta$, then $a \notin F_{\delta'}A$ and hence $a \notin F_{(\lambda',\delta')}A$. On the other hand, if $\delta' = \delta$ and $\lambda' < \lambda$, then $a \in F_{(\lambda',\delta)}A$ implies $a + F_\delta^\circ A \in F_{\lambda'} G_\delta$ which is a contradiction because of our choice of λ. We also see that we have $\alpha(v_\Gamma(a)) = v_\Delta(a)$ and so $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is a double structure. It remains to show that $\mathcal{U}_\alpha$ is inf-limited. For this we need

Claim 1. For all $\lambda \in \Lambda$ and $\delta \in \Delta$,

$$F_\lambda^\circ G_\delta = F_{(\lambda,\delta)}^\circ A / F_\delta^\circ A.$$

Proof. First note that for $\lambda \in \Lambda$ and $\delta \in \Delta$ we have

$$F_{(\lambda,\delta)}^\circ A = \bigcup_{(\lambda',\delta')<(\lambda,\delta)} F_{(\lambda',\delta')}A = \Big(\bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A\Big) \cup \Big(\bigcup_{\delta'<\delta} (\cup_{\lambda''\in\Lambda} F_{(\lambda'',\delta')}A)\Big)$$

$$= \Big(\bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A\Big) \cup \Big(\bigcup_{\delta'<\delta} F_{\delta'}A\Big) = \Big(\bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A\Big) \cup F_\delta^\circ A = \bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A.$$

On the other hand

$$F_\lambda^\circ G_\delta = \bigcup_{\lambda'<\lambda} F_{\lambda'} G_\delta = \bigcup_{\lambda'<\lambda} (F_{(\lambda',\delta)}A / F_\delta^\circ A) = \Big(\bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A\Big) / F_\delta^\circ A,$$

and hence $F^\circ_\lambda G_\delta = F^\circ_{(\lambda,\delta)}A/F^\circ_\delta A$ as desired.

Claim 2. $\Gamma_0 = \bigcup_{\delta\in\Delta_0} \Lambda_0^{(\delta)} \times \{\delta\}$.

Proof. Let $(\lambda,\delta) \in \Gamma_0$, and let $0 \neq a \in F_{(\lambda,\delta)}A - F^\circ_{(\lambda,\delta)}A$. Then, using Claim 1, $a + F^\circ_\delta A \in F_\lambda G_\delta - F^\circ_\lambda G_\delta$. Thus $\delta \in \Delta_0$, and $\lambda \in \Lambda_0^{(\delta)}$. Conversely, if $(\lambda,\delta) \in \Lambda_0^{(\delta)} \times \{\delta\}$, then $F^\circ_\lambda G_\delta \subsetneqq F_\lambda G_\delta$. Let $a + F^\circ_\delta A \in F_\lambda G_\delta - F^\circ_\lambda G_\delta$. Note that $a + F^\circ_\delta A \neq 0$ and that $a + F^\circ_\delta A \in G_\delta$. Thus $v_\Delta(a) = \delta$. Therefore $v_\Gamma(a) = (\lambda,\delta)$, and hence $(\lambda,\delta) \in \Gamma_0$.

Since $\mathcal{A}_\Delta$ is inf-limited, we use Claim 2, **P4** and Lemma 3.3 to conclude that $\mathcal{A}_\Gamma$ is inf-limited. Therefore $\mathcal{U}_\alpha$ is an inf-limited double structure. It is clearly split. □

As a particular case of the above construction, we consider the following. Let $\mathcal{A}_\Delta = (A, \Delta, \mathcal{F}_\Delta)$ be an inf-limited filtered structure, and let $\mathcal{A}_\Lambda = (A, \Lambda, \mathcal{F}_\Lambda)$ be any other filtered structure on the same ring A. For $\lambda \in \Lambda$ and $\delta \in \Delta$ we define

$$F_\lambda G_\delta = (F_\lambda A \cap F_\delta A + F^\circ_\delta A)/F^\circ_\delta A.$$

Then

$$F_{(\lambda,\delta)}A = F_\lambda A \cap F_\delta A + F^\circ_\delta A,$$

and by Claim 1 in Proposition 3.10,

$$F^\circ_{(\lambda,\delta)}A = \bigcup_{\lambda'<\lambda} F_{(\lambda',\delta)}A = \bigcup_{\lambda'<\lambda} (F_{\lambda'}A \cap F_\delta A + F^\circ_\delta A) = F^\circ_\lambda A \cap F_\delta A + F^\circ_\delta A.$$

Properties **P1** and **P2** are automatically satisfied. In order to satisfy properties **P3** and **P4**, we need to assume

P3′ For all $\delta \in \Delta_0$ and for all $a \in F_\delta A - F^\circ_\delta A$, there exists a least $\lambda \in \Lambda$ such that $a \in F_\lambda A \cap F_\delta A + F^\circ_\delta A$;

P4′ For all $\delta \in \Delta_0$, the set $\{\lambda \in \Lambda \mid F^\circ_\lambda A \cap F_\delta A + F^\circ_\delta A \subsetneqq F_\lambda A \cap F_\delta A + F^\circ_\delta A\}$ is well-ordered.

We observe that **P3′** and **P4′** follow from the single condition that for all $\delta \in \Delta$, the set $\{\lambda \in \Lambda \mid F_\lambda A \not\subset F^\circ_\delta A\}$ is well-ordered. Examples of this construction will be given in Sections 5.2 and 5.3.

