Relations between Nested Harmonic Sums

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- Introduction
- Algebraic Relations
- Structural Relations
- Representation of some Observables
- Factorial Series
- The Basis
- Conclusions

Refs. J.B. DESY 07–042, and in preparation.

1. Introduction

Why are no- and single scale quantities in Quantum Field Theory related to ζ-values, nested harmonic sums and related objects ?

- The former quantities can be obtained from the latter putting the Mellin variable N either to fixed values or $N \to \infty$.

Perturbation Theory [fixed order]

Scalar Propagators:

$$\frac{i}{p^2 + i\epsilon}$$

Combine Momenta using the Feynman Trick

$$\frac{1}{A \cdot B} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Feynman parameter integrals substitute all non-trivial angular integrals in $D = 4 + \varepsilon$–dimensions.

Introduction

Momentum Integrals yield rational functions of Γ–functions

$$\left.\frac{\Gamma(n_1+\alpha_1\varepsilon)...\Gamma(n_k+\alpha_k\varepsilon)}{\Gamma(m_1+\beta_1\varepsilon)...\Gamma(m_l+\beta_k\varepsilon)}\right|_{m_i,n_i\epsilon\mathbf{Z},\alpha_i,\beta_i\epsilon\mathbf{Q}}$$

- The scale-ratio in the diagrams factors form the Feynman parameter integrals for single scale processes

The Feynman parameter integrals can be transformed into Mellin-Barnes Integrals

$$\frac{1}{(A+B)^q}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\sigma A^{\sigma}B^{-q-\sigma}\frac{\Gamma(-\sigma)\Gamma(q+\sigma)}{\Gamma(q)}$$

Then all Feynman parameter integrals can again be integrated $\Longrightarrow$ rational function of Γ–functions

One or more Feynman parameters contain N as power $x_i^N, \dots$ in the numerators. $\Longrightarrow$ the Γ–functions contain N as argument.

The Mellin-Barnes Integrals can be carried out using the Residue Theorem.
$\Longrightarrow$ several infinite sums over the rational functions of Γ–functions.

Introduction

- Seek compact representations for these in terms of

(Generalizations of) generalized hypergeometric functions.

- The Mellin variable N is a discrete quantity in the first place for physical reasons

$\Longrightarrow$ Light-cone expansion; cut vertex method + dispersion relations

- Perform the ε–expansion.
- The respective coefficients obey Difference Equations of finite order.
- The ε–expansion of Pochhammer-Symbols & Γ–functions leads to products of single finite harmonic sums and MZV's.
- The infinite sums over the Mellin–Barnes parameters lead to the respective Nesting.
- Observation: Most of the sums occurring are Nested Finite Harmonic Sums.
- However, other related sums are possible too for individual Feynman diagrams. [Vermaseren et al. (2005)]
- General solution formalisms like, `Sigma`, will reveal this uniquely. cf. C. Schneider.

Introduction

- Single scale processes in massless Quantum Field Theories or being considered in the limit $m^2/Q^2 \to 0$ exhibit significant simplifications when calculated in Mellin space.
- This is, to some extent, due to structure of Feynman parameter intergrals which posess a Mellin symmetry.
- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \epsilon \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients

