Mathematical structure of heavy flavor operator matrix elements at $O(a_s^2)$ and beyond

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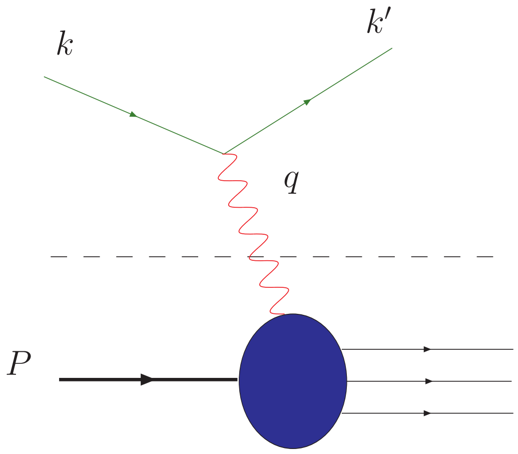
- Introduction
- Heavy OMEs
- 2–Loop
- Fixed Moments at 3–Loops
- Conclusions

based on:

- I. Bierenbaum, J. Blümlein, S. K., and C. Schneider
  `arXiv:0707.4759 [math-ph];`
  `arXiv:0803.0273 [hep-ph].`
- I. Bierenbaum, J. Blümlein, and S. K.,
  Phys. Lett. **B648** (2007) 195;
  Nucl. Phys. **B780** (2007) 40;
  arXiv:0706.2738; 0806.4613 [hep-ph];
  Acta Phys. Polon. B **38** (2007) 3543;
- J. Blümlein, A. De Freitas, W.L. van Neerven, and S. K.,
  Nucl. Phys. **B755** (2006) 272.

1. Introduction

Deep–Inelastic Scattering (DIS):



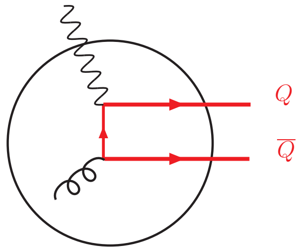
$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

LO:



Hadronic tensor for heavy quark production via single photon exchange:

$$W_{\mu\nu}^{Q\bar{Q}}(q,P) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P\rangle_{Q\bar{Q}}$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L^{Q\bar{Q}}(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2^{Q\bar{Q}}(x,Q^2) \; .$$

Need for the calculation:

- Heavy flavor (charm) contributions to DIS structure functions are rather large [20–40 % at lower values of x] .
- Increase in accuracy of the perturbative description of DIS structure functions.
- $\Longleftrightarrow$ QCD analysis and determination of $\Lambda_{\rm QCD}$, resp. $\alpha_s(M_Z^2)$, from DIS data: $\delta\alpha_s/\alpha_s \; < 1$ %.
- $\Longleftrightarrow$ Precise determination of the gluon and sea quark distributions.
- $\Longleftrightarrow$ Derivation of variable flavor number scheme for heavy quark production to $O(a_s^3)$.

Goal: Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,{\rm GeV}^2$ [sufficient in many applications] .

Previous calculations:

Unpolarized DIS :

- LO : [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979]
- NLO : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
  asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, Blümlein, Klein, 2007]
- Observation: $F_2^{c\bar{c}}(x, Q^2)$ is very well described by $F_2^{c\bar{c}}(x, Q^2)|_{Q^2 \gg m^2}$ for $Q^2 \gtrsim 10\, m_c^2$.

Polarized DIS :

- LO : [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]
- NLO :
  asymptotic: [Buza, Matiounine, Smith, van Neerven, 1997; Bierenbaum, Blümlein, Klein, 2008, to appear]

Mellin–space expressions: [Alekhin, Blümlein, 2003].

Variable flavor number scheme at $O(a_s^2)$: [Buza, Matiounine, Smith, van Neerven, 1998]

In the following, we report on results for unpolarized and polarized Heavy Flavor Wilson coefficients beyond NLO.

