Applications of Computer Algebra (ACA'08)

Session 12: Symbolic computation and quantum field theory

Difference field algorithms for Quantum Field Theory

Carsten Schneider
RISC-Linz, Austria

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Starting point: bonus problem 6.69 in "Concrete Mathematics"

FIND a closed form for

$$\sum_{k=1}^{n} k^2 S_1(n+k) = \text{?},$$

where $S_1(n) := \sum_{k=1}^{n} \frac{1}{k} (= H_n)$.

D.E. Knuth:

"It would be nice to automate the derivation of formulas such as this."

Telescoping

GIVEN $f(k) = k^2 S_1(n+k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

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$$\boxed{f(k)=g(k+1)-g(k)}$$

Sigma computes

$$g(k)=\frac{1}{36}(k(-4k^2+(6n+3)k-12n(n+1)+1)$$
$$+6(2k^3-3k^2+k+n(n+1)(2n+1))S_1(n+k)).$$

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Summing the telescoping equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 S_1(n+k)} = \boxed{g(n+1) - g(1)}$$

$$= -\frac{1}{36} n(n+1)\Big(10n + 6(2n+1)S_1(n) - 12(2n+1)S_1(2n) - 1\Big).$$

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A difference field for the **summand:**

Take the rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)$$

Karr 1981

and the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{n+k+1}, \qquad \mathcal{S}\, S_1(n+k) = S_1(n+k) + \frac{1}{n+k+1}.$$

Telescoping

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FIND $g(k)$:

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$$\downarrow$$

GIVEN $f := k^2 h \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\boxed{f = \sigma(g) - g}$$

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$\downarrow$ Sigma

$$g = \frac{1}{36}\big(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) \\ + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))h\big)$$

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FIND $g \in \mathbb{F}$:

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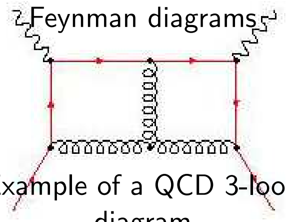
$\downarrow$ Sigma

$\uparrow$ h$\equiv S_1(n+k)$

$$g = \frac{1}{36}(k(-4k^2 + (6n+3)k - 12n(n+1) + 1) + 6(2k^3 - 3k^2 + k + n(n+1)(2n+1))h)$$

Evaluation of Feynman integrals

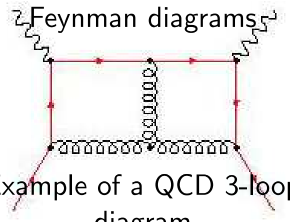
Feynman diagrams



Example of a QCD 3-loop diagram

Evaluation of Feynman integrals

Feynman diagrams



Example of a QCD 3-loop diagram

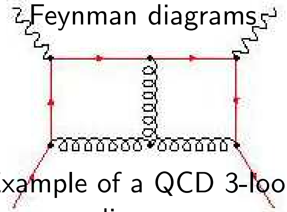
$\longrightarrow$

$$\int \Phi(x)dx$$

Task: Evaluation of Feynman integrals

Evaluation of Feynman integrals

Feynman diagrams



Example of a QCD 3-loop diagram

$\longrightarrow$

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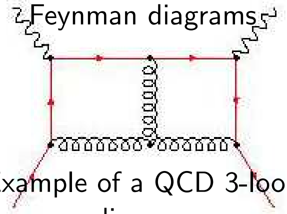
Task: Evaluation of Feynman integrals

$\downarrow$ Reduction (J. Blümlein; DESY)

multi-sums/ recurrences

Evaluation of Feynman integrals

Feynman diagrams



Example of a QCD 3-loop diagram

$\longrightarrow$

$$\int \Phi(x)dx$$

Task: Evaluation of Feynman integrals

Reduction
(J. Blümlein; DESY)

multi-sums/
recurrences

Sigma $\longleftarrow$

Simplified expressions/
solutions

Example 1: I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

GIVEN $F(N) =$

$$= \underbrace{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\right)}_{f(N,k,j)}.$$

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FIND the ϵ-expansion

$$F(N) = F_0(N) + \varepsilon F_1(N) + \dots$$

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Step 1: FIND the ϵ-expansion

$$f(N,k,j) = \qquad f_0(N,k,j) + \qquad \varepsilon f_1(N,k,j) + \dots$$

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Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) + \varepsilon \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) + \dots$$

$$\Big\| \qquad F(N)$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) =$$ ▸ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{a} f_0(N,k,j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!}$$
$$+\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}+\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \sum_{k=1}^{\infty}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$= \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}$$

where

$$S_2(N)=\sum_{i=1}^{N}\frac{1}{i^2}$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}\right.$$
$$\left.+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\right).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)+\dots,$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(N,k,j)=\frac{S_1(N)^2+3S_1(N)}{2N(N+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(N,k,j)=\frac{-S_1(N)^3-3S_2(N)S_1(N)-8S_3(N)}{6N(N+1)!}.$$

Similarly, we obtain simplifications like

$$\sum_{i,j=1}^{\infty}\frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)}=6\frac{S_1(N)}{N}\zeta_3+\zeta_2\Big(2\frac{S_1^2(N)}{N}+\frac{S_2(N)}{N}\Big)$$
$$+\frac{S_1^4(N)}{6N}+\frac{S_1^2(N)S_2(N)}{N}-\frac{S_2^2(N)}{N}+4\frac{S_{2,1,1}(N)}{N}$$
$$+S_1(N)\Big(-3\frac{S_{2,1}(N)}{N}+4\frac{S_3(N)}{3N}\Big)-2\frac{S_{3,1}(N)}{N}-\frac{S_4(N)}{2N}$$

where

$$S_a(N)=\sum_{i=1}^{N}\frac{1}{i^a}\qquad S_{2,1}(N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{1}{j^1}}{i^2}$$

$$S_{3,1}(N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{1}{j^1}}{i^3}\qquad S_{2,1,1}(N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k^1}}{j^1}}{i^2}$$

Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$ (I. Bierenbaum, J. Blümlein, S. Klein, C.S.)

Example 2: In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\boxed{F_{-1}(n)}\varepsilon^{-1}+\dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

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$$\downarrow$$

Initial values $F_{-1}(i)$, $i=1,\dots,450$

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$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + \boxed{F_{-1}(n)}\varepsilon^{-1} + \dots$$

$$\downarrow$$

Initial values $F_{-1}(i)$, $i = 1, \dots, 450$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_{-1}(n) + a_1(n)F_{-1}(n+1) + \dots + a_7(n)F_{-1}(n+7) = 0$$

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