We note that the expression $F_{(\lambda,\delta)}A = F_\lambda A \cap F_\delta A + F^\circ_\delta A$ gives an algebraic formulation inside A for a filtration given by the elimination order, since $F_\lambda A \cap F_\delta A$ corresponds to those elements whose Δ value is less than or equal to δ and Λ value is less than or equal to λ, and $F^\circ_\delta A$ corresponds to those elements whose Δ value is strictly less than δ which have any Λ value.

4 The Lifting Theorem

Throughout this section, we assume that $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is an inf-limited double structure. Since we are dealing with two filtrations on the same ring, we will denote by $\hat{A}_\Gamma$ and by $\hat{A}_\Delta$ the completions of A with respect to the topologies induced be $\mathcal{F}_\Gamma$ and $\mathcal{F}_\Delta$ respectively.

Since, by Corollary 3.4 the Γ topology is a refinement of the Δ topology, we have immediately

Lemma 4.1 *If $\{b_n\}_{n\in\mathbf{N}}$ is a Γ-Cauchy sequence in A, then it is also a Δ-Cauchy sequence. Moreover, if $b \in A$ is the limit of $\{b_n\}_{n\in\mathbf{N}}$ as a Γ-Cauchy sequence in A, then b is also the limit of $\{b_n\}_{n\in\mathbf{N}}$ as a Δ-Cauchy sequence in A.*

The next result, a special case of which appears in [15], shows that if $\{a_1,\ldots,a_r\}$ is a std-basis of an ideal K with respect to a filtration, it is also a std-basis of the same ideal with respect to a coarser filtration.

Theorem 4.2 *If $\{a_1,\ldots,a_r\}$ is a Γ-std-basis for the ideal K then it is a Δ-std-basis for K.*

Proof. Let $a \in K$. Then there exist $\hat{c}_1,\ldots,\hat{c}_r \in \hat{A}_\Gamma$ such that

$$a = \sum_{i=1}^{r} \hat{c}_i a_i, \text{ and } v_\Gamma(a) = \max_{\hat{c}_i\neq 0}(v_\Gamma(a_i) + v_\Gamma(\hat{c}_i)).$$

Let $\{c_{i,n}\}_{n\in\mathbf{N}}$ be a Γ-Cauchy sequence of elements of A converging to $\hat{c}_i$. Then $\{c_{i,n}\}_{n\in\mathbf{N}}$ is a Δ-Cauchy sequence by Lemma 4.1, and hence this sequence converges to an element $\hat{c}'_i \in \hat{A}_\Delta$. But then by the second statement in Lemma 4.1 we see that $a = \sum_{i=1}^{r} \hat{c}'_i a_i$, also. Finally, since $v_\Gamma(a) = \max_{\hat{c}_i\neq 0}(v_\Gamma(a_i) + v_\Gamma(\hat{c}_i)) = \max_{\hat{c}_i\neq 0}(v_\Gamma(a_i) + v_\Gamma(c_{i,n}))$, for n large enough, we have

$$v_\Delta(a) = \alpha(v_\Gamma(a)) = \alpha(\max_{\hat{c}_i\neq 0}(v_\Gamma(a_i) + v_\Gamma(c_{i,n})))$$

$$= \max_{\hat{c}_i\neq 0}(\alpha(v_\Gamma(a_i)) + \alpha(v_\Gamma(c_{i,n}))) = \max_{\hat{c}_i\neq 0}(v_\Delta(a_i) + v_\Delta(c_{i,n}))$$

$$= \max_{\hat{c}_i\neq 0}(v_\Delta(a_i) + v_\Delta(\hat{c}'_i)) \geq \max_{\hat{c}'_i\neq 0}(v_\Delta(a_i) + v_\Delta(\hat{c}'_i)),$$

since by Lemma 4.1 $\hat{c}_i = 0$ implies $\hat{c}'_i = 0$. But $a = \sum_{\hat{c}'_i\neq 0} \hat{c}'_i a_i$ implies, by **A4** and **A7** of Section 2, that $v_\Delta(a) \leq \max_{\hat{c}'_i\neq 0}(v_\Delta(a_i) + v_\Delta(\hat{c}'_i))$, and so the result follows. □

In view of the previous result, the lifting question can now be put in the following form:

> given a Δ-std-basis for K, can we lift that basis to a Γ-std-basis of K?

The key to this lifting is the fact that the graded ring $gr_\Delta(A)$ can be filtered by Γ, and that its associated graded ring is isomorphic to $gr_\Gamma(A)$.