Order α_s^2 contributions to the deep inelastic Wilson coefficient

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$$C_2^{(2),(+)}(x,1) = C_F^2 \Big[\frac{1+x^2}{1-x} \{ 4\ln^3(1-x) - (14\ln x + 9)\ln^2(1-x)$$
$$- [4\,\mathrm{Li}_2(1-x) - 12\ln^2 x - 12\ln x + 16\zeta(2) + \tfrac{27}{2}]\ln(1-x) - \tfrac{4}{3}\ln^3 x - \tfrac{3}{2}\ln^2 x$$
$$+ [-24\,\mathrm{Li}_2(-x) + 24\zeta(2) + \tfrac{61}{2}]\ln x + 12\,\mathrm{Li}_3(1-x) - 12 S_{1,2}(1-x)$$
$$+ 48\,\mathrm{Li}_3(-x) - 6\,\mathrm{Li}_2(1-x) + 32\zeta(3) + 18\zeta(2) + \tfrac{51}{4} \}$$
$$+ (1+x)\{ 2\ln x \ln^2(1-x) + 4[\mathrm{Li}_2(1-x) - \ln^2 x]\ln(1-x)$$
$$- 4[\mathrm{Li}_2(1-x) + \zeta(2)]\ln x + \tfrac{5}{3}\ln^3 x - 4\,\mathrm{Li}_3(1-x) \}$$
$$+ \Big(40 + 8x - 48x^2 - \tfrac{72}{5}x^3 + \frac{8}{5x^2}\Big)[\mathrm{Li}_2(-x) + \ln x \ln(1+x)]$$
$$+ (-8+40x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (5+9x)\ln^2(1-x)$$
$$+ \tfrac{1}{2}(-91+141x)\ln(1-x) - (28+44x)\ln x\ln(1-x) - (14+30x)\,\mathrm{Li}_2(1-x)$$
$$+ (\tfrac{29}{2} + \tfrac{25}{2}x + 24x^2 + \tfrac{36}{5}x^3)\ln^2 x + \frac{1}{10}\Big(13 - 407x + 144x^2 - \frac{16}{x}\Big)\ln x + (-10 + 6x - 48x^2 - \tfrac{72}{5}x^3)\zeta(2)$$
$$+ \tfrac{407}{20} - \tfrac{1917}{20}x + \tfrac{72}{5}x^2 + \frac{8}{5x} + [6\zeta(2)^2 - 78\zeta(3) + 69\zeta(2) + \tfrac{331}{8}]\delta(1-x) \Big]$$
$$+ C_A C_F \Big[\frac{1+x^2}{1-x} \{ -\tfrac{11}{3}\ln^2(1-x) + [4\,\mathrm{Li}_2(1-x) + 2\ln^2 x + \tfrac{44}{3}\ln x - 4\zeta(2) + \tfrac{367}{18}]\ln(1-x)$$
$$- \ln^3 x - \tfrac{55}{6}\ln^2 x + [4\,\mathrm{Li}_2(1-x) + 12\,\mathrm{Li}_2(-x) - \tfrac{239}{6}]\ln x - 12\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x) - 24\,\mathrm{Li}_3(-x)$$
$$+ \tfrac{22}{3}\mathrm{Li}_2(1-x) + 2\zeta(3) + \tfrac{22}{3}\zeta(2) - \tfrac{3155}{108} \}$$
$$+ 4(1+x)[\mathrm{Li}_2(1-x) + \ln x\ln(1-x)] + \Big(-20 - 4x + 24x^2 + \tfrac{36}{5}x^3 - \frac{4}{5x^2}\Big)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)]$$
$$+ (4-20x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (\tfrac{133}{6} - \tfrac{1113}{18}x)\ln(1-x)$$
$$+ (-2 + 2x - 12x^2 - \tfrac{18}{5}x^3)\ln^2 x + \frac{1}{30}\Big(13 + 1753x - 216x^2 + \frac{24}{x}\Big)\ln x + (-2 - 10x + 24x^2 + \tfrac{36}{5}x^3)\zeta(2)$$
$$- \tfrac{9687}{540} + \tfrac{59157}{540}x - \tfrac{36}{5}x^2 - \frac{4}{5x} + [\tfrac{71}{5}\zeta(2)^2 + \tfrac{140}{3}\zeta(3) - \tfrac{251}{3}\zeta(2) - \tfrac{5465}{72}]\delta(1-x) \Big]$$
$$+ n_f C_F \Big(\frac{1+x^2}{1-x} [\tfrac{2}{3}\ln^2(1-x) - (\tfrac{8}{3}\ln x + \tfrac{29}{9})\ln(1-x) - \tfrac{4}{3}\mathrm{Li}_2(1-x) + \tfrac{5}{3}\ln^2 x + \tfrac{19}{3}\ln x - \tfrac{4}{3}\zeta(2) + \tfrac{247}{54}]$$
$$+ \tfrac{1}{3}(1+13x)\ln(1-x) - \tfrac{1}{3}(7+19x)\ln x - \tfrac{23}{18} - \tfrac{27}{2}x + [\tfrac{4}{3}\zeta(3) + \tfrac{38}{3}\zeta(2) + \tfrac{457}{36}]\delta(1-x) \Big), \qquad (9)$$

