Chapter 8

Rational Parametrization of Curves

Most of the results in this chapter are obvious for lines. For this reason, and for simplicity in the explanation, we exclude lines from our treatment of rational parametrizations.

Throughout this chapter, unless explicitly stated otherwise, we use the following notation. K is an algebraically closed field of characteristic 0. In addition, if $\mathcal{C}$ is an affine rational curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$ over K (see Definition 8.1.1.), we write its components either as

$$\mathcal{P}(t) = \left(\frac{\chi_{1\,1}(t)}{\chi_{1\,2}(t)}, \frac{\chi_{2\,1}(t)}{\chi_{2\,2}(t)} \right),$$

where $\chi_{i,j}(t) \in K[t]$, or as

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t)),$$

where $\chi_i(t) \in K(t)$. Similarly, rational projective parametrizations (see Definition 8.1.2.) are expressed as

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t), \chi_3(t)),$$

where $\chi_i(t) \in K[t]$ and $\gcd(\chi_1, \chi_2, \chi_3) = 1$.

Furthermore, associated with a given parametrization $\mathcal{P}(t)$ we consider the polynomials

$$G_1^{\mathcal{P}}(s,t) = \chi_{1\,1}(s)\chi_{1\,2}(t) - \chi_{1\,2}(s)\chi_{1\,1}(t), \quad G_2^{\mathcal{P}}(s,t) = \chi_{2\,1}(s)\chi_{2\,2}(t) - \chi_{2\,2}(s)\chi_{2\,1}(t)$$

as well as the polynomials

$$H_1^{\mathcal{P}}(t,x) = x\chi_{1\,2}(t) - \chi_{1\,1}(t), \quad H_2^{\mathcal{P}}(t,y) = y\chi_{2\,2}(t) - \chi_{2\,1}(t).$$

Polynomials $G_1^{\mathcal{P}}$ will play an important role in Section 8.3 in deciding whether a parametrization $\mathcal{P}(t)$ is proper; i.e. in studing whether the parametrization is injective for almost all parameter values. Polynomials $H_i^{\mathcal{P}}$ will be used in Section 8.5 for the implicitization problem.

8.1 Rational Curves and Parametrizations

Some plane algebraic curves can be expressed by means of rational parametrizations, i.e. pairs of univariate rational functions that, except for finitely many exceptions, represent all the points on the curve. For instance, the parabola $y = x^2$ can also be described as the set $\{(t, t^2) \,|\, t \in K\}$; in this case, all affine points on the parabola are given by the parametrization (t, t^2). Or compare Example 6.2.2. Also, the tacnode curve (see Figure 8.1.) defined in $\mathbb{A}^2(\mathbb{C})$ by the polynomial

$$f(x, y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4$$

can be represented, for instance, as

$$\left\{ \left(\frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \quad \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \right) \,\middle|\, t \in \mathbb{C} \right\}$$

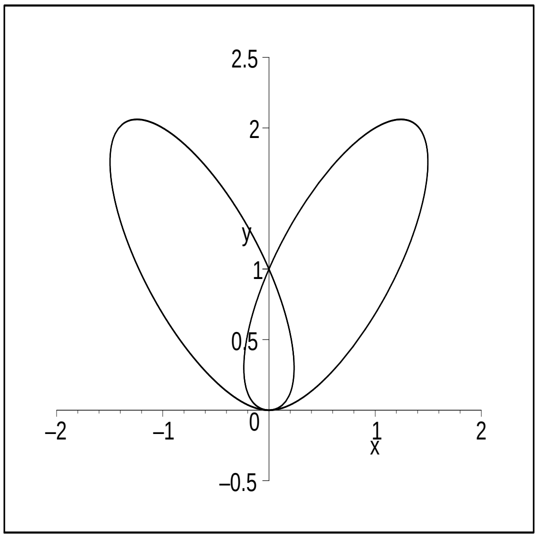

Figure 8.1: Tacnode curve

However, not all plane algebraic curves can be rationally parametrized, as we will see in Example 8.1.1. In this section we introduce the notion of rational or parametrizable curve and we study the main properties and characterizations of this type of curves. In the next sections we will show how to check the rationality by algorithmic methods and how to actually compute rational parametrizations of algebraic curves.

In Definition 6.2.5 we have introduced the notion of rationality for an arbitrary variety by means of rational isomorphisms. Now, we give a particular definition for the case of plane curves. Later, in Theorem 8.1.7, we prove that both definitions are equivalent.

Definition 8.1.1. The affine curve $\mathcal{C}$ in $\mathbb{A}^2(K)$ defined by the square–free polynomial $f(x,y)$ is *rational* (or *parametrizable*) if there are rational functions $\chi_1(t), \chi_2(t) \in K(t)$ such that

1. for almost all (i.e. for all but a finite number of exceptions) $t_0 \in K$, $(\chi_1(t_0), \chi_2(t_0))$ is a point on $\mathcal{C}$, and
2. for almost every point $(x_0, y_0) \in \mathcal{C}$ there is a $t_0 \in K$ such that $(x_0, y_0) = (\chi_1(t_0), \chi_2(t_0))$.