That is, for $\gamma \in \Gamma$, define

$$F_\gamma gr_\Delta(A) = \left(\bigoplus_{\delta<\alpha(\gamma)} G_\delta\right) \bigoplus F_\gamma A / F^\circ_{\alpha(\gamma)} A. \tag{2}$$

We summarize the properties of this structure in the following

Proposition 4.3 *1. $\mathcal{G}_\Gamma = (gr_\Delta(A), \Gamma, \mathcal{F}_\Gamma)$ where $\mathcal{F}_\Gamma = \{F_\gamma gr_\Delta(A)\}_{\gamma\in\Gamma}$ is an inf-limited filtered structure. Its associated graded ring $gr_\Gamma(gr_\Delta(A))$ is isomorphic to $gr_\Gamma(A)$.*

2. *Let $v_{\Delta\Gamma}$ and $in_{\Delta\Gamma}$ denote the value and initial term, respectively, associated with the filtered structure $\mathcal{G}_\Gamma$. Let $0 \neq g \in gr_\Delta(A)$ be Δ homogeneous, say, $g = a + F^\circ_\delta A$, for $a \in A$ with $v_\Delta(a) = \delta$, then $v_{\Delta\Gamma}(g) = v_\Gamma(a)$. Moreover, $in_{\Delta\Gamma}(g) = in_\Gamma(a)$, that is, for any non zero $a \in A$, $in_{\Delta\Gamma}(in_\Delta(a)) = in_\Gamma(a)$ (these last equations assume the identification of $gr_\Gamma(gr_\Delta(A))$ and $gr_\Gamma(A)$.*

3. *Let $\mathcal{G}_\Delta = (gr_\Delta(A), \Delta, \mathcal{F}_\Delta)$ be the filtered structure naturally associated with $gr_\Delta(A)$. Then $(\mathcal{G}_\Gamma, \mathcal{G}_\Delta, \alpha)$ is an inf-limited double structure.*

Proof. First we suppose that γ and γ' are in Γ and $\gamma < \gamma'$. Then if $\alpha(\gamma) < \alpha(\gamma')$ we have by Lemma 3.2 that $F_\gamma A/F^\circ_{\alpha(\gamma)}A \subseteq G_{\alpha(\gamma)}$ and so $F_\gamma gr_\Delta(A) \subseteq F_{\gamma'} gr_\Delta(A)$, while if $\alpha(\gamma) = \alpha(\gamma')$ we have the same result since $F_\gamma A \subseteq F_{\gamma'}A$. We next verify the second condition in Definition 2.1. So let $g \in F_\gamma gr_\Delta(A)$ and $g' \in F_{\gamma'} gr_\Delta(A)$. It suffices to assume that g and g' are homogeneous, that is, let δ and δ' be in Δ with $g \in G_\delta$ and $g' \in G_{\delta'}$ with $\delta \leq \alpha(\gamma)$ and $\delta' \leq \alpha(\gamma')$. If $\delta \leq \alpha(\gamma)$ and $\delta' < \alpha(\gamma')$, then $\delta + \delta' < \alpha(\gamma) + \alpha(\gamma') = \alpha(\gamma + \gamma')$ and so $gg' \in G_{\delta+\delta'} \subseteq F_{\gamma+\gamma'} gr_\Delta(A)$. The other case is where $\delta = \alpha(\gamma)$ and $\delta' = \alpha(\gamma')$. Then writing $g = a + F^\circ_\delta A$ and $g' = a' + F^\circ_{\delta'}A$, we see that $aa' \in F_{\delta+\delta'}A \subseteq F_{\gamma+\gamma'}A$, and so $gg' \in F_{\gamma+\gamma'}A/F^\circ_{\alpha(\gamma+\gamma')}A$. We now verify the third condition in Definition 2.1. Let $g = (g_\delta)$ be a non-zero element of $gr_\Delta(A)$, where $g_\delta \in G_\delta$. Choose δ_0 largest such that $g_{\delta_0} \neq 0$ and write $g_{\delta_0} = a_{\delta_0} + F^\circ_{\delta_0}A$. Set $\gamma_0 = v_\Gamma(a_{\delta_0})$, so that $\alpha(\gamma_0) = \delta_0$. Then one verifies easily that $g \in F_{\gamma_0} gr_\Delta(A)$ and $g \notin F_\gamma gr_\Delta(A)$ for all $\gamma < \gamma_0$. It further follows that

$$F^\circ_\gamma gr_\Delta(A) = \bigoplus_{\delta < \alpha(\gamma)} G_\delta \bigoplus F^\circ_\gamma A/F^\circ_{\alpha(\gamma)}A, \tag{3}$$

and thus from Equations (2) and (3) we have

$$F_\gamma gr_\Delta(A)/F^\circ_\gamma gr_\Delta(A) \cong F_\gamma A/F^\circ_\gamma A,$$

and hence the graded ring $gr_\Gamma(gr_\Delta(A))$ associated with $gr_\Delta(A)$ is isomorphic to $gr_\Gamma(A)$. The Γ_0 associated with $\mathcal{G}_\Gamma$ is the same as the one associated with $\mathcal{A}_\Gamma$, and hence $\mathcal{G}_\Gamma$ is an inf-limited filtered structure. Parts 2. and 3. of the Proposition are straight forward. $\square$

We note that, in the construction discussed in the previous section before Proposition 3.10, the Γ-filtration on $gr_\Delta(A)$ defined is simply given by

$$F_{(\lambda,\delta)} gr_\Delta(A) = \left(\bigoplus_{\delta' < \delta} G_{\delta'} \right) \bigoplus F_\lambda G_\delta.$$

We are now in a position to state and prove our Lifting Theorem. *We say that elements in A or $gr_\Delta(A)$ are a std-basis if they are a std-basis of the ideal they generate.*

Theorem 4.4 *Let $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ be an inf-limited double structure. Let $a_1, \ldots, a_t$ be elements of A. Then the following statements are equivalent.*

1. *$a_1, \ldots, a_t$ is a Γ-std basis.*

2. (a) *$a_1, \ldots, a_t$ is a Δ-std basis;*

(b) $in_\Delta(a_1), \ldots, in_\Delta(a_t)$ is a Γ-std basis in $gr_\Delta(A)$.