where C_A, C_F denote the colour factors and n_f stands for the number of flavours. Here we have put $\mu^2 = Q^2$. The more general case ($\mu^2 \neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by

$$C_2^{(2),G}(x,1) = n_f C_F \Big[8(1+x)^2$$
$$\times [-4 S_{1,2}(-x) - 4\ln(1+x)\,\mathrm{Li}_2(-x) - 2\zeta(2)\ln(1+x) - 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)]$$
$$+ 4(1-x)^2\{ \tfrac{5}{6}\ln^3(1-x) - (2\ln x + \tfrac{13}{4})\ln^2(1-x) + [2\,\mathrm{Li}_2(1-x) + 2\ln^2 x + 4\ln x + \tfrac{7}{2}]\ln(1-x) - \tfrac{5}{12}\ln^3 x$$
$$+ [\mathrm{Li}_2(1-x) - 4\,\mathrm{Li}_2(-x) + 3\zeta(2)]\ln x - 4\,\mathrm{Li}_3(1-x) - S_{1,2}(1-x) + 12\,\mathrm{Li}_3(-x) + 13\zeta(3) + \tfrac{13}{2}\zeta(2) \}$$
$$+ x^2\{ \tfrac{10}{3}\ln^3(1-x) - 12\ln x\ln^2(1-x) + [16\ln^2 x - 16\zeta(2)]\ln(1-x) - 5\ln^3 x$$
$$+ [12\,\mathrm{Li}_2(1-x) + 20\zeta(2)]\ln x - 8\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x) \}$$
$$+ \Big(48 + \tfrac{64}{3}x + \tfrac{96}{5}x^3 + \frac{8}{15x^2}\Big)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + (14x - 23x^2)\ln^2(1-x)$$
$$+ (-12x + 10x^2)\ln(1-x) + (-24x + 56x^2)\ln x\ln(1-x) + 64x\,\mathrm{Li}_3(-x) + (-10 + 24x)\mathrm{Li}_2(1-x)$$
$$+ (-\tfrac{3}{2} + \tfrac{22}{3}x - 36x^2 - \tfrac{48}{5}x^3)\ln^2 x + \frac{1}{15}\Big(-236 + 339x - 648x^2 - \frac{8}{x}\Big)\ln x + (64x + 36x^2)\zeta(3)$$
$$+ (-\tfrac{20}{3}x + 46x^2 + \tfrac{96}{5}x^3)\zeta(2) - \tfrac{647}{15} + \tfrac{239}{5}x - \tfrac{36}{5}x^2 + \frac{8}{15x} \Big]$$
$$+ n_f C_A \Big\{ 4(1+x)^2[S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) + 4 S_{1,2}(-x) - 2\ln x\,\mathrm{Li}_2(1-x) + 4\ln(1+x)\,\mathrm{Li}_2(-x)$$
$$+ 2\ln x\,\mathrm{Li}_2(-x) + 2\zeta(2)\ln(1+x) + 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)]$$
$$+ 8(1+2x+2x^2)\Big[\mathrm{Li}_3\Big(\frac{1-x}{1+x}\Big) - \mathrm{Li}_3\Big(-\frac{1-x}{1+x}\Big) - \ln(1-x)\,\mathrm{Li}_2(-x) - \ln x\ln(1-x)\ln(1+x)\Big]$$
$$+ \Big(-24 + \tfrac{80}{3}x^2 - \frac{16}{3x}\Big)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + x^2[-4 S_{1,2}(1-x) + 16\,\mathrm{Li}_3(-x) + 8\ln x\,\mathrm{Li}_2(1-x)$$
$$+ 8\ln^2 x\ln(1+x)] + \tfrac{2}{3}(1 - 2x + 2x^2)\ln^3(1-x) + (24x - 8x^2)\ln x\ln^2(1-x)$$
$$+ \Big(-2 + 36x - \tfrac{122}{3}x^2 + \frac{8}{3x}\Big)\ln^2(1-x) + (-4 - 32x + 8x^2)\ln^2 x\ln(1-x)$$
$$+ (8 - 144x + 148x^2)\ln x\ln(1-x) + (4 + 40x - 8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x)$$
$$+ (-20 + 24x - 32x^2)\zeta(2)\ln(1-x) + \frac{1}{9}\Big(-186 - 1362x + 1570x^2 + \frac{104}{x}\Big)\ln(1-x)$$
$$+ (-4 - 72x + 8x^2)\,\mathrm{Li}_3(1-x) + \frac{1}{3}\Big(12 - 192x + 176x^2 + \frac{16}{x}\Big)\mathrm{Li}_2(1-x) + \tfrac{1}{3}(10 + 28x)\ln^3 x$$
$$+ (-1 + 88x - \tfrac{194}{3}x^2)\ln^2 x + (-48x + 16x^2)\zeta(2)\ln x + (58 + \tfrac{584}{3}x - \tfrac{2090}{9}x^2)\ln x - (10 + 12x + 12x^2)\zeta(3)$$
$$+ \frac{1}{3}\Big(12 - 240x + 268x^2 - \frac{32}{x}\Big)\zeta(2) + \tfrac{239}{9} + \tfrac{1072}{9}x - \tfrac{4493}{27}x^2 + \frac{344}{27x} \Big\}, \qquad (5)$$

