Algorithmic Lie Theory for Solving Ordinary Differential Equations

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In the second half of the 19^{th} century Sophus Lie developed a theory for solving differential equations in analogy to Galois's theory of solving algebraic equations in terms of radicals. As an auxiliary device he established his theory of continuous groups, known as Lie-groups today, and its corresponding Lie-algebras.

In principle Lie's theory applies to differential equations of arbitrary order. However if the goal is to design solution algorithms based on this theory that may be implemented in a computer algebra system, several steps have to be worked out in more detail. To this end it is supplemented by two basic concepts:

1. Janet bases for systems of linear partial differential equations that determine the symmetry generators of a given differential equation and the transformation to canonical forms.
2. The decomposition of these Janet bases into irreducible components by Loewy's theory in analogy to the decomposition of ordinary linear differential equations.

Proceeding in this way completely algorithmic procedures are obtained for obtaining closed form solutions of equations with symmetries the solutions of which are contained in well-defined function fields. The basic steps involved are reviewed. Complete results are presented for second- and third order ordinary differential equations. Various possible extension are discussed, e.g. equations of order four or higher, and certain types of partial differential equations.

Software Demo: On top of the algebraic type system ALLTYPES, software for working with differential equations has been developed with special emphasis on Lie's symmetry theory. It is shown how the use of this software makes available many of these concepts that could not be used by conventional pencil-and-paper methods.