Present and Future of Proving Termination of Rewriting

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Overview of the talk

Overview of the talk

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  - Rewriting
  - Termination
  - Motivation

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- A bit of history

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- A bit of history
- Reduction Orderings
  - Polynomial Interpretations
  - Path Orderings

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- Powerful automatable methods
  - Dependency Pairs
  - Monotonic Semantic Path Ordering

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- Potential sources of improvement

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- Conclusions

Rewriting

Computing the product of positive integers

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Computing the product of positive integers

$\mathrm{prod}(2, 4) = 8$ $\qquad$ $\mathrm{prod}(s(s(0)), s(s(s(s(0))))) = s(s(s(s(s(s(s(s(0))))))))$

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prod(2, 4) = 8 $\qquad$ prod($s(s(0))$, $s(s(s(s(0))))$) = $s(s(s(s(s(s(s(s(0))))))))$

int prod (*int* n, *int* m) {
 if (n==0) *return*(0);
 else *return*(prod(n-1,m)+m);
 }

$$prod(0, m) \to 0$$
$$prod(s(n'), m) \to prod(n', m) + m$$

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Computing the product of positive integers

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```
int prod (int n, int m) {
 if (n==0) return(0);
 else      return(prod(n-1,m)+m);
 }
int prod ( int n,  int m) {
 int p= 0;
 while (n!=0) {
   p+=m;
   n- -;
 }
 return(p);
}
```

$$
\begin{array}{lcl}
prod(0, m) & \to & 0 \\
prod(s(n'), m) & \to & prod(n', m) + m
\end{array}
$$

$$
\begin{array}{lcl}
prod(n, m) & \to & prod1(n, m, 0) \\
prod1(0, m, p) & \to & p \\
prod1(s(n'), m, p) & \to & prod1(n', m, p + m)
\end{array}
$$

Rewriting (cont.)

