A Systematic Treatment of Growth Terms Appearing in Continuum Mechanics Formulations for Biological Materials

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Irschik, H., On the Necessity of Surface Growth Terms for the Consistency of Jump Relations at a Singular Surface. *Acta Mechanica* Vol.162 (2003).
Irschik, H., A Treatise on the Equations of Balance and on the Jump Relations in Continuum Machanics. In: *Advanced Dynamics and Control of Structures and Machines* (H. Irschik, K. Schlacher, eds.), CISM Courses and Lectures No. 444, Springer-Verlag Wien New York 2004.
Irschik, H., Holl, H. J., Mechanics of variable-mass systems. Part 1: Balance of mass and linear momentum. *ASME*, *Applied Mechanics Reviews*, 57 (2004).
Irschik, H., Über Wachstumsterme in den Bilanzgleichungen der Kontinuumsmechanik, speziell beim Wachstum von biologischen Materialien. *Sitzungsberichte der Österreichischen Akademie der Wissenschaften* (2005)

I. Local form of the general equation of balance: (Eulerian description, inertial frame)

$$\frac{d}{dt}(\rho\Psi)+\rho\Psi\,\mathrm{div}\,\dot{p}=s[\Psi]+\pi[\Psi]-\mathrm{div}\,i[\Psi]$$

ρ...mass density, p...position vector,

$\dot{p}$...(absolute) velocity, Ψ...scalar or vector;

$\rho\Psi$...entity (per unit volume) to be balanced, for example:

$\Psi = 1$: Balance of mass,

$\Psi = \dot{p}$: Balance of linear momentum.

$$i[\Psi]=i_0[\Psi]+i_+[\Psi] \;\dots\; \text{influx}$$

$\Psi = \dot{p}$: $i_0[\dot{p}] = -\sigma$ … stress tensor

Index 0: “Classical” theory, for reference see:

Liu, I.-S., Continuum Mechanics, Springer 2002.
Greve, R., Kontinuumsmechanik, Springer 2003.

Index +: “Non-classical” theory, e.g. theory of mixtures, porous media, growth of biological materials, for reference see:

Kelly, P.D., A Reacting Continuum, *Int. J. Eng. Sc.*, 1964.
Truesdell, C. *Rational Thermodynamics*, Springer 1984.
Morland, L. W., Sellers, S., Multiphase Mixtures and Singular Surfaces, *Int. J. Non-Linear Mech.*, 2001.
De Boer, R., Contemporary Progress in Porous Media Theory, *Applied Mechanics Reviews*, 2000.
Epstein, M., Maugin, G., Thermomechanics of Growth in Uniform Bodies, *Int. J. Plasticity*, 2000.

Skalak, R., Dasgupta, et al., Analytical Description of Growth, *Journal of Theoretical Biology*, 1982
Lubarda, V.A., Hoger, A., On the Mechanics of Solids with a Growing Mass, *Int. J. Solids Struct.*, 2002.
Zaixing Huang, The Equilibrium Equations and Constitutive Equations of the Growing Deformable Body in the Framework of Continuum Theory. *Int. J. Non-Linear Mechanics*, 2004.
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$s[\Psi] = s_0[\Psi] + s_+[\Psi]$ ... supply (action at a distance)

$s_+[\Psi] = 0$ … vanishes in general
(action at a distance accounted for in classical theory)

$\Psi = 1$: $s_0[1] = 0$… no supply of mass

$\Psi = \dot{p}$: $s_0[\dot{p}] = b$…body force per unit volume

$\pi[\Psi] = \pi_0[\Psi] + \pi_+[\Psi]$... production

$\pi_0[\Psi] = 0$ in the fundamental equations of balance, except for balance of energy and balance of entropy

$\pi_+[\Psi]$…volumetric growth

$\pi_+[1]$...production (growth) of mass

Effective equation of balance: $\frac{d}{dt}(\rho\Psi) + \rho\Psi \operatorname{div}\dot{p} = S_0[\Psi] + P_+[\Psi]$