2. Heavy OMEs

- massless RGE and light–cone expansion in Bjorken–limit $\{Q^2, \ \nu\} \ \rightarrow \ \infty, \ x$ fixed:

$$\lim_{\xi^2\rightarrow 0}\Big[J(\xi), J(0)\Big] \propto \sum_{i,N,\tau} c^N_{i,\tau}(\xi^2,\mu^2)\xi_{\mu_1}...\xi_{\mu_N} O^{\mu_1\cdots\mu_m}_{i,\tau}(0,\mu^2) \ .$$

- RGE for collinear singularities

  $\Longrightarrow$ mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x,Q^2) = \sum_j \underbrace{C_i^j\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent

$$C^{\text{fl}}_{(2,L);i}\left(\frac{Q^2}{\mu^2}\right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C^{\text{fl},(l)}_{(2,L),i}, \qquad i = q, g$$

  $\Longrightarrow$ Known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2005.]

- Heavy quark contributions given by heavy quark Wilson coefficient, $H_{(2,L),i}^{\rm S,NS}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)$.

- In the limit $Q^2 \gg m_Q^2$ [$Q^2 \approx 10\ m_Q^2$ for F_2]:
massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H_{(2,L),i}^{\rm S,NS}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\rm S,NS}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}}.$$

- holds for polarized and unpolarized case. OMEs obey expansion

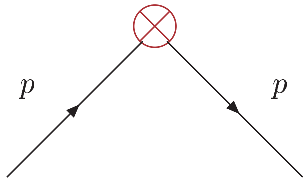
$$A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2}\right) = \langle i|O_k^{\rm S,NS}|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\rm S,NS,(l)}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

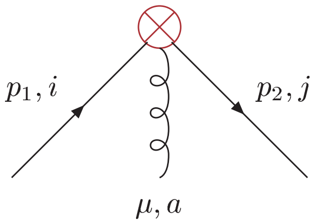
Operator insertions in light–cone expansion

E.g. singlet heavy quark operator:

$$O_Q^{\mu_1....\mu_N}(z) = \frac{1}{2} i^{N-1} S[\overline{q}(z)\gamma^{\mu_1} D^{\mu_2} .... D^{\mu_N} q(z)] - \mathsf{TraceTerms}\ .$$



$$\not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\ ,$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\ ,$$

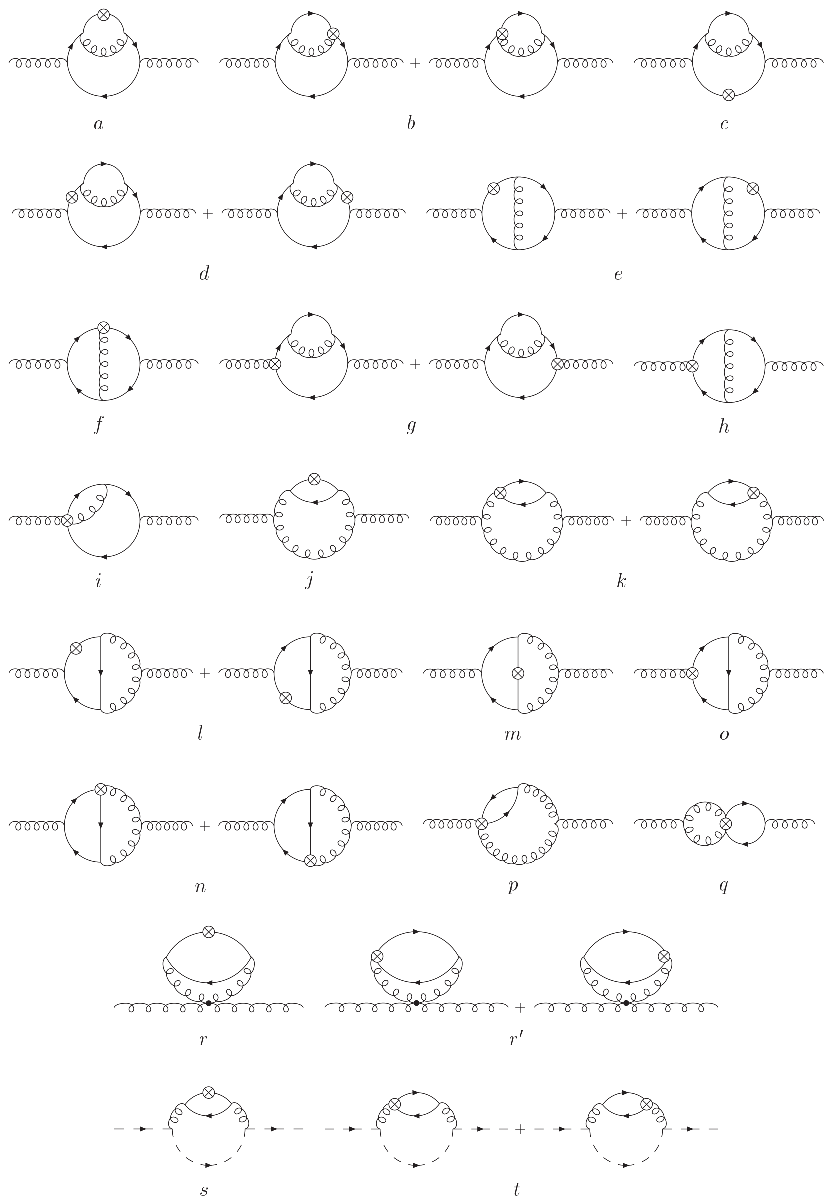
$$\gamma_+ = 1\ , \quad \gamma_- = \gamma_5\ .$$