Proof. The implication (1) $\Longrightarrow$ (2-a) is Theorem 4.2. For the implication (1) $\Longrightarrow$ (2-b), let $h \in \langle in_\Delta(a_1), \ldots, in_\Delta(a_t)\rangle \subseteq gr_\Delta(A)$. Since $\langle in_\Delta(a_1), \ldots, in_\Delta(a_t)\rangle$ is a Δ-homogeneous ideal of $gr_\Delta(A)$, every homogeneous component of h is also in that ideal, so we may assume that h is Δ-homogeneous, and we may write $h = a + F^\circ_\delta A$, for some $a \in F_\delta A$ such that $v_\Delta(a) = \delta$. In view of Theorem 2.8, it suffices to show that $in_{\Delta\Gamma}(h)$ is in the ideal $\langle in_{\Delta\Gamma}(a_1), \ldots, in_{\Delta\Gamma}(a_t)\rangle$ of $gr_\Gamma(gr_\Delta(A))$, or equivalently, using Proposition 4.3, we need to show that $in_\Gamma(a)$ is in the ideal $\langle in_\Gamma(a_1), \ldots, in_\Gamma(a_t)\rangle$ of $gr_\Gamma(A)$. Now, since $h \in \langle in_\Delta(a_1), \ldots, in_\Delta(a_t)\rangle$, and since h is Δ-homogeneous, we may write

$$h = \sum_{i=1}^{t} g_i in_\Delta(a_i),$$

where, if $g_i \neq 0$, g_i is Δ-homogeneous, say $g_i = b_i + F^\circ_{\delta_i} A \in G_{\delta_i}$, and $\delta = \delta_i + v_\Delta(a_i)$. Consider the element $a^* = \sum_{g_i \neq 0} b_i a_i \in F_\delta A$. Since $a^* \in \langle a_1, \ldots, a_t\rangle$, we have by hypothesis $in_\Gamma(a^*) \in \langle in_\Gamma(a_1), \ldots, in_\Gamma(a_t)\rangle$. To conclude the proof, it is sufficient to show that $in_\Gamma(a) = in_\Gamma(a^*)$. By definition of a^*, we have $a - a^* \in F^\circ_\delta A = F^\circ_{v_\Delta(a)} A$, and therefore, $a - a^* \in F^\circ_{v_\Gamma(a)} A$, by Lemma 3.2, and hence $in_\Gamma(a) = in_\Gamma(a^*)$.

To prove the implication (2) $\Longrightarrow$ (1), let $a \in \langle a_1, \ldots, a_t\rangle$. By (2-a), we have

$$in_\Delta(a) \in \langle in_\Delta(a_1), \ldots, in_\Delta(a_t)\rangle \subseteq gr_\Delta(A).$$

Then by (2-b), we have $in_{\Delta\Gamma}(a) \in \langle in_{\Delta\Gamma}(a_1), \ldots, in_{\Delta\Gamma}(a_t)\rangle$, so in view of Proposition 4.3, $in_\Gamma(a) \in \langle in_\Gamma(a_1), \ldots, in_\Gamma(a_t)\rangle$ as desired. $\square$

We now give an algorithm that inputs a Δ-std-basis and outputs a Γ-std-basis for an ideal K of A under the assumption that we have an algorithm to compute Γ-standard bases in $gr_\Delta(A)$ and that we can solve linear equations in $gr_\Delta(A)$ arbitrarily accurately, that is we can find the $\bar{c}_{ij}$ in the algorithm below.

LIFTING ALGORITHM

```
input: a_1,...,a_t a Δ-std-basis for K
output: a Γ-std-basis for K
        compute a Γ-std-basis G = {b̄_1,...,b̄_s} of Δ-homogeneous
                elements of ⟨in_Δ(a_1),...,in_Δ(a_t)⟩,
                where b̄_i = b_i + F°_{v_Δ(b_i)} A.
        compute Δ-homogeneous elements c̄_ij ∈ gr_Δ(A) such that
```

$$\bar{b}_i - \sum_{j=1}^{t} \bar{c}_{ij} in_\Delta(a_j) \in F^\circ_{v_\Gamma(b_i)} A.$$

```
                where, if c̄_ij ≠ 0,
                c̄_ij = c_ij + F°_{v_Δ(c_ij)} A, and  v_Δ(b_i) = v_Δ(c_ij) + v_Δ(a_j).
        add the following polynomials to the original list:
                for each i = 1,...,s,
```

$$a_i^* = \sum_{\bar{c}_{ij} \neq 0} c_{ij} a_j.$$

We have to show that this algorithm actually gives a Γ-std-basis for the ideal $\langle a_1, \ldots, a_t\rangle$. Clearly, $\langle a_1, \ldots, a_t, a_1^*, \ldots, a_s^*\rangle = \langle a_1, \ldots, a_t\rangle$. Statement (2-a) of Theorem 4.4 is satisfied by $\{a_1, \ldots, a_t\}$ alone. Also, Statement (2-b) of Theorem 4.4 is satisfied by $\{a_1^*, \ldots, a_s^*\}$, since $in_{\Delta\Gamma}(in_\Delta(a_i^*)) = in_{\Delta\Gamma}(in_\Delta(b_i)) = in_{\Delta\Gamma}(\bar{b}_i)$. To see this, note that $in_\Delta(c_{ij}a_j) = in_\Delta(c_{ij})in_\Delta(a_j)$, since $v_\Delta(b_i) = v_\Delta(c_{ij}) + v_\Delta(a_j)$, and hence $in_\Delta(a_i^*) = \sum_{j=1}^t \bar{c}_{ij} in_\Delta(a_j)$, which gives us the result. (The a_i^*'s generalize the T-polynomials in [9] and [2].)