Effective classical supply: $S_0[\Psi] = s_0[\Psi] - \operatorname{div} i_0[\Psi] \{\underbrace{+s_+[\Psi]}_{=0}\}$

$\Psi = \dot{p}$: $S_0[\dot{p}] = b + \operatorname{div}\sigma$ … resultant force density

Effective additional production:

$$P_+[\Psi] = \pi_+[\Psi] - \operatorname{div} i_+[\Psi] + \pi_0[\Psi]$$

II. Fundamental equations of balance:

1: Balance of mass: $\Psi = 1$; $S_0[1] = 0$:

$\dot{\rho} + \operatorname{div} \rho = P_+[1]$ … continuity equation in the presence of an (effective) mass growth

General equation of balance: $\boxed{\rho\dot{\Psi} + P_+[1]\Psi = S_0[\Psi] + P_+[\Psi]}$

2: Balance of linear momentum: $\Psi = \dot{p}$:

$$s_0[\dot{p}] = b, \quad i_0[\dot{p}] = -\sigma, \quad S_0[\dot{p}] = b + \operatorname{div}\sigma:$$

$\rho\ddot{p} + P_+[1]\dot{p} = S_0[\dot{p}] + P_+[\dot{p}]$ … equation of motion

3: Balance of angular momentum:

$\Psi = \alpha = p \times \dot{p} + \ell$

ℓ…intrinsic spin (polar and micromorphic media; e.g. Cosserat theory)

$$s_0[\alpha] = p \times b + c, \quad i_0[\alpha] = \sigma \times p - \mu,$$

$$S_0[\alpha] = p \times b + c - \operatorname{div}(\sigma \times p - \mu);$$

c…body couple per unit volume,

μ…couple stress:

$$\rho\dot{\alpha} + P_+[1]\alpha = S_0[\alpha] + P_+[\alpha]$$

4: Balance of total energy:

$\Psi = \varepsilon = k_{tr} + k_{rot} + u$ … energy per unit mass

$k_{tr} = \frac{1}{2}\dot{p}\cdot\dot{p}$…translational kinetic energy,

$k_{rot} = \frac{1}{2}\ell\cdot\omega$ … rotatory kinetic energy,

ω …intrinsic angular velocity,

u…internal energy (per unit volume) ,

$s_0\left[\varepsilon\right] = \dot{p}\cdot b + \omega\cdot c + r$,

r…non-mechanical supply,

$i_0\left[\varepsilon\right] = -\sigma\cdot\dot{p} - \mu\cdot\omega + q$ (q…non-mechanical supply),

$S_0\left[\varepsilon\right] = \dot{p}\cdot b + \omega\cdot c + r + \mathrm{div}(\sigma\cdot\dot{p} + \mu\cdot\omega - q)$;

$$\rho\dot{\varepsilon} + P_+\left[1\right]\varepsilon = S_0\left[\varepsilon\right] + P_+\left[\varepsilon\right]$$

5: Balance of entropy:

$\Psi = \eta$…entropy per unit mass

$s_0\left[\eta\right] = \vartheta r$, $\vartheta = T^{-1}$…coldness, $i_0\left[\eta\right] = \vartheta q$,

$S_0\left[\eta\right] = \vartheta r - \mathrm{div}(\vartheta q)$

$$\rho\dot{\eta} + P_+\left[1\right]\eta = S_0\left[\eta\right] + P_+\left[\eta\right]$$

III. Consequences of fundamental equations:

1: Vector product of p and balance of linear momentum:

$$p\times\rho\ddot{p}+p\times P_+[1]\dot{p}=p\times S_0[\dot{p}]+p\times P_+[\dot{p}]$$

$\Rightarrow$ Balance of moment of momentum:

$$\Psi=\beta=p\times\dot{p};\quad \dot{\beta}=\dot{p}\times\dot{p}+p\times\ddot{p}=p\times\ddot{p},$$

$$p\times\operatorname{div}\sigma=-\operatorname{div}(\sigma\times p)-\sigma_\times,$$

