Resultants and Neighborhoods of a Polynomial

Valentina Marotta
Department of Mathematics and Computer Science
Viale A. Doria, 6 - 95125, Catania, Italy.
E-mail: marotta@dmi.unict.it

In Computer Algebra and Symbolic Computation we deal with exact data, but in many real life situations data come either from physical measurements or observations or they are the output of numerical computations. Moreover computer works with Floating Point, i.e. with finite arithmetic, so a real number is represented approximately because of the limited number of available digits. For these reasons it is growing up the necessity of combining methods in computer algebra and numerical analysis. This led to a new branch of classic polynomial algebra, the *numerical polynomial algebra*, whose aim is to accommodate the presence of inaccurate data and inexact computation. Since a polynomial is not exact, a family of polynomials called *neighborhood* is considered as in the papers of Kaltofen and Stetter. We focus our attention on the univariate case. Let $p(x)=\sum_{j=0}^{n} a_j x^j \in \mathbb{R}[x]$ be a polynomial, the *tolerance* e associated with $p(x)$ is a non negative vector $e=\{e_0,\ldots,e_n\}$, such that $e_j \in \mathbb{R}$ and $e_j \geq 0$ for $j=0,\ldots,n$. The neighborhood $\mathcal{N}(p,e)$ of a polynomial p with tolerance e is the family of polynomials $\tilde{p} \in \mathbb{R}[x]$, $\tilde{p}=\sum_{j=0}^{n} \tilde{a}_j x^j$ such that $|\tilde{a}_j - a_j| \leq e_j$; $e_j=0$ means a_j is exact.
In this paper we give a new approach based on the idea of Resultant in order to discover the common factors between a polynomial and the polynomials in its neighborhood. Sometimes beside the acceptable variations for the coefficients of the uncertain polynomial $p(x)$, we know (often experimentally) more properties of the exact polynomial $\tilde{p}(x)$. For instance, we can know a zero of the exact polynomial, that we are looking for. So we construct $p(x)$, such that it has the same zero but it differs from the exact $\tilde{p}(x)$ in some coefficients.

Definition 1 *Let $p(x) \in \mathbb{R}[x]$ be a polynomial and let e be a vector of tolerance. The polynomial $\tilde{p}(x)$ is a* k common-factor perturbed *if $\tilde{p}(x) \in \mathcal{N}(p,e)$ and it has a common factor of degree at least k with $p(x)$. A* k common-factor neighborhood $\mathcal{N}_k(p,e)$ *of a polynomial* $p(x)$ *is the set of all* k *common-factor perturbed.*

$Res(p,\tilde{p})$ is a homogeneous polynomial of degree $2n$ in the δ_j's and a_j's and a homogeneous polynomial of degree n as polynomial in the only δ_j's, hence by using definition as above we have the algebraic conditions on the perturbed polynomial $\tilde{p}(x)$.
Moreover given a polynomial, the Square Free property of the polynomials in its neighborhood is investigated. By using the resultant we can find the square free conditions on a polynomial $\tilde{p}(x)$ in the neighborhood.
It is also useful to see all the conditions discussed above by a geometric point of view, where the neighborhood represents a polytope and the found algebraic conditions represent hyperplanes and hypersurfaces in the space of the δ_j's (or $\tilde{a}_j$'s).