5 Applications

In this section, we apply the results of the previous sections to three important cases:

1. lifting Gröbner bases from $(k[y_1, \ldots, y_m])[x_1, \ldots, x_n]$ (polynomials are viewed in the x variables with coefficients polynomials in the y variables) to $k[y_1, \ldots, y_m, x_1, \ldots, x_n]$;
2. lifting standard bases with respect to partial orderings on the monomials to standard bases with respect to finer partial orderings in $k[x_1, \ldots, x_n]$ (this process can be iterated to obtain Gröbner bases);
3. lifting standard bases in the I-adic case when we refine the I-adic filtration by another ideal (this procedure can also be iterated).

These examples have already been introduced in Examples 2.2 and 2.7.

5.1 Lifting Gröbner Bases from $(k[\mathbf{y}])[\mathbf{x}]$ to $k[\mathbf{y}, \mathbf{x}]$

In [1], [2] and [3] the question of lifting Gröbner bases from the polynomial ring in n variables $(k[y_1, \ldots, y_m])[x_1, \ldots, x_n]$ to the polynomial ring in $n+m$ variables $k[y_1, \ldots, y_m, x_1, \ldots, x_n]$ was addressed. We now show that this is a special case of the results proved in the previous sections. The results below, in fact, generalize those in [2].

Let $A = R[x_1, \ldots, x_n]$ be as in Example 2.2 (1) (except now we use Δ for $\mathbf{N}^n$ instead of Γ, since Γ is used below for another purpose). In addition we assume that the ring R is filtered by a monoid Λ such that Λ_0 is well-ordered (in [2] the ring R was only assumed to be graded). Then we can filter each homogeneous direct summand G_δ of $gr_\Delta(A)(= A)$ by Λ as follows. For $\delta \in \Delta$ and $\lambda \in \Lambda$, we define

$$F_\lambda G_\delta = \{f \in G_\delta \mid lc_x(f) \in F_\lambda R\}.$$

(Note that in the definition of $F_\lambda G_\delta$ we have identified A and $gr_\Delta(A)$.) We set $\Gamma = \Lambda \times \Delta$, where Γ is endowed with the elimination order with Δ largest, and we define α to be the projection map from Γ to Δ.

It is easy to see that this construction satisfies the conditions of Proposition 3.10 and hence we have a Γ-filtration on A which gives rise to an inf-limited filtered structure $\mathcal{A}_\Gamma = (A, \Gamma, \{F_\gamma A\}_{\gamma\in\Gamma})$ and to a split double structure $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$.

Using the fact that A is isomorphic to $gr_\Delta(A)$, we can restate Theorem 4.4 to obtain

Theorem 5.1 *Let $f_1, \ldots, f_t$ be elements of A. Then the following statements are equivalent.*

1. *$\{f_1, \ldots, f_t\}$ is a Γ-std-basis ;*
2. *(a) $\{f_1, \ldots, f_t\}$ is a Δ-Gröbner basis;*

   (b) $\{lt_x(f_1), \ldots, lt_x(f_t)\}$ is a Γ-std-basis in A.

We now would like to translate Condition $2(b)$ of the above result into a statement about the ring R alone. First, let $lt_x(f_i) = r_i\mathbf{X}_i$, where $lc_x(f_i) = r_i \in R$, and $\mathbf{X}_i$ is a power product in the x variables. As in [2], we define for a subset S of $\{1, \ldots, t\}$, $D_S = lcm_{i \in S}\mathbf{X}_i$, and $F_S = \{f_i \mid \mathbf{X}_i \text{ divides } D_S\}$. Also we let $S^* = \{i \mid f_i \in F_S\}$. We say that S is *saturated* if $S = S^*$.

Proposition 5.2 *The following statements are equivalent.*

1. *$\{lt_x(f_1), \ldots, lt_x(f_t)\}$ is a Γ-std-basis in A;*
2. *for every saturated subset S of $\{1, \ldots, t\}$, $\{lc_x(f_i) \mid i \in S\}$ is a Λ-std-basis in R.*