Δ: light–like momentum, $\Delta^2 = 0$.

$\Longrightarrow$ additional vertices with 2 and more gluons at higher orders.

- Diagrams contain two scales: the mass m and the Mellin–parameter N.
- 2–point functions with on–shell external momentum, $p^2 = 0$.
  $\longrightarrow$ reduce for $N = 0$ to massive tadpoles.
- E.g. diagrams contributing to the gluonic OME

$$\hat{A}_{Qg}^{(2)} \qquad \Longrightarrow$$



Renormalization

$$\hat{\hat{A}}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{\hat{A}}_{ij}^{(k)}$$

need for:

- Mass renormalization (on–mass shell scheme)
- Charge renormalization

$\longrightarrow$ use $\overline{\rm MS}$ scheme $(D = 4 + \varepsilon)$ and
decoupling formalism [Ovrut, Schnitzer, 1981; Bernreuther, Wetzel, 1982].

- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities
  $\Longrightarrow$ are factored into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.
  For massless quarks it would hold: $\Gamma = Z^{-1}$.
  Here: Γ–matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} \quad = \quad Z^{-1}_{il} \hat{A}_{lm} \Gamma^{-1}_{mj} \ .$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

3. 2–Loops

- Calculation in Mellin-space for space–like q^2, $Q^2 = -q^2$: $0 \le x \le 1$
- use of generalized hypergeometric functions for general analytic results
- use of Mellin-Barnes integrals for numerical checks (MB, [Czakon, 2006]) and some analytic results
- Summation of lots of new infinite one-parameter sums into harmonic sums. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1\left(-3S_{2,1} + \frac{4S_3}{3}\right) - \frac{S_4}{2}$$

$$- S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1\zeta_3 + \zeta_2\left(2S_1^2 + S_2\right) .$$

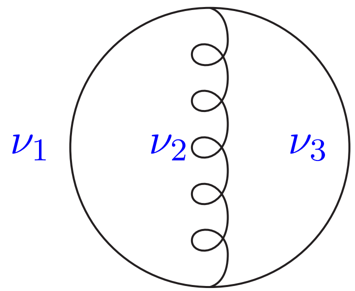
use of integral techniques and the Mathematica package SIGMA [C. Schneider, 2007], [I. Bierenbaum, J. Blümlein, S. K., C. Schneider, arXiv:0707.4659 [math-ph]; arXiv:0803.0273 [hep-ph]]

- Partial checks for fixed values of N using SUMMER, [Vermaseren, Int. J. Mod. Phys. A **14** (1999)].
- Algebraic and structural simplification of the harmonic sums [J. Blümlein, 2003, 2007].