$\sigma_\times\ldots$Gibssian cross-vector of σ,

$$\rho\dot{\beta}+P_+[1]\beta=S_0[\beta]+P_+[\beta]$$

On the Equations of Balance

$$S_0[\beta]=p\times b-\operatorname{div}(\sigma\times p),$$

$$s_0[\beta]=p\times b,\ i_0[\beta]=\sigma\times p;$$

$$\boxed{P_+[\beta]=p\times P_+[\dot{p}]-\sigma_\times}$$

$$\pi_0[\beta]=-\sigma_\times$$

2: Substitute balance of moment of momentum into balance of angular momentum:

$\Rightarrow$ Balance of intrinsic spin:

$$\Psi=\ell=\alpha-\beta;$$

$$\rho\dot{\ell}+P_+[1]\ell=S_0[\ell]+P_+[\ell]$$

$$S_0\left[\ell\right]=S_0\left[\alpha\right]-S_0\left[\beta\right]=c+\operatorname{div}\mu$$

$$s_0\left[\ell\right]=c,\ i_0\left[\ell\right]=-\mu;$$

$$P_+\left[\ell\right]=P_+\left[\alpha\right]-P_+\left[\beta\right]:$$

$$\boxed{P_+\left[\ell\right]=P_+\left[\alpha\right]-p\times P_+\left[\dot{p}\right]+\pi_0[\ell]}$$

$$\pi_0[\ell]=-\pi_0\left[\beta\right]=\sigma_\times$$

Non-polar case:

$$\boxed{\sigma_{\times\text{non-polar}}=-P_+\left[\alpha\right]+p\times P_+\left[\dot{p}\right]}$$

On the Equations of Balance

3: Scalar product of velocity $\dot{p}$ and balance of linear momentum:

$$\dot{p}\cdot\rho\ddot{p}+\dot{p}\cdot P_+\left[1\right]\dot{p}=\dot{p}\cdot S_0\left[\dot{p}\right]+\dot{p}\cdot P_+\left[\dot{p}\right]$$

$\Rightarrow$ Balance of translatory kinetic energy:

$$\Psi=k_{tr}=\frac{1}{2}\dot{p}\cdot\dot{p};\ \ \dot{k}_{tr}=\dot{p}\cdot\ddot{p},$$

$$\dot{p}\cdot\operatorname{div}\sigma=\operatorname{div}\left(\dot{p}\cdot\sigma\right)-\sigma:\operatorname{grad}\dot{p},$$

$$\rho\dot{k}_{tr}+P_+\left[1\right]k_{tr}=S_0\left[k_{tr}\right]+P_+\left[k_{tr}\right]$$

$$S_0\left[k_{tr}\right]=\dot{p}\cdot b+\operatorname{div}\left(\dot{p}\cdot\sigma\right),$$

$$s_0\left[k_{tr}\right]=\dot{p}\cdot b,\ i_0\left[k_{tr}\right]=-\dot{p}\cdot\sigma;$$

$$\boxed{P_+\left[k_{tr}\right] = \dot{p}\cdot P_+\left[\dot{p}\right] - k_{tr}\, P_+\left[1\right] + \pi_0[k_{tr}]}$$

$$\pi_0[k_{tr}] = -\sigma : \operatorname{grad}\dot{p} \ldots \text{stress power}$$

4: Scalar product of angular velocity ω and balance of intrinsic spin:

$$\omega\cdot\rho\dot{\ell} + \omega\cdot P_+\left[1\right]\ell = \omega\cdot S_0\left[\ell\right] + \omega\cdot P_+\left[\ell\right]$$