W.L. van Neerven et al.: (1992) 79 functions   80 objects would be maximal.

- The high complexity is partly caused applying the the IBP–Method.
- x–space usually is not the best space to work in.

3 Loop Anomalous Dimensions & Wilson Coefficients

- $\Longrightarrow$ Harmonic Sums in linear representation.
- Still high complexity of terms.
- Compactification possible applying algebraic and structural relations.
- Observation : In all single scale calculations the same Basic Functions occur in the resp. weight.

- $\Longrightarrow$ Derive these Universal Functions and their complex analysis.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1}.

Harmonic sums form a quasi-shuffle algebra through $\sqcup\!\sqcup$. (M.E. Hoffman, J. Algebraic Combin. **11** (2000) 49)

$$\begin{aligned} S_{a_1,a_2} \sqcup\!\sqcup S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &+ S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of n_1 a's, n_2 b's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)!...(n_d/d)!}, \qquad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	2	5	13	31	79	195

Algebraic Relations

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums exhibit never any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)^w\right]$$

$$N_{\neg\{-1\}}^{\text{basic}}(\mathsf{w}) \; = \; \frac{2}{\mathsf{w}} \sum_{d|\mathsf{w}} \mu\left(\frac{\mathsf{w}}{d}\right) N_{\neg\{-1\}}^{\text{basic}}(d) \; .$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

- Here $\#_c$ is smaller than $\#_r$ in the general case.

Algebraic Relations

Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.B., Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \qquad \& \qquad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E \quad = \quad -\psi(N) - \gamma_E + \beta(N) + \ln(2); \qquad \beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \;\epsilon\; \mathbf{Q}$

Structural Relations

$N \ \epsilon \ \mathbf{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \qquad \text{(etc.)}$$

For $N \ \epsilon \ \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

Harmonic sums $\cup$ $\zeta_{k_1,\ldots k_n}$ are closed under differentiation.

w = 2:

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta(N) + \beta'(N)$$

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \rightarrow S_{1,-1}(N)$$

Structural Relations

The relations for w = 2 were explored by N. Nielsen (1906).

$$\xi(N) = \mathbf{M}\left[\left(\frac{\ln(1-x)}{1+x}\right)_+\right](N); \qquad \eta(N) = \mathbf{M}\left[\frac{\ln(1+x)-\ln(2)}{1-x}\right](N)$$

$$\xi_1(N) = \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N); \qquad -\xi_2(N) = \mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N)$$

$$\begin{aligned}
[\psi(z)+\gamma_E][\psi(1-z)+\gamma_E] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\
\beta(z)[\psi(z)+\gamma_E] &= \beta'(z) + \beta(z)\ln(2) - \xi_1(z) + \xi_2(z) \\
\beta(z)\beta(1-z) &= \eta(z) + \eta(1-z) \\
\beta^2(z) &= \psi'(z) - 2\eta(z)
\end{aligned}$$