Use of hypergeometric functions for general analytic results

$$ {}_PF_Q\left[\begin{array}{c}(a_1)...(a_P)\\(b_1)...(b_Q)\end{array};z\right] = \sum_{i=0}^{\infty}\frac{(a_1)_i...(a_P)_i}{(b_1)_i...(b_Q)_i}\frac{z^i}{\Gamma(i+1)}, \quad {}_1F_0[a;z]=\frac{1}{(1-z)^a}. $$

Consider the massive 2–loop tadpole diagram with arbitrary exponents ν_i and $\nu_{i...j} := \nu_i + ... + \nu_j$ etc.



Using Feynman–parameters, this integral can be cast into the general form

$$ I_1 = C_1 \int\!\!\int_0^1 dxdy\, \frac{x^a(1-x)^b y^c (1-y)^d}{(1-xy)^e}. $$

Thus one obtains

$$ I_1 = C_1\Gamma\left[\begin{array}{c}\nu_{123}-4-\varepsilon, \varepsilon/2-\nu_2, \nu_{23}-2-\varepsilon/2, \nu_{12}-2-\varepsilon/2\\ \nu_1, \nu_2, \nu_3, \nu_{123}-2-\varepsilon/2\end{array}\right] {}_3F_2\left[\begin{array}{c}\nu_{123}-4-\varepsilon, \varepsilon/2+2-\nu_2, \nu_3\\ \nu_3, \nu_{123}-2-\varepsilon/2\end{array};1\right]. $$

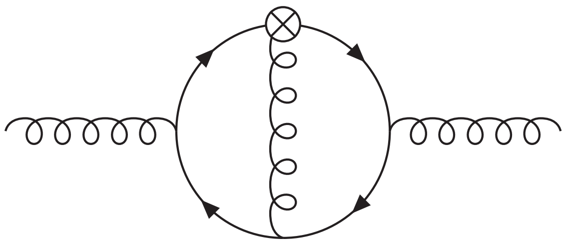
For any diagram deriving from the 2–loop tadpole topology, one obtains as a general integral

$$I_2 = C_2 \iint_0^1 dx dy \, \frac{x^a(1-x)^b y^c (1-y)^d}{(1-xy)^e} \int_0^1 dz_1 ... \int_0^1 dz_i \; \mathsf{P}\Big(x, y, z_1, ..., z_i, N\Big) \; .$$

Here P is a rational function of x, y and possibly more parameters $z_1 ... z_i$. N is the Mellin–parameter and occurs in some exponents.

$\Longrightarrow$ for fixed values of N, one obtains for all diagrams a finite sum over integrals of the type I_1.

Consider e.g the scalar Integral of the Diagram



$$I_3 = C_3 \exp\left\{\sum_{l=2}^{\infty} \frac{\zeta_l}{l} \varepsilon^l\right\} \frac{2\pi}{N \sin(\frac{\pi}{2}\varepsilon)} \sum_{j=1}^{N} \left\{ \binom{N}{j} (-1)^j + \delta_{j,N} \right\}$$
$$\times \left\{ \frac{\Gamma(j)\Gamma(j+1-\frac{\varepsilon}{2})}{\Gamma(j+2-\varepsilon)\Gamma(j+1+\frac{\varepsilon}{2})} - \frac{B(1-\frac{\varepsilon}{2}, 1+j)}{j} \; {}_3F_2\Big[1-\varepsilon, \frac{\varepsilon}{2}, j+1; 1, j+2-\frac{\varepsilon}{2}; 1\Big] \right\}$$
$$= C_3 \left\{ \frac{4}{N}\left[S_2 - \frac{S_1}{N}\right] + \frac{\varepsilon}{N}\left[-2S_{2,1} + 2S_3 + \frac{4N+1}{N} S_2 - \frac{S_1^2}{N} - \frac{4}{N} S_1\right]\right\} + O(\varepsilon^2) \; .$$