$\Rightarrow$ Balance of rotatory kinetic energy: $\Psi = k_{rot} = \frac{1}{2}\omega\cdot\ell;$

$$\dot{k}_{rot} = \frac{1}{2}(\dot{\omega}\cdot\ell + \omega\cdot\dot{\ell}) = \omega\cdot\dot{\ell} + \frac{1}{2}(\dot{\omega}\cdot\ell - \omega\cdot\dot{\ell}),$$

$$\omega\cdot\operatorname{div}\mu = \operatorname{div}\left(\omega\cdot\mu\right) - \mu : \operatorname{grad}\omega.$$

$$\rho\dot{k}_{rot} + P_+\left[1\right]k_{rot} = S_0\left[k_{rot}\right] + P_+\left[k_{rot}\right]$$

$$S_0\left[k_{rot}\right] = \omega\cdot c + \operatorname{div}\left(\omega\cdot\mu\right),$$

$$s_0\left[k_{rot}\right] = \omega\cdot c,\; i_0\left[k_{rot}\right] = -\omega\cdot\mu;$$

$$\boxed{P_+\left[k_{rot}\right] = \omega\cdot P_+\left[\ell\right] - k_{rot}\, P_+\left[1\right] + \pi_0[k_{rot}]}$$

$$\pi_0[k_{rot}] = -\mu : \operatorname{grad}\omega + m_0[k_{rot}]$$

$$m_0[k_{rot}] = \frac{1}{2}\rho(\dot{\omega}\cdot\ell - \omega\cdot\dot{\ell}) \;\ldots\; \text{micromorphic spin production}$$

micropolar case: $\dot{\omega}\cdot\ell - \omega\cdot\dot{\ell} = 0.$

5: Substitute balance of translatory and rotatory inertia into balance of total energy: $\Rightarrow$ Balance of internal energy: $\Psi = u = \varepsilon - k_{tr} - k_{rot}$;

$$\rho \dot{u} + P_+[1]u = S_0[u] + P_+[u]$$

$$S_0[u] = S_0[\varepsilon] - S_0[k_{tr}] - S_0[k_{rot}] = r - \operatorname{div} q$$

$$s_0[u] = r,\ i_0[u] = q;$$

$$P_+[u] = P_+[\varepsilon] - P_+[k_{tr}] - P_+[k_{rot}]:$$

$$\boxed{\begin{aligned} P_+[u] &= P_+[\varepsilon] - \dot{p}\cdot P_+[\dot{p}] - \omega\cdot(P_+[\alpha] - p\times P_+[\dot{p}]) \\ &\quad + (k_{tr} + k_{rot})P_+[1] + \pi_0[u] \end{aligned}}$$

$$\pi_0[u] = \sigma : \operatorname{grad}\dot{p} + \mu : \operatorname{grad}\omega - m_0[k_{rot}] - \omega\cdot\sigma_\times$$

6. Substitute balance of internal energy into balance of entropy:

$\Rightarrow$ Balance of free Helmholtz energy:

$$\Psi = \varphi = u - T\eta; \quad \dot{\varphi} = \dot{u} - T\dot{\eta} - \dot{T}\eta$$

$$\rho\dot{\varphi} + P_+[1]\varphi = S_0[\varphi] + P_+[\varphi]$$

$$S_0[\varphi] = S_0[u] - T S_0[\eta] = 0$$

$$s_0[\varphi] = 0,\ i_0[\varphi] = 0;$$

$$\boxed{P_+[\varphi] = P_+[u] - T P_+[\eta] + \pi_0[\varphi]}$$

$$\pi_0[\varphi] = \pi_0[u] - T\pi_0[\eta] = T q\cdot\operatorname{grad}\vartheta - \dot{T}\eta$$

IV. A model of growth (Irschik, 2005, Sitzungsberichte der Österreichischen Akademie der Wissenschaften):

Non-polar case: $\ell = 0, \omega = 0, c = 0, \mu = 0$

Motivated by the Meshchersky (1897) reactive force for particles with a variable mass (simple rocket model):

Meshchersky, I.V., Works on the Mechanics of Bodies with Variable Mass (in Russian), Leningrad 1949.