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Higher-Order rewriting

Combining rewriting and β-reduction.

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With lambdas:

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Combining rewriting and β-reduction.

With lambdas:

- rewriting union β-reduction
- rewriting over β-normalized terms

Termination

Absence of infinite computations

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Example of non-termination of higher-order rewriting

Termination

Absence of infinite computations

Let $f : \alpha \times \alpha \Rightarrow \alpha$ and $g : (\alpha \to \alpha) \Rightarrow \alpha$ in

$$\{x : \alpha \to \alpha\} \vdash f(g(x), g(x)) : \alpha \longrightarrow @(x, g(x)) : \alpha$$

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- Knuth-Bendix completion and theorem proving
- Termination of some particular TRS
  - rule-based algorithms
  - rule-based specification

A bit of history

The beginings

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- 1979, Dershowitz Manna. Multiset ordering.
- 1979, Lankford. One of the first references on polynomial interpretations.
- 1980, Kamin and Levy. Lexicographic Path Ordering (LPO) and Semantic Path Ordering (SPO).

Reduction Orderings

$$s(s(\overbrace{s(\underbrace{s(0)}_{n}) + \underbrace{s(s(s(s(0))))}_{m}}))\ \rightarrow\ s(s(\overbrace{s(\underbrace{s(0)}_{n} + \underbrace{s(s(s(s(0))))}_{m})}))$$

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Reduction Orderings (cont.)

$\succ$ is a *reduction ordering* if it's

- *well-founded*

$$\nexists t_1 \succ t_2 \succ t_3 \succ \ldots$$

- *stable under sustitution*

$$s \succ t \quad \text{implies} \quad s\sigma \succ t\sigma \quad \text{for every substitution } \sigma$$

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$$s \succ t \quad \text{implies} \quad u[s] \succ u[t] \quad \text{for every context } u[\ ]$$