Unpolarized case, Singlet, O(ε)

$$
\begin{aligned}
\overline{a}_{Qg}^{(2)} = T_F C_F \Bigg\{ & \frac{2}{3} \frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)} S_1^2 \\
& + \frac{N^2+N+2}{N(N+1)(N+2)} \Big(16 S_{2,1,1} - 8 S_{3,1} - 8 S_{2,1} S_1 + 3 S_4 - \frac{4}{3} S_3 S_1 - \frac{1}{2} S_2^2 - S_2 S_1^2 - \frac{1}{6} S_1^4 + 2\zeta_2 S_2 - 2\zeta_2 S_1^2 - \frac{8}{3} \zeta_3 S_1 \Big) \\
& - 8 \frac{N^2-3N-2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2 S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
& + \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \Bigg\} \\
+ T_F C_A \Bigg\{ & \frac{N^2+N+2}{N(N+1)(N+2)} \Big(16 S_{-2,1,1} - 4 S_{2,1,1} - 8 S_{-3,1} - 8 S_{-2,2} - 4 S_{3,1} - \frac{2}{3} \beta''' + 9 S_4 - 16 S_{-2,1} S_1 \\
& + \frac{40}{3} S_1 S_3 + 4\beta'' S_1 - 8\beta' S_2 + \frac{1}{2} S_2^2 - 8\beta' S_1^2 + 5 S_1^2 S_2 + \frac{1}{6} S_1^4 - \frac{10}{3} S_1 \zeta_3 - 2 S_2 \zeta_2 - 2 S_1^2 \zeta_2 - 4\beta' \zeta_2 - \frac{17}{5} \zeta_2^2 \Big) \\
& + \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2} \Big(-4 S_{-2,1} + \beta'' - 4\beta' S_1 \Big) - \frac{2}{3} \frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3} \beta' \\
& + 2 \frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2} S_2 S_1 - \frac{16}{3} \frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2+N-1}{(N+1)^2(N+2)^2} \zeta_2 S_1 \\
& - \frac{2}{3} \frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
& - \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \Bigg\} .
\end{aligned}
$$

- Representation in terms of hypergeometric series allows feasible computation of higher orders in ε & automatized check for fixed values of N.
- For genuine scalar 2–Loop Integrals see [Bierenbaum, Blümlein and S. K., (2007).]
- Structure of expression is given by:

$$\beta(N+1) = (-1)^N[S_{-1}(N) + \ln(2)] \ ,$$
$$\beta^{(k)}(N+1) = \Gamma(k+1)(-1)^{N+k}[S_{-k-1}(N) + (1-2^{-k})\zeta_{k+1}] \ , \ k \geq 2 \ .$$

- New: Calculation of the $O(\varepsilon)$–terms in the polarized and the unpolarized case.
- Appearing functions:

$$\{S_1, \ S_2, \ S_3, \ S_4, \ S_{-2}, S_{-3}, S_{-4}\}, \quad S_{2,1}, \quad S_{-2,1}, \quad S_{-3,1}, \quad S_{2,1,1}, \quad S_{-2,1,1}$$

$S_{-2,2}$ depends on $S_{-2,1}$, $S_{-3,1}$

$S_{3,1}$ depends on $S_{2,1}$

$\Longrightarrow$ 6 basic objects

- $\Longrightarrow$ harmonic sums with index $\{-1\}$ cancel (holds even for each diagram) [cf. Blümlein, 2004; Blümlein and Ravindran, 2005; Blümlein and Moch, in preparation].