Structural Relations

If half-integer arguments in N are allowed
$\mathbf{M}[\mathrm{Li}_k(-x)/(x \pm 1)](N)$ are not independent functions :

$$\frac{1}{2^{k-2}}\frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$

$$\begin{aligned}
(-1)^N \mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) &= -S_{2,-1}(N) - \ln(2)[S_2(N) - S_{-2}(N)] \\
&\quad -\frac{1}{2}\zeta_2 S_{-1}(N) + \frac{1}{4}\zeta_3 - \frac{1}{2}\zeta_2 \ln(2) \\
(-1)^N \mathbf{M}\left[\frac{-\mathrm{Li}_2(x) - \ln(x)\ln(1-x) + \zeta_2}{1+x}\right](N) &= -S_{-1,2}(N) + \zeta_2 S_{-1}(N) - \zeta_3 + \frac{3}{2}\zeta_2 \ln(2) \\
S_{-1,2}(N) + S_{2,-1}(N) &= S_{-1}(N) S_2(N) + S_{-3}(N)
\end{aligned}$$

Structural Relations

$$(-1)^{(N+1)}\mathbf{M}\left[\frac{\mathrm{Li}_3(-x)}{1+x}\right](N) = -S_{3,-1}(N) - \ln(2)[S_3(N) - S_{-3}(N)] - \frac{1}{2}\zeta_2 S_{-2}(N) + \frac{3}{4}\zeta_3 S_{-1}(N) - \frac{1}{8}\zeta_2^2 + \frac{3}{4}\ln(2)\zeta_3$$

$$(-1)^{N}\mathbf{M}\left[\frac{S_{1,2}(1-x)}{1+x}\right](N) = -S_{-1,3}(N) + \zeta_3 S_{-1}(N) - \frac{19}{40}\zeta_2^2 + \frac{7}{4}\zeta_3\ln(2)$$

$$S_{1,2}(1-x) = -\mathrm{Li}_3(x) + \log(x)\mathrm{Li}_2(x) + \frac{1}{2}\log(1-x)\log^2(x) + \zeta_3$$

$$S_{-1,3}(N) + S_{3,-1}(N) = S_{-1}(N)S_3(N) + S_{-4}(N)$$

- At even w there exists an algebraic relation

$$S_{\mathsf{w}/2,\mathsf{w}/2}(N) = \frac{1}{2}\left[S^2_{\mathsf{w}/2}(N) + S_{\mathsf{w}}(N)\right]$$

which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.

w = 3:

$$\rightarrow \frac{\mathrm{Li}_2(x)}{x\pm 1}, \qquad \frac{\ln^2(1+x)}{x\pm 1}$$

Double Sums in General

- Applying differential operators one may show :

For $N \epsilon \mathbf{R}$ double harmonic sums can always be represented
by one basic function for even weight and two basic functions for odd weight.

$$\Longrightarrow \quad \frac{\mathrm{Li}_k(x)}{1+x}, \qquad \frac{\mathrm{Li}_k(x)}{1 \pm x}$$

Examples, which reduce :

$$S_{2,3}(N) = \mathbf{M}\left[\left(\frac{\ln(x)\left[S_{1,2}(1-x)-\zeta_3\right]+3\left[S_{1,3}(1-x)-\zeta_4\right]}{x-1}\right)_+\right](N)+3\zeta_4 S_1(N)$$

$$S_{-4,-2}(N) = -\mathbf{M}\left[\left(\frac{4\mathrm{Li}_5(-x)-\ln(x)\mathrm{Li}_4(-x)}{x-1}\right)_+\right](N)$$
$$+\frac{1}{2}\zeta_2\left[S_4(N)-S_{-4}(N)\right]-\frac{3}{2}\zeta_3 S_3(N)+\frac{21}{8}\zeta_4 S_2(N)-\frac{15}{4}\zeta_5 S_1(N)$$

$$S_{1,3}(1-x) = -\mathrm{Li}_4(x)+\log(x)\mathrm{Li}_3(x)-\frac{1}{2}\log^2(x)\mathrm{Li}_2(x)-\frac{1}{6}\log^3(x)\log(1-x)+\zeta_4$$