Mass is supplied by fictitious particles with an own velocity and internal energy :

$$P_+[\dot{p}] = v_f \, P_+[1]$$

$$P_+[\alpha] = p \times v_f \, P_+[1]$$

$$P_+[\varepsilon] = (k_f + u_f) \, P_+[1]$$

v_f, k_f, u_f ... *velocity, kinetic and internal energy of fictitious particles adhering to the particles of the continuum under consideration before impacting*

Fundamental equations of balance:

$$\rho \, \ddot{p} = S_0[\dot{p}] + (v_f - \dot{p}) \, P_+[1]$$

$$\rho \, \dot{\varepsilon} = S_0[\varepsilon] + (k_f + u_f - \varepsilon) \, P_+[1]$$

$$k_f = \frac{1}{2} v_f \cdot v_f$$

Remark:

$\sigma_\times = 0$: Cauchy stress is symmetric

Supply terms in derived equations of balance:

$$P_+[k] = \dot{p} \cdot \left(-\frac{1}{2}\,\dot{p} + v_f \right) P_+[1]$$

$$P_+[u] = \left(\frac{1}{2}\,(\dot{p} - v_f) \cdot (\dot{p} - v_f) + u_f \right) P_+[1]$$

Classical formulation for systems with varying mass:

$v_f = 0$:

Truesdell, C. and Toupin, R., The Classical Field Theories, in: S. Flügge (Hsg.): Handbuch der Physik, Band III/1: Prinzipien der Klassischen Mechanik und Feldtheorie, Springer-Verlag, Berlin 1960, p. 226-793.

More recent biomechanical formulations:

$v_f = \dot{p}$:

Ambrosi, D. and Mollica, F., On the Mechanics of a Growing Tumor. *Int. J. Engineering Sciences*, Vol. 40, 2002.
Lubarda, V.A. and Hoger, A., On the Mechanics of Solids with a Growing Mass, *Int. J. Solids Structures*, Vol. 39, 2002.

$v_f = \frac{\dot{p}}{2}$:

Zaixing Huang, The Equilibrium Equations and Constitutive Equations of the Growing Deformable Body in the Framework of Continuum Theory. *Int. J. Non-Linear Mechanics*, Vol. 39, 2004.

V. General equation of jump at a singular surface (non-polar case):

Extended Kotchine (Rankine-Hugoniot) jump conditions:

$$n_\Sigma \cdot \{ i_\Sigma [\Psi] + [\![i_0 [\Psi] + (\dot{p} - w_\Sigma) \rho \Psi]\!] \} = 0$$

n_Σ ... unit normal vector at singular surface Σ

Jump of influx: $[\![i_0 [\Psi]]\!] = i_0^+ [\Psi] - i_0^- [\Psi]$

Surface growth term (additional influx): $i_\Sigma[\Psi]$

$i_\Sigma [\Psi] = 0$ ... *classical form* of Kotchine jump conditions

1: Balance of linear momentum:

$\Psi = \dot{p}$: $\quad i_0 \left[\dot{p} \right] = -\sigma$ ... stress tensor

Cauchy fundamental theorem on stresses:

$n \cdot \sigma = = t_{(n)}$ ... surface traction

$t_{(n_\Sigma)}$ ... traction at singular surface

$$n_\Sigma \cdot i_\Sigma [\dot{p}] = [\![t_{(n_\Sigma)} - n_\Sigma \cdot (\dot{p} - w_\Sigma) \rho \dot{p}]\!]$$

... Jump of linear momentum at a singular surface with an equivalent surface growth term of momentum

2: Balance of kinetic energy:

$$\Psi = k = \frac{1}{2}\dot{p}\cdot\dot{p}: \quad i_0\left[k\right] = -\dot{p}\cdot\sigma, \quad \dot{p}\cdot\sigma\cdot n = \dot{p}\cdot t_{(n_\Sigma)}$$

$$n_\Sigma \cdot i_\Sigma\,[k] = \left[\!\left[t_{(n_\Sigma)}\cdot\dot{p} - \frac{1}{2}\, n_\Sigma\cdot(\dot{p} - w_\Sigma)\,\rho\,\dot{p}\cdot\dot{p} \right]\!\right]$$