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$$R \text{ terminates} \quad \text{if and only if} \quad l \succ r \text{ for all } l \to r \in R$$

for some reduction ordering $\succ$.

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- *functional*

$$s \longrightarrow_{\beta} t \quad \text{implies} \quad s \succ t$$

Reduction Orderings (cont.)

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for some higher-order reduction ordering $\succ$.

Polynomial Interpretations

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Interprets every function symbol as a (linear) polynomial:
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Then a reduction ordering is obtained by
$s \succ t$ if $Pol(s) - Pol(t) > 0$
if all coeficients are strictly positive.

If there are zero coeficients then it is just weakly monotonic.

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Let $\succ_{\mathcal{F}}$ be a (well-founded) ordering on the function symbols
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The recursive path ordering

compares terms by extending an ordering on function symbols.

Let $\succ_{\mathcal{F}}$ be a (well-founded) ordering on the function symbols
For instance: $prod \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$

$$s = f(s_1, \ldots, s_n) \succ_{rpo} t \quad \text{if and only if}$$

1. $s_i \succeq_{rpo} t$ for some $1 \leq i \leq n$.
2. $t = g(t_1, \ldots, t_m)$, $f \succ_{\mathcal{F}} g$ and $s \succ_{rpo} t_j$ for all $1 \leq j \leq m$.
3. $t = f(t_1, \ldots, t_m)$ and $\{s_1, \ldots, s_n\} \succ\!\succ_{rpo} \{t_1, \ldots, t_m\}$

The *recursive path ordering* (cont.)

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3. $t = f(t_1, \ldots, t_m)$ and $\{s_1, \ldots, s_n\} \succ\succ_{rpo} \{t_1, \ldots, t_m\}$

Let $prod \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$

$$prod(s(n), m) \succ_{rpo} s(prod(n, m))$$

The *recursive path ordering* (cont.)