Structural Relations

<u>$\mathrm{w} = 4;\ i \neq -1\ \ :$</u>

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

The Mellin transform of

$$\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+$$

reads

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left\{\frac{d}{dN}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)+\zeta_2}{x-1}\right)_+\right](N) \right.$$
$$\left. -S_{2,2}(N-1) + \zeta_2 S_2(N-1) + 2\zeta_3 S_1(N-1)\right\}$$

and can be traced back to that of $(\mathrm{Li}_2(x)/(x-1))_+$

Structural Relations

$\mathrm{w} = 5;\ i \neq -1$:

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \quad \frac{S_{1,3}(x)}{x+1} \quad \frac{S_{2,2}(x)}{x \pm 1} \quad \frac{\mathrm{Li}_2^2(x)}{x+1} \quad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \quad \frac{S_{1,3}(x)}{x-1} \quad [\mathrm{Occur\ in\ the\ 3-loop\ Wils.\ Coeff.\ only}]$$

$\mathrm{w} = 6;\ i \neq -1$:

$$\frac{\mathrm{Li}_5(x)}{x+1} \quad \frac{S_{3,2}(x)}{x \pm 1} \quad \frac{S_{2,3}(x)}{x \pm 1} \quad \frac{S_{1,4}(x)}{x \pm 1} \quad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \quad \frac{A_1(x)}{x+1} \quad \frac{A_2(x)}{x \pm 1} \quad \frac{A_3(x)}{x+1} \quad \frac{H_{0,-1,0,1,1}(x)}{x-1}$$

$$\frac{A_1(-x) + N_\alpha(x)}{x+1}\Big|_{\alpha=1..3}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

Harmonic Polylogarithms

- iterated integrals over the alphabet

$$f_a(x) = \frac{1}{x}, \qquad \frac{1}{1-x}, \qquad \frac{1}{1+x}$$

$$H_0(x) = \int_0^x \frac{dx}{x}, \qquad H_1(x) = \int_0^x \frac{dx}{1-x}, \qquad H_{-1}(x) = \int_0^x \frac{dx}{1+x}$$

$$H_{a,\vec{b}}(x) = \int_0^x dz f_a(z) H_{\vec{b}}(z)$$

Structural Relations

$\underline{\mathrm{w} = 5;\ i \neq -1\ \ :}$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \qquad \frac{S_{1,3}(x)}{x-1} \qquad [\text{Occur in the } 3-\text{loop Wils. Coeff. only}]$$

$\underline{\mathrm{w} = 6;\ i \neq -1\ \ :}$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_2(x)}{x \pm 1} \qquad \frac{A_3(x)}{x+1} \qquad \frac{H_{0,-1,0,1,1}(x)}{x-1}$$

$$\frac{A_1(-x) + N_\alpha(x)}{x+1}\Big|_{\alpha=1..3}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

Representation of Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$,
  w $= 4$ JB and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^2)$, w $= 4$ JB and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$, w $= 5, 6$,

  from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 → J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$, w $= 4$ J.B. and S. Moch
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$, w $=$ 4,5,

  J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026,

  DESY 07-027; DESY-08-029;
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$, w $= 4$,

  J.B. and S. Klein, arXiv:0706.2426 [hep-ph]