4. Fixed moments at 3–loops

Contributing OMEs:

Singlet	A_{Qg}	$A_{qg,Q}$	$A_{gg,Q}$	$A_{gq,Q}$	mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$	$A_{qq,Q}^{\rm NS,-}$	$A_{qq,Q}^{\rm NS,v}$		

- All 2–loop $O(\varepsilon)$–terms in the unpolarized case are known:

$$\overline{a}_{Qg}^{(2)}, \quad \overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \overline{a}_{gg,Q}^{(2)}, \quad \overline{a}_{gq,Q}^{(2)}, \quad \overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]

  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–Loop heavy OMEs are present.

  $\Longrightarrow$ Calculation will provide first independent checks on $\gamma_{qg}^{(3)}$, $\gamma_{qq}^{(3),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(3),\rm NS\pm,v}$, $\gamma_{gg}^{(3)}$ and $\gamma_{gq}^{(3)}$.

- Calculation proceeds in the same way in the polarized case. Known so far :

$$\Delta\overline{a}_{Qg}^{(2)}, \quad \Delta\overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \Delta\overline{a}_{qq,Q}^{(2),\mathbf{NS}} = \overline{a}_{qq,Q}^{(2),\mathbf{NS}} \ .$$

First step: Calculation of fixed moments of $A_{ij}^{(3)}(N)$, $N = 2, 4, 6, \ldots$

Fixed moments using MATAD

- three–loop "self-energy" type diagrams with an operator insertion
- Extension: additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1}...\Delta^{\mu_n} \langle p|O_{\mu_1...\mu_n}|p\rangle = \Delta^{\mu_1}...\Delta^{\mu_n} \langle p|S\,\bar{\Psi}\gamma_{\mu_1}D_{\mu_2}...D_{\mu_n}\Psi|p\rangle = A(N)\cdot(\Delta p)^N$$

$$D_\mu = \partial_\mu - igt_a A_\mu^a \quad , \qquad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ-contraction]
- 3–loop OMEs are generated with QGRAF [Nogueira, 1993]
- Color factors are calculated with [Ritbergen, Schellekens, Vermaseren, 1998]
- Translation to suitable input for MATAD [Steinhauser, 2001]

Tests performed: Various 2–loop calculations for $N = 2, 4, 6, ...$ were repeated
$\rightarrow$ agreement with our previous calculation.

First Results

Non-singlet terms : $O(T_F^2 C_F)$ $\quad A_{qq,Q}^{(3),\text{NS}^\pm}(N)$: 2 heavy quark loops

$$\hat{\hat{A}}_{qq,QQ}^{(3),\mathrm{NS}} = \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2}\left(-\frac{8\beta_{0,Q}^2\gamma_{qq}^{(0),\mathrm{NS}}}{3\varepsilon^3} - \frac{4\beta_{0,Q}\gamma_{qq,Q}^{(1),\mathrm{NS}}}{3\varepsilon^2} + \frac{\gamma_{qq,QQ}^{(2),\mathrm{NS}} - 12\beta_{0,Q}a_{qq,Q}^{(2),\mathrm{NS}}}{3\varepsilon} + a_{qq,QQ}^{(3),\mathrm{NS}}\right).$$

$$A_{qq,QQ}^{(3),\mathrm{NS}} = \frac{1}{6}\ln^3\left(\frac{m^2}{\mu^2}\right)\beta_{0,Q}^2\gamma_{qq}^{(0),\mathrm{NS}} + \frac{1}{2}\ln^2\left(\frac{m^2}{\mu^2}\right)\beta_{0,Q}\gamma_{qq,Q}^{(1),\mathrm{NS}} + \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\gamma_{qq,QQ}^{(2),\mathrm{NS}}$$
$$+a_{qq,QQ}^{(3),\mathrm{NS}} + 4\beta_{0,Q}\overline{a}_{qq,Q}^{(2),\mathrm{NS}}.$$

$$\gamma_{qq,Q}^{(0),\mathrm{NS}} = 4C_F\left[2S_1 - \frac{3N^2+3N+2}{2N(N+1)}\right]$$

