Approximate implicitization

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Methods for finding the exact algebraic representation of rational parametric curves and surfaces are denoted implicitization. A number of well establish methods for exact implicitization exists such at resultant based methods and Groebner based method.

Floating point arithmetic is used in most industrial computer software. Exact implicitization methods implemented in floating point arithmetic gives approximate implicit curves and surfaces. In the region of computational interest the exact representation can have singular or near singular point that complicate the use of the algebraic representation. Often the use of the implicit representation is simplified if singular or near singular points are well separated from the region of computational interest. In other applications such as test for selfintersections it is important to have accurate reproduction of singularities.

Let l and g be integers with $1 \leq g < l$, and let $\mathbf{p}(\mathbf{s})$, $\mathbf{s} \in \Omega \subset \mathbb{R}^g$ be a manifold of dimension g in $\mathbb{R}^l$. The nontrivial algebraic hypersurface $q(\mathbf{x}) = 0$, $q \in P_m(\mathbb{R}^l)$, is an approximate implicitization of $\mathbf{p}(\mathbf{s})$ within the tolerance $\epsilon \geq 0$ if we can find a continuous function $\|\mathbf{g}(\mathbf{s})\|_2 = 1$ describing the direction for error measurement and a error function $|\eta(\mathbf{s})| \leq \epsilon$ such that $q(\mathbf{p}(\mathbf{s}) + \eta(\mathbf{s})\mathbf{g}(\mathbf{s})) = 0, \quad \mathbf{s} \in \Omega.$

The approximate implicitization approach is based on expressing the combination $q(\mathbf{p}(\mathbf{s}))$ as a matrix vector product $q(\mathbf{p}(\mathbf{s})) = (\mathbf{Db})^T \alpha(\mathbf{s})$. Here $\mathbf{D}$ is a matrix, $\mathbf{b}$ contains the coefficients of q and $\alpha(\mathbf{s})$ contains the basis functions related to the coordinate functions of $\mathbf{p}(\mathbf{s})$. This means that if $\mathbf{b}$ is in the null space of $\mathbf{D}$, then $q(\mathbf{p}(\mathbf{s})) = 0$. By assuming that the basis is a partition of unity we have $\|\alpha(\mathbf{s})\|_2 \leq 1$ and thus $|q(\mathbf{p}(\mathbf{s}))| \leq \|\mathbf{Db}\|_2$. The matrix $\mathbf{D}$ has desirable numeric properties if the coefficients of $\mathbf{p}(\mathbf{s})$ are contained in a simplex S, and this simplex is used for the description of q

in barycentric coordinates. When $\sigma_1 \geq 0$ is the smallest singular value of $\mathbf{D}$ we show that

$$\min_{||\mathbf{b}||_2=1} \max_{\mathbf{s}\in\Omega} \mid q(\mathbf{p}(\mathbf{s})) \mid \leq \sigma_1.$$

Singular value decomposition of $\mathbf{D}$ can thus be used for finding small singular values and thus an algebraic approximation of a parametric represented manifold. Constraints can be added to the algebraic approximation to control the behavior. The convergence rate of the approximation is higher than what is normal in approximation theory. Successful use of approximate algebraic surfaces depends a proper control of the gradient. Our first use of approximate implicitization was to separate near intersecting curves and surface. In the European project GAIA, IST-1999.29010, (www.math.sintef.no/gaia) we look into the potential of using approximate implicitization for detecting selfintersection. Examples of both these uses will be given.