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$prod(s(n), m) \succ_{rpo} s(prod(n, m))$ Case 2

$prod(s(n), m) \succ_{rpo} prod(n, m)$

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Let $prod \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$

$prod(s(n), m) \succ_{rpo} s(prod(n, m))$ Case 2

$prod(s(n), m) \succ_{rpo} prod(n, m)$ Case 3

$\{s(n), m\} \succ\succ_{rpo} \{n, m\}$

The *recursive path ordering* (cont.)

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Let $prod \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$

$prod(s(n), m) \succ_{rpo} s(prod(n, m))$ Case 2

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Powerful automatable methods

Methods that can be fully automated

Can handle hard examples.

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Only two will be considered in this talk:

- The Dependency Pair Method
- The Monotonic Semantic Path Ordering

The Dependency Pair Method (cont.)

(Arts and Giesl 1997)

Basically, it transforms the rewrite system into a new reduction system which is simpler to analyze and (dis)prove termination.

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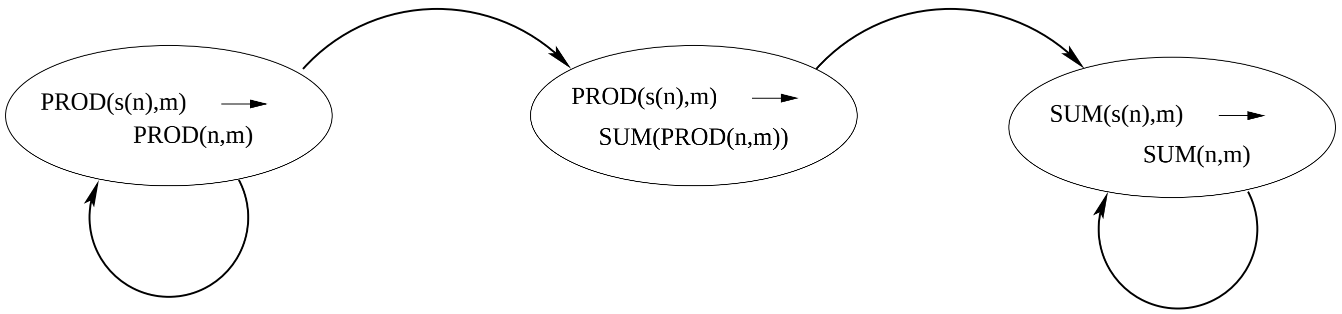

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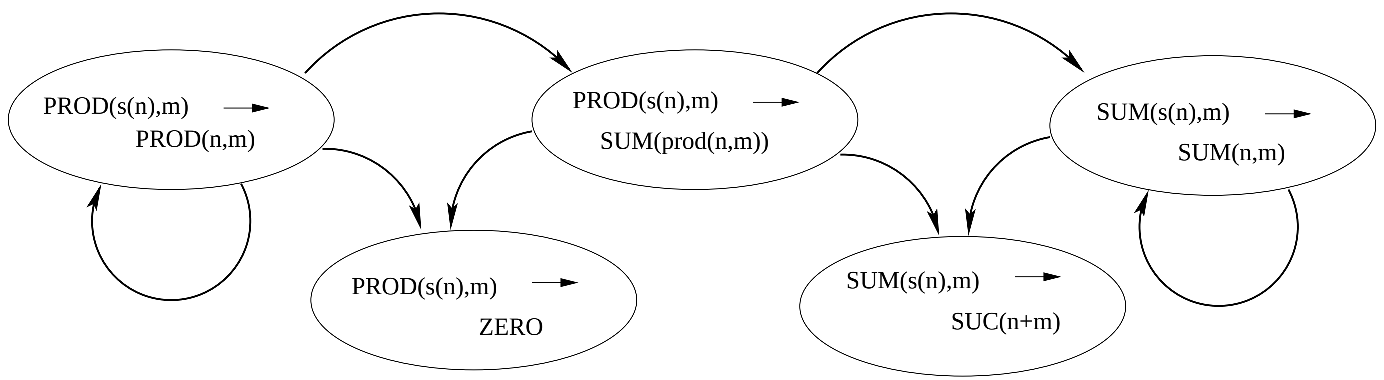

Does not change the potential cycles!!!

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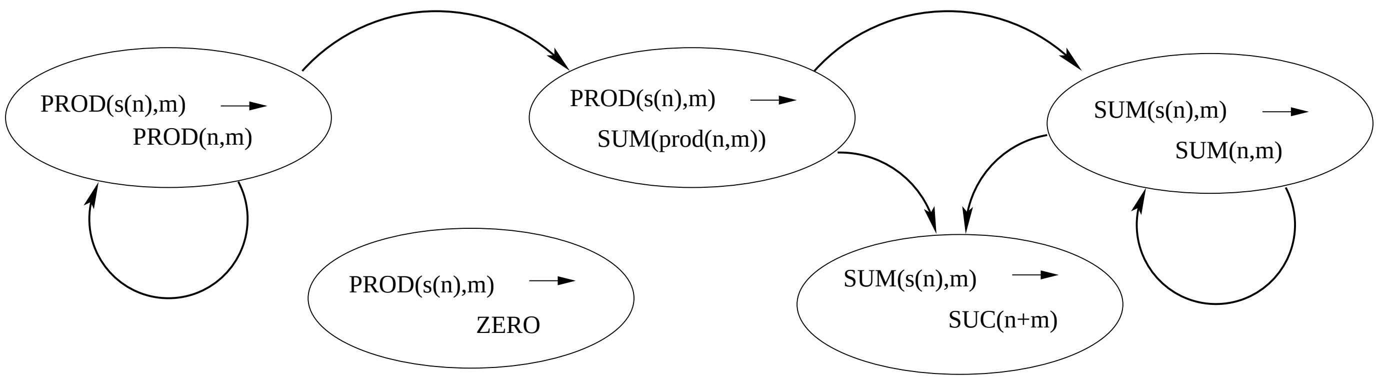

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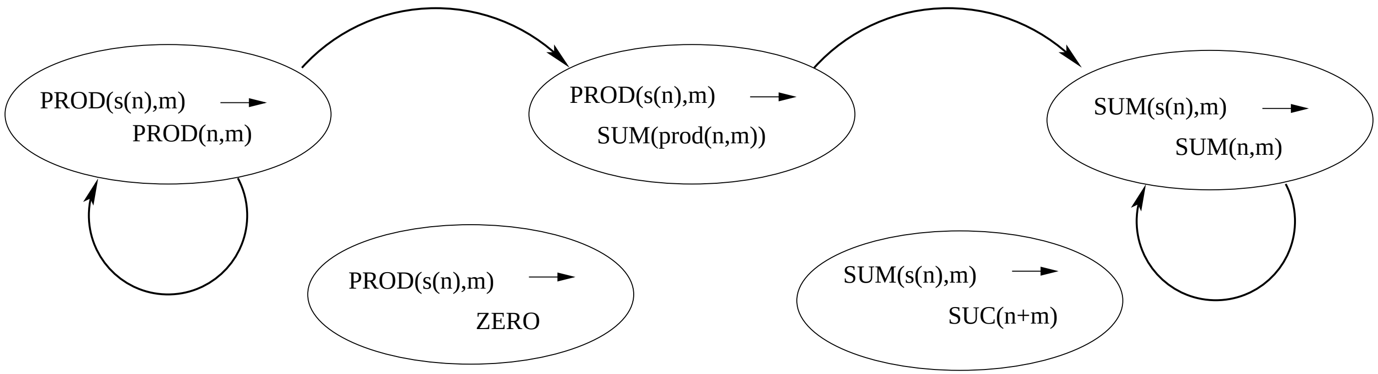

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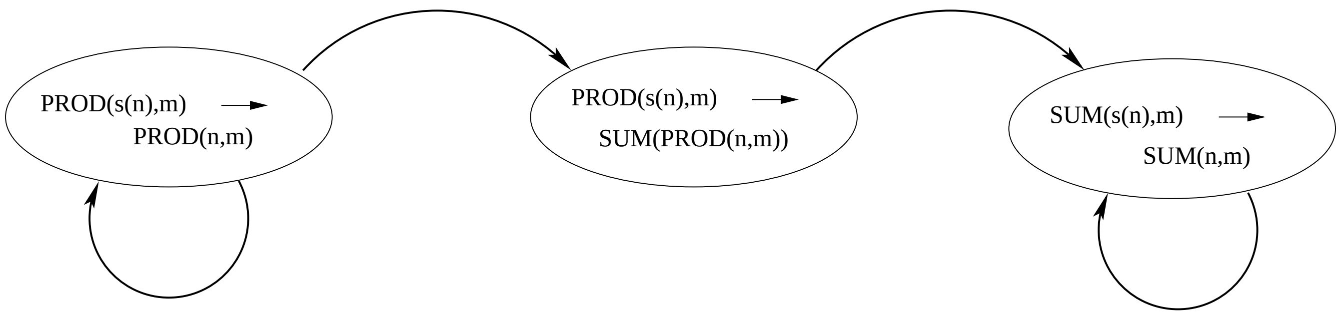

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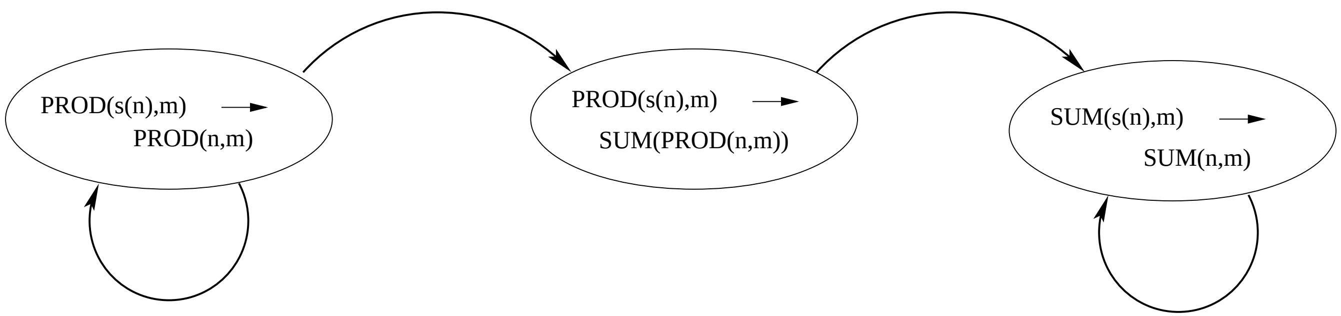

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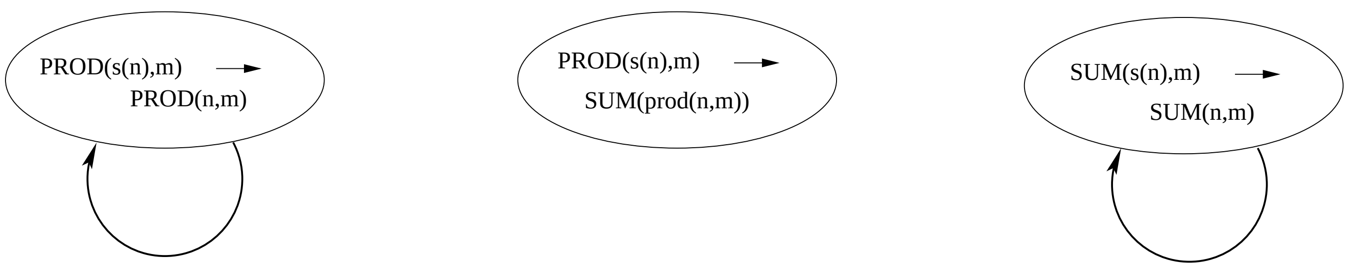

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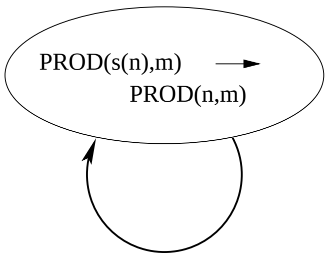