Example: Bhabha s+v

$$
\begin{aligned}
T_0 \quad = \quad & \frac{248+15\,N^2+N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{1,1,1,1}(N)+\frac{-2}{(N-1)(N+1)}\mathbf{S_{2,1,1}}(N)\\
&+\frac{-340+120\,N+17\,N^2+18\,N^3-31\,N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{3,1}(N)+\frac{1344-502\,N-69\,N^2-2\,N^3+57\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_4(N)\\
&+\frac{304-328\,N-500\,N^2+330\,N^3-6\,N^4+6\,N^5-2\,N^6+4\,N^7}{(N-2)^2(N-1)^2N^2(N+1)(N+2)}\mathbf{S_{2,1}}(N)\\
&+\frac{-112-4\,N^2-4\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{2,1}(N)\mathbf{S_1}(N)+\frac{-48+8\,N+6\,N^2+7\,N^3}{(N-1)N(N+1)(N+2)}S_3(N)S_1(N)\\
&+\frac{-1840+292\,N+5532\,N^2+827\,N^3-1978\,N^4-274\,N^5+36\,N^6+19\,N^7-22\,N^8}{4(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_{1,1,1}(N)\\
&+\frac{128-56\,N-252\,N^2+54\,N^3+177\,N^4-91\,N^5+19\,N^6+9\,N^7}{2(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_3(N)\\
&+\frac{4032-2048\,N-14200\,N^2+5036\,N^3+23610\,N^4+2521\,N^5-12342\,N^6}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-3365\,N^7+2148\,N^8+903\,N^9+14\,N^{10}-167\,N^{11}+50\,N^{12}}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-124+16\,N+24\,N^2-4\,N^3-14\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{1,1}(N)\zeta(2)+\frac{424-118\,N+9\,N^2-2\,N^3+23\,N^4}{4(N-2)(N-1)N(N+1)(N+2)}S_2(N)S_{1,1}(N)\\
&+\frac{224+144\,N-1216\,N^2-56\,N^3+1786\,N^4+641\,N^5-406\,N^6}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)\\
&+\frac{+17\,N^7-308\,N^8+141\,N^9-56\,N^{10}+N^{11}}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)+\frac{58+21\,N+N^2+15\,N^3+10\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_2(N)\zeta(2)
\end{aligned}
$$

Example: Bhabha s+v

$$+\frac{232-384N^2-17N^3+286N^4-128N^5-14N^6+N^7}{4(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_2(N)S_1(N)$$
$$+\frac{-560-26N-31N^2-10N^3-33N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_2(N)^2$$
$$+\frac{576+1088N-3280N^2-5136N^3+11764N^4+20392N^5-17385N^6-30114N^7}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{+5984N^8+17228N^9-1228N^{10}-2754N^{11}-112N^{12}-8N^{13}+33N^{14}-24N^{15}}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{-56+336N+522N^2+424N^3-53N^4-500N^5+60N^6+28N^7-5N^8}{2(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_1(N)\zeta(2)$$
$$+\frac{64+6N^2+N^3}{(N-2)(N-1)N(N+1)}S_1(N)\zeta(3)+\frac{2112+608N+76N^2-140N^3+107N^4}{10(N-2)(N-1)N(N+1)(N+2)}\zeta(2)^2$$
$$+\frac{-224-136N+1688N^2+1290N^3-1998N^4-1997N^5+198N^6}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{+405N^7+376N^8-119N^9+56N^{10}+5N^{11}}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{-552+144N+1654N^2-370N^3-361N^4+19N^5+35N^6-25N^7}{2(N-2)^2(N-1)^2N^2(N+1)^2}\zeta(3)$$
$$+\frac{320-64N-1920N^2+1600N^3+6524N^4-14872N^5-19036N^6+31543N^7-43960N^8-13935N^9}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+\frac{+65372N^{10}+26822N^{11}-44576N^{12}-9558N^{13}+9840N^{14}+339N^{15}+428N^{16}-371N^{17}+128N^{18}}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+4\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,2}+(-2)\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,1}^2$$

z-space: A. Penin, (2005)

$\Longrightarrow$ 3 basic sums only; no alternating sums.

5. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt\ t^{z-1}\ \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\dots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \,\epsilon\, \mathbf{C}$, obeys a recursion $z \to z+1$ and has an analytic asymptotic representation.
- The poles are situated at the non-positive integers.