$$\gamma_{qq,Q}^{(1),\mathrm{NS}} = 4C_FT_F\left\{\frac{8}{3}S_2 - \frac{40}{9}S_1 + \frac{3N^4+6N^3+47N^2+20N-12}{9N^2(N+1)^2}\right\}$$

$$\gamma_{qq,QQ}^{(2),\mathrm{NS}} = C_FT_F^2\Bigg(\frac{128}{9}S_3 - \frac{640}{27}S_2 - \frac{128}{27}S_1$$
$$+8\frac{51N^6+153N^5+57N^4+35N^3+96N^2+16N-24}{27N^3(N+1)^3}\Bigg).$$

$$\hat{\hat{A}}_{qq,QQ}^{(3),\mathsf{NS}}(2) = \frac{1}{\varepsilon^3}\Bigl(-\frac{2048}{81}\Bigr) + \frac{1}{\varepsilon^2}\Bigl(-\frac{4096}{243}\Bigr) + \frac{1}{\varepsilon}\Bigl(-\frac{20224}{729} - \frac{256}{27}\zeta_2\Bigr) - \frac{28736}{2187} - \frac{2048}{81}\zeta_3 - \frac{512}{81}\zeta_2 \;,$$

$$\hat{\hat{A}}_{qq,QQ}^{(3),\mathsf{NS}}(8) = \frac{1}{\varepsilon^3}\Bigl(-\frac{632512}{8505}\Bigr) + \frac{1}{\varepsilon^2}\Bigl(-\frac{144967772}{2679075}\Bigr) + \frac{1}{\varepsilon}\Bigl(-\frac{285344205403}{3375634500} - \frac{79064}{2835}\zeta_2\Bigr) - \frac{740566685766263}{17013197880000} - \frac{632512}{8505}\zeta_3 - \frac{36241943}{1786050}\zeta_2 \;,$$

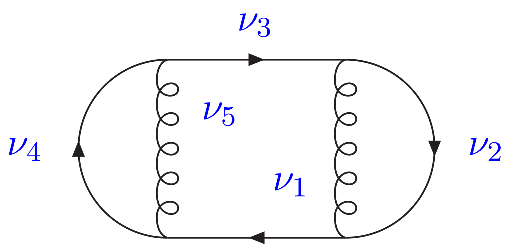
$$A_{qq,QQ}^{(3),\mathsf{NS}}(2) = T_F^2 C_F\Bigl[\frac{128}{81}\ln^3\Bigl(\frac{m^2}{\mu^2}\Bigr) + \frac{512}{81}\ln^2\Bigl(\frac{m^2}{\mu^2}\Bigr) - \frac{896}{243}\ln\Bigl(\frac{m^2}{\mu^2}\Bigr) + \frac{25024}{2187} - \frac{1792}{81}\zeta_3\Bigr] \;,$$

$$A_{qq,QQ}^{(3),\mathsf{NS}}(8) = T_F^2 C_F\Bigl[\frac{39532}{8505}\ln^3\Bigl(\frac{m^2}{\mu^2}\Bigr) + \frac{36241943}{1786050}\ln^2\Bigl(\frac{m^2}{\mu^2}\Bigr) - \frac{38920977797}{4500846000}\ln\Bigl(\frac{m^2}{\mu^2}\Bigr) + \frac{112863804957506 3}{34026395760000} - \frac{79064}{1215}\zeta_3\Bigr] \;.$$

All ζ_2 terms vanish after renormalization.