Proof. (1) $\Longrightarrow$ (2). Let S be a saturated subset of $\{1, \ldots, t\}$ and let $K = \langle r_i \mid i \in S\rangle \subseteq R$. By Theorem 2.8, we need to show that $\{in_\Lambda(r_i) \mid i \in S\}$ generates the ideal $\langle in_\Lambda(r) \mid r \in K\rangle$ of $gr_\Lambda(R)$. So let r be an element of K, say $r = \sum_{i \in S} a_i r_i$. Then

$$rD_S = \sum_{i \in S} a_i r_i \mathbf{X}_i(\frac{D_S}{\mathbf{X}_i}) = \sum a_i \frac{D_S}{\mathbf{X}_i} lt_x(f_i)$$

is in the ideal $\langle lt_x(f_i), i = 1, \ldots, t\rangle$ of A. By hypothesis, $in_\Gamma(rD_S)$ is in the ideal $\langle in_\Gamma(lt_x(f_i)), i = 1, \ldots, t\rangle$ of $gr_\Gamma(A)$. Now note that $in_\Gamma(lt_x(f_i)) = in_\Lambda(r_i)\mathbf{X}_i$, and $in_\Gamma(rD_S) = in_\Lambda(r)D_S$. Therefore

$$in_\Lambda(r)D_S = \sum_{i=1}^{t} c_i \mathbf{X}'_i in_\Lambda(r_i)\mathbf{X}_i,$$

where c_i is a Λ-homogeneous element of $gr_\Lambda(R)$ and $\mathbf{X}'_i$ is a power product in the x variables. Then, by homogeneity, if $c_i \neq 0$, $\mathbf{X}_i\mathbf{X}'_i = D_S$, and $\mathbf{X}_i$ must divide D_S, so that $i \in S$, since S is saturated. Thus

$$in_\Lambda(r)D_S = \sum_{i \in S} c_i \mathbf{X}'_i in_\Lambda(r_i)\mathbf{X}_i,$$

and hence $in_\Lambda(r) = \sum_{i \in S} c_i in_\Lambda(r_i)$.

(2) $\Longrightarrow$ (1). Let J be the ideal $\langle lt_x(f_i), i = 1, \ldots, n\rangle$ of A. By Theorem 2.8, and using the fact that $in_\Gamma(lt_x(f_i)) = in_\Lambda(r_i)\mathbf{X}_i$, we need to show that $\{in_\Lambda(r_i)\mathbf{X}_i, i = 1, \ldots, t\}$ generates the ideal $\langle in_\Lambda(r)\mathbf{X} \mid in_\Lambda(r)\mathbf{X} = in_\Gamma(f), \text{ for some } f \in J\rangle$ of $gr_\Gamma(A)$. So let $in_\Lambda(r)\mathbf{X} = in_\Gamma(f)$, $f \in J$. Note that it is enough to assume that f is Δ homogeneous, i.e., only one power product in the x variables appears in f. Let S be the saturation of the set $\{i \in \{1, \ldots, t\} \mid \mathbf{X}_i \text{ divides } \mathbf{X}\}$. Note that S is not empty, since f is in J. Moreover, r is in the ideal $\langle r_i \mid i \in S\rangle$ of R. By

hypothesis, $in_\Lambda(r) \in \langle in_\Lambda(r_i) \mid i \in S\rangle$. Therefore $in_\Lambda(r) = \sum_{i\in S} c_i in_\Lambda(r_i)$, where $c_i \in gr_\Lambda(R)$ is Λ-homogeneous. Then

$$in_\Lambda(r)\mathbf{X} = \sum_{i\in S} c_i in_\Lambda(r_i)\mathbf{X}_i(\frac{\mathbf{X}}{\mathbf{X}_i}) = \sum_{i\in S} c_i \frac{\mathbf{X}}{\mathbf{X}_i} in_\Gamma(lt_x(f_i)).\square$$

In case R is Λ-graded, i.e., $gr_\Lambda(R)$ is isomorphic to R, we obtain Theorem 3.2 in [2] as a corollary of the above theorem.

As in Corollary 3.4 in [2], we can apply this result to the case when $R = k[y_1, \ldots, y_m]$ is graded by a term order $\Lambda = \mathbf{N}^m$ to obtain

Corollary 5.3 *For a subset $F = \{f_1, \ldots, f_t\}$ of $k[\mathbf{y}, \mathbf{x}]$ the following statements are equivalent.*

1. *$F = \{f_1, \ldots, f_t\}$ is a Gröbner basis in $k[\mathbf{y}, \mathbf{x}]$ over k;*
2. *(a) F is a Gröbner basis in $(k[\mathbf{y}])[\mathbf{x}]$ over $k[\mathbf{y}]$ and*
   (b) For all saturated subsets S of $\{1, ..., t\}$, $\{lc_x(f_i) \mid i \in S\}$ is a Gröbner basis in $k[\mathbf{y}]$ over k.

An algorithm and an example illustrating this result are given in [3].

5.2 Lifting standard bases with respect to partial orderings on the monomials to Gröbner Bases in $k[\mathbf{x}]$

Let $A = k[x_1, \ldots, x_n]$,and let $u_1, \ldots, u_s, u_{s+1}, \ldots, u_{s+r}$ be vectors in $\mathbf{R}^n$. As indicated in Example 2.2 (2), these vectors define a Γ filtration on A, where Γ is the submonoid of $\mathbf{R}^{s+r}$ of all vectors $(\eta\cdot u_1, \ldots, \eta\cdot u_{s+r})$, as η ranges through $\mathbf{N}^n$. In a similar fashion, the vectors $u_1, \ldots, u_s$ determine a Δ filtration on A, where Δ is the submonoid of $\mathbf{R}^s$ of all vectors $(\eta\cdot u_1, \ldots, \eta\cdot u_s)$, as η ranges through $\mathbf{N}^n$. We assume that (Γ, Γ_0) is inf-limited. Let $\alpha\colon\Gamma \longrightarrow \Delta$ be the projection $\mathbf{R}^{s+r} \longrightarrow \mathbf{R}^s$ restricted to Γ. We then have an inf-limited double structure on A.