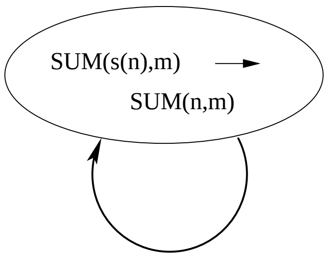

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The first phase its a simple rewriting trace analysis.

This method was a breakthrough in the development of automated termination provers for term rewriting

The Dependency Pair Method (cont.)

After simplifying the graph we have to show that all pontential cycles are finite.

Solve the ordering constraints ensuring that all cycles in the graph are strictly decreasing.

Find an ordering satisfying the constraint:

- Polynomial Interpretations.
- RPO-like orderings with argument filterings.
- ...

The (Monotonic) Semantic Path Ordering

(Borralleras, Ferreira and Rubio 2000)

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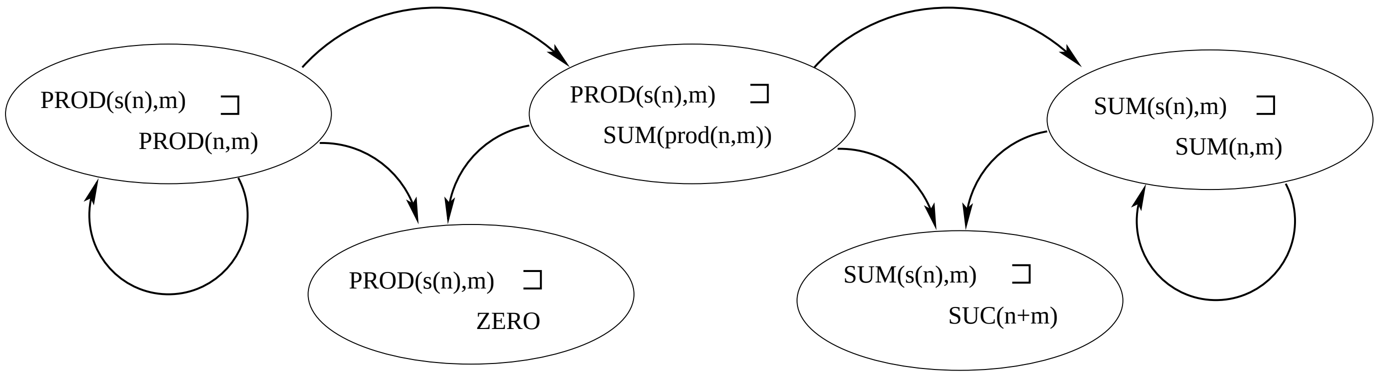

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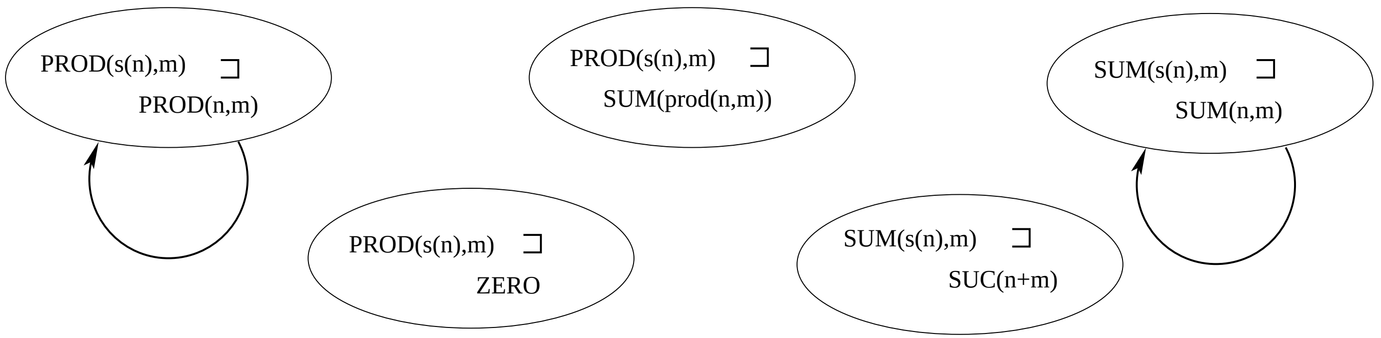

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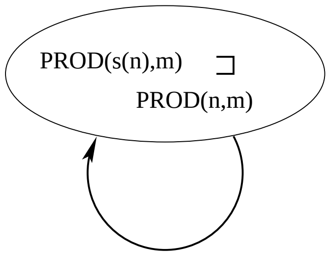

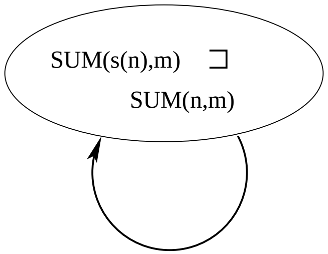

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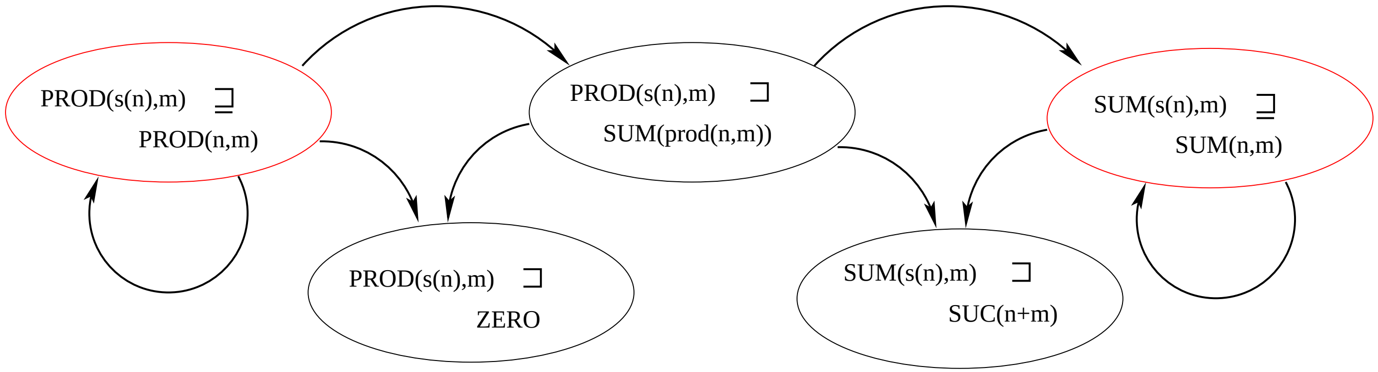

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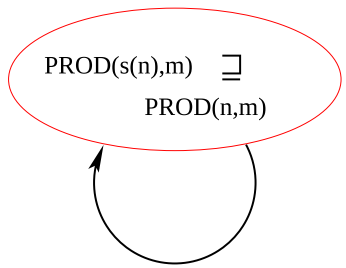

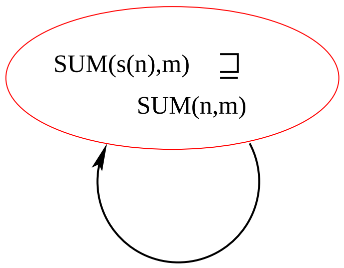

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0+m & \to & m & & & \\
s(n)+m & \to & s(n+m) & SUM(s(n),m) & \sqsupset & SUM(n,m) \\
 & & & SUM(s(n),m) & \sqsupseteq & SUC(n+m)
\end{array}
$$

The (Monotonic) Semantic Path Ordering

(Borralleras, Ferreira and Rubio 2000)

Different pathes provide different constraints to be solved.

The Dependency Pair constraint is one of them

It is unkown whether there are, in general, better constraints than the one used by the Dependency Pair Method.

$$
\begin{array}{lcllcl}
prod(0,m) & \rightarrow & 0 & prod(0,m) & \sqsupset & ZERO \\
prod(s(n),m) & \rightarrow & prod(n,m)+m & PROD(s(n),m) & \sqsupset & SUM(prod(n,m),m) \\
