NONLINEAR WAVES IN SHALLOW WATER

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The scaled surface wave equations for horizontal and vertical velocities $u(x,z,t)$ and $w(x,z,t)$ read:

$$\mu^2 u_x + w_z = 0 \quad , \quad -sx < z < \varepsilon\eta(x,t) \tag{1}$$

$$\eta_t + \varepsilon u \eta_x - \mu^{-2} w = 0 \quad , \quad z = \varepsilon\eta(x,t) \quad , \tag{2}$$

$$u_t + \varepsilon\left(u u_x + \mu^{-2} w w_x\right) + \eta_x = 0 \quad , \quad z = \varepsilon\eta(x,t) \quad , \tag{3}$$

$$w = -\mu^2 h_x u \quad , \quad z = -sx. \tag{4}$$

where $\varepsilon = a_0'/h_0'$, $\mu = h_0'/l_0'$ and a_0', h_0', l_0' are a characteristic wave amplitude, water depth, and wavelength, respectively. For the depth-averaged velocity $U \equiv \left(\int_{-h}^{\varepsilon\eta} u dz\right) / (h + \varepsilon\eta)$, Boussinesq-type equations retaining terms of orders $O(\mu^2)$ and $O(\varepsilon)$ were derived by Peregrin (1967). The work by Madsen & Schäffer (1998) contains an algorithm for constructing a series of the Boussinesq-type equations $\mathcal{B}_m$ retaining ε^m, $\varepsilon^{m-1}\mu^2$, ... , μ^{2m} terms. We consider the equations $\mathcal{B}_2$ for the case of sloping bottom in some area, excluding the deep water region where the shallow water restrictions are violated.

$$\begin{aligned}\mathcal{B}_2 : U_t + \eta_x + \varepsilon U U_x + \mu^2\left(-s^2 x U_{xt} - \frac{1}{3}s^2x^2U_{xxt}\right) + \\ \varepsilon\mu^2\left(-sU_t\eta_x - sU_{xt}\eta - sxU_{xt}\eta_x - s^2xUU_{xx} + \frac{1}{3}s^2x^2U_{xx}U_x - \frac{2}{3}sx\eta U_{xxt} - \frac{1}{3}s^2x^2U_{xxx}U\right) + \\ \mu^4\left(-\frac{4}{9}s^4x^2U_{xxt} - \frac{2}{9}s^4x^3U_{xxxt} - \frac{1}{45}s^4x^4U_{xxxxt}\right) = 0\end{aligned} \tag{5}$$

$$\eta_t + ((h + \varepsilon\eta)U)_x = 0 \tag{6}$$

The depth-averaged velocity U and surface elevation η are periodic and expanded in Fourier series with frequency ω. The major finding is the explicit expressions, found by computer, for the coefficients of the first four harmonics of the Fourier series calculated up to the orders ε^3, $\varepsilon\mu^2$, and μ^4 inclusively. They are polynomials of Bessel functions $J_0\left(2\omega\sqrt{\frac{x}{s}}\right)$, $Y_0\left(2\omega\sqrt{\frac{x}{s}}\right)$, $J_1\left(2\omega\sqrt{\frac{x}{s}}\right)$, $Y_1\left(2\omega\sqrt{\frac{x}{s}}\right)$ whose coefficients are polynomials of $x^{\frac{1}{2}}$ and $x^{-\frac{1}{2}}$. This result is closely related to note [3], where case of standing waves is considered

We conjecture that periodic solutions to $\mathcal{B}_m$ over a slope can be found as expansions of the form:

$$C^0(x) + S^1(x)\sin(\omega t) + C^1(x)\cos(\omega t) + ... + S^m(x)\sin(m\omega t) + C^m(x)\cos(m\omega t) + ... \tag{7}$$

where $S^m(x)$ and $C^m(x)$ are polynomials of Bessel functions $J_0\left(2\omega\sqrt{\frac{x}{s}}\right)$, $Y_0\left(2\omega\sqrt{\frac{x}{s}}\right)$, $J_1\left(2\omega\sqrt{\frac{x}{s}}\right)$, $Y_1\left(2\omega\sqrt{\frac{x}{s}}\right)$ whose coefficients are polynomials of $x^{\frac{1}{2}}$ and $x^{-\frac{1}{2}}$.

Velocities $u(x,z,t)$ and $w(x,z,t)$ can be expressed in terms of U, η, and their derivatives which permits to interpret the result as a periodic solution to classical wave problem (1) - (4) over a slope found up to the orders ε^2, $\varepsilon\mu^2$, and μ^4. This allows to conjecture that exact periodic solutions to the problem (1) - (4) can be described as a power series in z with coefficients of the form (7).

Numerous numeric results illustrating the presented method are included

References:

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