Fixed moments using Feynman–parameters

Consider e.g the 3–loop tadpole diagram

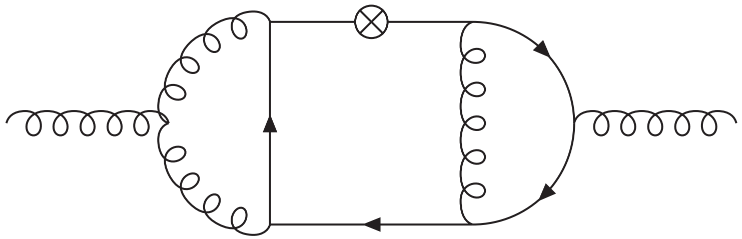


Using Feynman–parameters, one obtains a representation in terms of a double sum

$$I_4 = C_4 \Gamma \left[\begin{array}{c} 2+\varepsilon/2-\nu_1, 2+\varepsilon/2-\nu_5, \nu_{12}-2-\varepsilon/2, \nu_{45}-2-\varepsilon/2, \nu_{1345}-4-\varepsilon, \nu_{12345}-6-3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2+\varepsilon/2, \nu_{345}-2-\varepsilon/2, \nu_{12345}-4-\varepsilon \end{array} \right]$$
$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345}-2-\varepsilon/2)_{n+m}(\nu_{12345}-6-3/2\varepsilon)_m(2+\varepsilon/2-\nu_1)_m(2+\varepsilon/2-\nu_5)_n(\nu_{45}-2-\varepsilon/2)_n}{m!n!(\nu_{12345}-4-\varepsilon)_{n+m}(\nu_{345}-2-\varepsilon/2)_m(\nu_{345}-2-\varepsilon/2)_n},$$

which derives from a Appell–function of the first kind, F_1 .

As in the 2–loop case, for any diagram deriving from the tadpole–ladder topology, one obtains for fixed values of N a finite sum over double sums of the type I_4. Consider e.g.



For the above scalar diagram, we calculated first moments using MATAD and a representation in terms of Feynman–parameters (the results of which agree).

N	Results for I_4
0	$\frac{1}{\varepsilon}\frac{2}{9}+\frac{79}{36}-2\zeta_3$
2	$\frac{1}{\varepsilon}\frac{2723}{64800}+\frac{2690023}{3888000}-\frac{11}{18}\zeta_3$
4	$\frac{1}{\varepsilon}\frac{1119191}{79380000}+\frac{2347050779}{6667920000}-\frac{137}{450}\zeta_3$
6	$\frac{1}{\varepsilon}\frac{12925711}{2074464000}+\frac{679812813653}{3136589568000}-\frac{363}{1960}\zeta_3$
8	$\frac{1}{\varepsilon}\frac{141196078589}{43568189280000}+\frac{71608151768113879}{483084082736640000}-\frac{7129}{56700}\zeta_3$

5. Conclusions

- The heavy flavor contributions to the structure function F_2 are rather large in the region of lower values of x.

- QCD precision analyses therefore require the description of the heavy quark contributions to 3–loop order.

- We calculate the heavy flavor DIS Wilson Coefficients in the asymptotic regime $[Q^2 \geq 10m^2]$ using massive operator matrix elements.

- We recently presented first contributions to these corrections:
  - $\bar{a}_{Qg}^{(2)}$, $\bar{a}_{Qq}^{PS,(2)}$, $\bar{a}_{gg,Q}^{(2)}$, $\bar{a}_{qq}^{NS,(2)} = \Delta\bar{a}_{qq,Q}^{NS,(2)}$
  - $\Delta\bar{a}_{Qg}^{(2)}$, $\Delta\bar{a}_{Qg}^{PS,(2)}$

  in the unpolarized and polarized case for general values of the Mellin variable. These terms contribute to $H_{ij}^{(3)}$, $\Delta H_{ij}^{(3)}$ respectively, through renormalization.

- The calculation is performed in Mellin space, which allows to obtain compact results.
  - The analytic results were obtained using representations in terms of generalized hypergeometric functions.
  - Numerical checks were performed applying Mellin–Barnes integrals.
- Integral techniques and the summation package SIGMA have been used for summation. The results are given in the form of nested harmonic sums.
- We developed a programme–chain to calculate the massive operator matrix elements $A_{ij}^{(3)}$ for fixed Mellin moments based on QGRAF and MATAD
- We checked this procedure for some diagrams at the 3–loop level using representations in terms of infinite sums and found agreement.
- We presented first partial 3–loop results.