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Comparison of both methods

Monotonic Semantic Path Ordering	Dependency Pairs
Positive	Positive
Easy to extend	Conceptually simple
AC-case [Borralleras Thesis 2003]*	
CS-case [Borralleras Thesis 2003]	
HO-case [Borralleras Thesis 2003]*	
Constraint Framework [Borralleras Thesis 2003]*	Constraint Framework [GTS2004]

Comparison of both methods

Monotonic Semantic Path Ordering	Dependency Pairs
Negative	Negative
Conceptualy more difficult	Harder to extend e.g. HO-case

XXI century motivation

Present and Future

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Making a lot of plublicity of performance results.
Certified Termination.
Keep existing successful known tools alive: e.g. CiME

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making it accessible for users, developers and researchers.
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- RTA 2008: one third of the papers are on termination.
- RTA 2005-2008: it is the 30 % of the papers (34 papers).
- RTA 2001-2004: it is the 18 % of the papers (18.5 papers).

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It's hard to compare the real progress in two periods, but, at least it's a bit surprising.

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Research on first-order theorem proving has had a different behavior

even though they also have a competition, a large list of problems and an undecidable problem.

Potential sources of improvement

Constraint Solving techniques

Finding orderings to ensure that all cycles decrease it's a key ingredient in all provers
Like, for instance, SAT solvers or SMT solvers in many verification tools.
There have been several recent results on finding:

- RPO, LPO, KBO
- polynomial interpretations (integers and reals)
- matrix interpretations

by translating the problem into SAT or integer programming.

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These solvers can be used as an ingredient for the termination tool, and it is possible to use them as a black box.
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reusing the work done in previous stages.

These solvers, in particular the ones on polynomial interpretations, may be useful in other contexts.

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For instance,

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The existing systems can provide the problems to be solved.
There is no need to implement a full competitive termination prover.
It can be atractive for new participants.
It can be done together with the termination competition or not

Potential sources of improvement

Trace analysis

In all currently existing methods potentially looping traces are detected by analyzing the dependency graph.

There exist many other techniques, for instance in program analysis, for approximating exectution traces, and so detecting potential loops.

I don't know about any work on termination of rewriting using, for instance, predicate abstraction techniques.

New techniques need to be scalable to large programs.

This would be an alternative way of obtaining ordering constraints to ensure termination of all potential loops.
These constraints can be solved by the same solvers we have just described.

Potential sources of improvement

Built-in theories

This is a crucial matter in other verification tools.

Only very recently [FK2008] there has been an attempt to handle built-in intergers and other theories when proving termination of rewriting.
Having built-in integers is mandatory for many aplications.

We need to study how it combines with

- existing and new trace analysis techniques and
- the constraint solving techniques.

Maybe constraint solving should be restricted to polynomial interpretations, but not necessarily.

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- and will see....