It was already noted in Example 2.2 (2), that Γ and Δ define gradings on A. Moreover, for all $\delta \in \Delta$, we have

$$A_\delta = \bigoplus_{\alpha(\gamma)=\delta} A_\gamma.$$

For $f \in A$, we denote by $lt_\Gamma(f)$ (resp. $lt_\Delta(f)$) the leading term of f with respect to $u_1, \ldots, u_{s+r}$ (resp. $u_1, \ldots, u_s$). Using the identifications of $gr_\Gamma(A)$, $gr_\Delta(A)$, and A, we see that $in_\Delta(f) = lt_\Delta(f)$ and $in_\Gamma(f) = lt_\Gamma(f)$. Thus a Γ-std-basis is a set of polynomials $f_1, \ldots, f_t$ such that for $K = \langle f_1, \ldots, f_t\rangle$ we have

$$\langle lt_\Gamma(f) \mid f \in K\rangle = \langle lt_\Gamma(f_1), \ldots, lt_\Gamma(f_t)\rangle.$$

In this context we can restate Theorem 4.4 to obtain

Theorem 5.4 *Let $f_1, \ldots, f_t$ be elements of A. Then the following statements are equivalent.*

1. *$\{f_1, \ldots, f_t\}$ is a std-basis with respect to $u_1, \ldots, u_{s+r}$;*
2. *(a) $\{f_1, \ldots, f_t\}$ is a std-basis with respect to $u_1, \ldots, u_s$;*
   (b) $\{lt_\Delta(f_1), \ldots, lt_\Delta(f_t)\}$ is a std-basis with respect to $u_1, \ldots, u_{s+r}$.

As a consequence of this theorem and the Lifting Algorithm in Section 4 one could construct a Gröbner basis of an ideal with respect to a term order given by vectors $u_1, \ldots, u_s$ as follows. First compute a std-basis with respect to the vector u_1, then lift it to a std-basis with respect to u_1, u_2, etc. However, it should be pointed out that this algorithm seems to be impractical. As far as we know, computing a std-basis with respect to even one vector is at least as difficult as applying Buchberger's algorithm to compute the Gröbner basis directly.

We now give an example illustrating the above algorithm. We will work in $k[x, y]$ using deglex with $y > x$. That is, we are using the vectors $u_1 = (1, 1)$ and $u_2 = (0, 1)$. We consider the ideal K generated by the polynomials $f_1 = x^2y + xy^2 + y$, $f_2 = y^3 + x + y$. Following the procedure described in Theorem 2.7 in [2], the std-basis with respect to u_1 is computed, as it was in [3], as follows. A homogeneous basis for the syzygy module of $(lt_{u_1}(f_1), lt_{u_1}(f_2))$ is given by $(-y^2, x^2 + xy)$. We compute $-y^2f_1 + (x^2 + xy)f_2 = -y^3 + x^3 + 2x^2y + xy^2$. Since the latter polynomial cannot be reduced by f_1 and f_2, we denote it by f_3 and add it to the basis. A homogeneous basis for the syzygy module of $(lt_{u_1}(f_1), lt_{u_1}(f_2), lt_{u_1}(f_3))$ is given by $(x + y, -y, -y)$ together with $(-y^2, x^2 + xy, 0)$. Since $(x + y)f_1 + (-y)f_2 + yf_3 = 0$, we have that $\{f_1, f_2, f_3\}$ is a std-basis for the ideal $\langle f_1, f_2\rangle$ with respect to u_1, i.e. a Macaulay basis. To lift this std-basis to a Gröbner basis of K we first compute a std-basis of $\langle lt_{u_1}(f_1), lt_{u_1}(f_2), lt_{u_1}(f_3)\rangle$ with respect to u_1, u_2. This is easily computed to be $g_1 = lt_{u_1}(f_1) = xy^2 + x^2y$, $g_2 = lt_{u_1}(f_2) = y^3$, $g_3 = lt_{u_1}(f_3) = -y^3 + xy^2 + 2x^2y + x^3$, $g_4 = x^4$, and $g_5 = x^2y + x^3$. Since $g_4 = (-2x + y)g_1 + xg_3$ and $g_5 = -g_1 + g_2 + g_3$, we add $f_4 = (-2x + y)f_1 + xf_3 = x^4 + y^2 - 2xy$ and $f_5 = -f_1 + f_2 + f_3 = x^2y + x^3 + x$ to the std-basis with respect to u_1, f_1, f_2, f_3 to obtain a std-basis with respect to (u_1, u_2), i.e. a Gröbner basis with respect to deglex.

We note that the result that Statement 1. implies Statement 2. in Theorem 5.4 is used in [10] to compute an explicit presentation of the associated graded ring of an affine algebra with respect to one of its ideals.

Finally we put this example in the framework of the constructions at the end of Section 3. With Δ and Γ defined as above, we also let Λ be the set of all $(\eta\cdot u_{s+1}, \ldots, \eta\cdot u_{s+r})$ as η ranges through $\mathbf{N}^n$. Then $\Gamma = \Lambda \times \Delta$, and the order on Γ is the elimination order with Δ largest. Moreover, $F_{(\lambda,\delta)}A = F_\lambda A \cap F_\delta A + F^\circ_\delta A$, as is easily verified.