Examples:

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1) &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\text{Asymp. ser. : } \mathrm{Li}_2(z) &\to -\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z) + \zeta_2 \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\sim \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6}\dots
\end{aligned}$$

Factorial Series

$$F_{13}(z) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(z)}{z-1}\right)_+\right](z)$$

$$F_{13}(z+1) = -F_{13}(z) + \frac{\zeta_2^2}{z} + \frac{4\zeta_3}{z^2} + \frac{2\zeta_2}{z^2} S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3}\left[S_1^2(z) + S_2(z)\right]$$

$$\text{Asymp. ser. : } \mathrm{Li}_2^2(z) \rightarrow \mathrm{Li}_2^2(1-z) + \ln^2(z)\ln^2(1-z) + \zeta_2^2 + 2\mathrm{Li}_2(1-z)\ln(z)\ln(1-z) + \ldots$$

$$\begin{aligned}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(1-z)}{z-1}\right)_+\right](N) \sim\; & \frac{1}{z^2} - \frac{7}{24}\frac{1}{z^4} + \frac{1}{12}\frac{1}{z^5} + \frac{223}{1080}\frac{1}{z^6} - \frac{7}{45}\frac{1}{z^7} - \frac{3767}{15120}\frac{1}{z^8} + \frac{38}{105}\frac{1}{z^9} \\ & + \frac{14327}{31500}\frac{1}{z^{10}} - \frac{198}{175}\frac{1}{z^{11}} - \frac{138673}{118800}\frac{1}{z^{12}} + \frac{3263}{693}\frac{1}{z^{13}} + \frac{5265804043}{1324323000}\frac{1}{z^{14}} \\ & - \frac{13399637}{525525}\frac{1}{z^{15}} - \frac{143341487}{8408400}\frac{1}{z^{16}} + \frac{25092}{143}\frac{1}{z^{17}} + \frac{34809672614}{402026625}\frac{1}{z^{18}} \\ & - \frac{5749693892}{3828825}\frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right)\end{aligned}$$

6. The Basis

$w=1$	$1/(x-1)_+$		
$w=2$	$\ln(1+x)/(x+1)$		
$w=3$	$\mathrm{Li}_2(x)/(x\pm 1)$		
$w=4$	$\mathrm{Li}_3(x)/(x+1)$	$S_{1,2}(x)/(x\pm 1)$	
$w=5$	$\mathrm{Li}_4(x)/(x\pm 1)$	$S_{1,3}(x)/(x\pm 1)$	$S_{2,2}(x)/(x\pm 1)$
	$\mathrm{Li}_2^2(x)/(x\pm 1)$	$[\ln(x)S_{1,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm 1)$	
$w=6$	$\mathrm{Li}_5(x)/(x+1)$	$S_{1,4}(x)/(x\pm 1)$	$S_{2,3}(x)/(x\pm 1)$
	$S_{3,2}(x)/(x\pm 1)$	$\mathrm{Li}_2(x)\mathrm{Li}_3(x)/(x\pm 1)$	$S_{1,2}(x)\mathrm{Li}_2(x)/(x+1)$
	$A_1(x)/(x+1)$	$A_2(x)/(x\pm 1)$	$A_3(x)/(x+1)$
	$H_{0,-1,0,1,1}(x)/(x-1)$	$[A_1(-x)+N_\alpha(x)]/(x+1)\vert_{\alpha=1..3}$	

- $O(\alpha)$ Wilson Coefficients/anom. dim. $\#1$
- $O(\alpha^2)$ Anomalous Dimensions $\#2$
- $O(\alpha^2)$ Wilson Coefficients $\# \leq 5$
- $O(\alpha^3)$ Anomalous Dimensions $\#15$
- $O(\alpha^3)$ Wilson Coefficients $\#35$

8. Conclusions

- We considered mathematical structures which determine no scale and single scale quantities in Quantum Field Theories.
- The former correspond to integrated cross sections, expansion coefficients of the β–function, or anomalous dimensions at fixed moments, etc.
- The latter correspond to differential scattering cross sections of one variable, N–dependent anomalous dimensions, coefficient functions, etc.
- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ w = 6 can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.
- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to algebraic relations [index structure], and structural relations $\mathrm{N} \epsilon \mathbf{Q}$, $\mathrm{N} \epsilon \mathbf{R}$.

- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.
- Up to $w = 6$ physical (pseudo-) observables are free of harmonic sums with index = {-1}. Up to $w = 5$ all numerator functions are Nielsen integrals.