... Jump of kinetic energy at a singular surface with an equivalent surface influx of kinetic energy

Again: Compare with scalar product of $\dot{p}$ with jump relation for linear momentum

Special assumption: $i_\Sigma\,[1] = 0, \quad i\,[1] = 0$:

$$\left[\!\left[n_\Sigma\cdot(\dot{p} - w_\Sigma)\,\rho \right]\!\right] = 0$$

... Jump of mass at a singular surface without an equivalent surface influx

Consistency relation I:

$$n_\Sigma\cdot i_\Sigma\,[k] - (n_\Sigma\cdot i_\Sigma\,[\dot{p}])\cdot\left\langle \dot{p}\right\rangle = \left\langle t_{(n_\Sigma)}\right\rangle\cdot\left[\!\left[\dot{p}\right]\!\right]$$

$\left\langle t_{(n_\Sigma)}\right\rangle = \frac{1}{2}\,(t^{+}_{(n_\Sigma)} + t^{-}_{(n_\Sigma)})$ ... mean value of traction across Σ

$\left\langle \dot{p}\right\rangle = \frac{1}{2}\,(\dot{p}^{+} + \dot{p}^{-})$ ... mean value of velocity across Σ

$\left[\!\left[\dot{p}\right]\!\right] = \dot{p}^{+} - \dot{p}^{-}$ ... jump of velocity across Σ

Useful relation:

$$[\![\Psi \cdot \Phi]\!] = [\![\Psi]\!] \cdot \langle\Phi\rangle + \langle\Psi\rangle \cdot [\![\Phi]\!]$$

3. Balance of total energy:

Consistency relation II:

$$i_\Sigma[k+u] = i_\Sigma[k] + i_\Sigma[u]$$

Assume vanishing surface growth terms for (total) energy and momentum:

Consistency relation III:

$$n_\Sigma \cdot i_\Sigma[u] = -n_\Sigma \cdot i_\Sigma[k] = -\langle t_{(n_\Sigma)}\rangle \cdot [\![\dot{p}]\!]$$

4. Balance of internal energy:

$\Psi = u$: $\quad i_0[u] = q$

$$n_\Sigma \cdot i_\Sigma[u] = n_\Sigma \cdot [\![q - (\dot{p} - w_\Sigma)\,\rho\, u]\!]$$

... Balance of internal energy at a singular surface with an equivalent surface influx of internal energy

Substitute $n_\Sigma \cdot i_\Sigma[u] = -\langle t_{(n_\Sigma)}\rangle \cdot [\![\dot{p}]\!]$:

Alternative jump relation for internal energy:

$$\langle t_{(n_\Sigma)}\rangle \cdot [\![\dot{p}]\!] + n_\Sigma \cdot [\![q - (\dot{p} - w_\Sigma)\,\rho\, u]\!] = 0$$

.... see Eq.(5.1.3-9) of Slattery for a different derivation.

Second formulation presented by Slattery:

$$\left[\!\left[\mathbf{t}_{(\mathbf{n}_\Sigma)} \cdot \dot{\mathbf{p}} \right]\!\right] + \mathbf{n}_\Sigma \cdot \left[\!\left[\mathbf{q} - (\dot{\mathbf{p}} - \mathbf{w}_\Sigma)\,\rho\, u \right]\!\right] = 0$$

Slattery: "... there is not a sufficient experimental evidence so far in order to distinguish between the two formulations."

Slattery, J.C.: *Advanced Transport Phenomena*. Cambridge University Press (1999)

Similar consistency relations can be derived for *polar* media.

Closing remark: Equations of balance and jump must be *closed* by means of constitutive relations for stress, couple stress and *growth terms*, as well as by kinematic relations and relations for non-mechanical terms. In this process, growth terms should be *consistently* taken into account in the equations of balance and jump, see arguments above.