5.3 Lifting I-adic Standard Bases

In this section, we consider I-adic std-bases in a general commutative ring A as in Example 2.2 (4). Recall that if I is an ideal of A such that $\cap_{n=1}^{\infty} I^n = \{0\}$, and if we let $\Delta = -\mathbf{N}$, then I defines a Δ-filtration on A as follows: for $\delta \in \Delta$,

$$F_\delta A = I^{-\delta}.$$

Then we have
$$G_\delta = I^{-\delta}/I^{-\delta+1}.$$
As noted in Example 2.2 (4), this defines an inf-limited filtered structure $\mathcal{A}_\Delta$.

We can refine the filtration given by I using another ideal J, provided that $J^n \subseteq I$ for some positive integer n (for example, provided that $\sqrt{J} \subseteq \sqrt{I}$ if the ring A is Noetherian). Let $\Lambda = -\mathbf{N}$ and $\Gamma = \Lambda \times \Delta$. For each $\delta \in \Delta, \lambda \in \Lambda$, we define
$$F_\lambda G_\delta = (I^{-\delta} \cap J^{-\lambda} + I^{-\delta+1})/I^{-\delta+1}.$$
We also define $F_{(\lambda,\delta)}A$ as in proposition 3.10. That is,
$$F_{(\lambda,\delta)}A = I^{-\delta} \cap J^{-\lambda} + I^{-\delta+1}.$$
(This is an example of the construction at the end of Section 3.) We define α to be the projection map from Γ to Δ. Since, for all $\delta \in \Delta$, $\alpha^{-1}(\delta)$ is finite ($J^n \subseteq I$) the hypotheses of Proposition 3.10 are satisfied. Therefore $\mathcal{U}_\alpha = (\mathcal{A}_\Gamma, \mathcal{A}_\Delta, \alpha)$ is an inf-limited split double structure.

Standard bases with respect to the refined filtration given by Γ will be called (J, I)-*adic std-bases.*

Note that, as pointed out in Section 4, $gr_\Delta(A)$ is filtered by Γ, and hence we have the concept of std-bases in $gr_\Delta(A)$ with respect to Γ. We will also call these bases (J, I)-adic std-bases.

In this context, we can restate Theorem 4.4 as follows

Theorem 5.5 *Let $a_1, \ldots, a_t$ be elements of A. Then the following statements are equivalent.*

1. *$\{a_1, \ldots, a_t\}$ is a (J, I)-adic std-basis ;*
2. *(a) $\{a_1, \ldots, a_t\}$ is a I-adic std-basis ;*
   (b) $\{in_I(a_1), \ldots, in_I(a_t)\}$ is a (J, I)-adic std-basis in $gr_\Delta(A)$.

Given another ideal K of A we can iterate this construction using the ideas at the end of Section 3. We let $\Omega = -\mathbf{N} \times \Gamma$ and for $\omega = (n, \gamma) \in \Omega$ we set
$$F_\omega A = K^{-n} \cap F_\gamma A + F^\circ_\gamma A.$$
As noted there, if for all $\gamma = (\lambda, \delta) \in \Gamma$ the set of all $n \in -\mathbf{N}$ such that
$$K^{-n} \not\subset F^\circ_\gamma A = J^{-\lambda+1} \cap I^{-\delta} + I^{-\delta+1}$$
is well-ordered, then the conditions $\mathbf{P3'}$ and $\mathbf{P4'}$ of Section 3 are satisfied. This set would be, in fact, finite provided that a power of K is contained in I, and, more interestingly provided a power of K is contained in J (the former case tends to be trivial).

To illustrate how these results can be used, we consider the computation of Hironaka bases in the power series ring.

Recall that given a term order, denoted $<$, on the monomials in the variables $x_1, \ldots, x_n$, and given $f \in k[[x_1, \ldots, x_n]]$, we define the leading term of f ($lt(f)$) to be the least term appearing in f.

Then we have the concept of leading term ideal $(Lt(K))$ for an ideal K, and hence the concept of standard bases with respect to this "negative" term order. These bases are also called Hironaka bases and were introduced in [7].

For example, we assume that $n = 2$ and that the order is negative degree lexicographic. Note that if we let $I = \langle x_1, x_2\rangle$, $\Delta = -\mathbf{N}$, then the completion of $A = k[[x_1, x_2]]$ with respect to the topology defined by I is A itself. Now let $f \in k[[x_1, x_2]]$, then $in_I(f)$ is the leading term of f with respect to the negative total degree (here we use the identification $k[x_1, x_2] \cong gr_I(k[[x_1, x_2]]))$. So an I-adic std-basis will be a std-basis with respect to the negative total degree ordering. Now we let $J = \langle x_1\rangle$. Since $J \subset I$, the hypotheses of Theorem 5.5 are satisfied, and we can refine the I-adic filtration using the ideal J. Using the identification of the graded rings with the polynomial ring $k[x_1, x_2]$, we have that $lt(f) = in_{(J,I)}(f)$. Therefore an (J, I)-adic std-basis is a Hironaka basis with respect to the negative degree lexicographic ordering. Also such bases can be computed using Theorem 5.5 and the Lifting Algorithm presented in Section 4.

In the case of three variables x_1, x_2, x_3 with the negative degree lexicographic ordering, we would use the iterated construction above with $I = \langle x_1, x_2, x_3\rangle$, $J = \langle x_1, x_2\rangle$ and $K = \langle x_1\rangle$.

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