In this case $(\chi_1(t), \chi_2(t))$ is called a (*rational affine*) *parametrization* of $\mathcal{C}$.
We say that $(\chi_1(t), \chi_2(t))$ is in *reduced form* if the rational functions $\chi_1(t), \chi_2(t)$ are in reduced form; i.e. if for $i = 1, 2$ the gcd of the numerator and the denominator of χ_i is trivial. □

Definition 8.1.2. The projective curve $\mathcal{C}$ in $\mathbb{P}^2(K)$ defined by the square–free homogeneous polynomial $F(x,y,z)$ is *rational* (or *parametrizable*) if there are polynomials $\chi_1(t), \chi_2(t), \chi_3(t) \in K[t]$, $\gcd(\chi_1, \chi_2, \chi_3) = 1$, such that

1. for almost all $t_0 \in K$, $(\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$ is a point on $\mathcal{C}$, and
2. for almost every point $(x_0 : y_0 : z_0) \in \mathcal{C}$ there is a $t_0 \in K$ such that $(x_0 : y_0 : z_0) = (\chi_1(t_0) : \chi_2(t_0) : \chi_3(t_0))$.

In this case, $(\chi_1(t), \chi_2(t), \chi_3(t))$ is called a (*rational projective*) *parametrization* of $\mathcal{C}$. □

Remark.

(1) Later we will introduce the notion of local parametrization of a curve over K, not necessarily rational. Rational parametrizations are also called *global parametrizations*, and can only be achieved for genus zero curves (see Theorem 8.1.8.). On the other hand, since $K(t) \subset K((t))$, it is clear that any global parametrization is a local parametrization. By interpreting the numerator and denominator of a global parametrization as formal power series, and formally dividing, we get exactly a local parametrization.

(2) The notion of rational parametrization can be stated by means of rational maps as we did in Definition 6.2.5. More precisely, let $\mathcal{C}$ be a rational affine curve and $\mathcal{P}(t) \in K(t)^2$ a parametrization of $\mathcal{C}$. If $t_0 \in K$ is such that the denominators of the rational functions in $\mathcal{P}(t)$ are defined, then $\mathcal{P}(t_0) \in \mathcal{C}$. Thus, the parametrization $\mathcal{P}(t)$ induces the rational map

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t), \end{array}$$

and $\mathcal{P}(\mathbb{A}^1(K))$ is a dense (in the Zariski topology) subset of $\mathcal{C}$.

(3) Every rational parametrization $\mathcal{P}(t)$ defines a monomorphism from the field of rational functions $K(\mathcal{C})$ to $K(t)$ as follows (see proof of Theorem 8.1.6.):

$$\begin{array}{rccc} \varphi: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

□

Example 8.1.1. An example of an irreducible curve which is not rational is the projective cubic $\mathcal{C}$, defined over $\mathbb{C}$, by $x^3 + y^3 = z^3$. Suppose that $\mathcal{C}$ is rational, and let $(\chi_1(t), \chi_2(t), \chi_3(t))$ be a parametrization of $\mathcal{C}$ in reduced form. Then

$$\chi_1^3 + \chi_2^3 - \chi_3^3 = 0.$$

Differentiating this equation by t we get

$$3 \cdot (\chi_1'\chi_1^2 + \chi_2'\chi_2^2 - \chi_3'\chi_3^2) = 0.$$

So $\chi_1^2, \chi_2^2, \chi_3^2$ are a solution of the system of homogeneous linear equations with coefficient matrix

$$\begin{pmatrix} \chi_1 & \chi_2 & -\chi_3 \\ \chi_1' & \chi_2' & -\chi_3' \end{pmatrix}.$$

Gaussian elimination reduces this coefficient matrix to

$$\begin{pmatrix} \chi_2\chi_1' - \chi_2'\chi_1 & 0 & \chi_2'\chi_3 - \chi_2\chi_3' \\ 0 & \chi_2\chi_1' - \chi_2'\chi_1 & \chi_3'\chi_1 - \chi_3\chi_1' \end{pmatrix}.$$

So

$$\chi_1^2 : \chi_2^2 : \chi_3^2 = -\chi_2\chi_3' + \chi_3\chi_2' : -\chi_3\chi_1' + \chi_1\chi_3' : \chi_1\chi_2' - \chi_2\chi_1'.$$

Since χ_1, χ_2, χ_3 are relatively prime, this proportionality implies

$$\chi_1^2 \,|\, (\chi_2\chi_3' - \chi_3\chi_2'), \quad \chi_2^2 \,|\, (\chi_3\chi_1' - \chi_1\chi_3'), \quad \chi_3^2 \,|\, (\chi_1\chi_2' - \chi_2\chi_1').$$

Suppose $\deg(\chi_1) \geq \deg(\chi_2), \deg(\chi_3)$. Then the first divisibility implies $2\deg(\chi_1) \leq \deg(\chi_2) + \deg(\chi_3) - 1$, a contradiction. Similarly we see that $\deg(\chi_2) \geq \deg(\chi_1), \deg(\chi_3)$ and $\deg(\chi_3) \geq \deg(\chi_1), \deg(\chi_2)$ are impossible. Thus, there can be no parametrization of $\mathcal{C}$. □

Definition 8.1.1. is given for affine (resp. Definition 8.1.2. for projective) plane curves without multiple components. However, in the next theorem we show that only irreducible curves can be parametrizable.

Theorem 8.1.1. *Any rational curve is irreducible.*

Proof: Let $\mathcal{C}$ be a rational affine curve (similarly if $\mathcal{C}$ is projective) parametrized by a rational parametrization $\mathcal{P}(t)$. First observe that the ideal of $\mathcal{C}$ consists of the polynomials vanishing at $\mathcal{P}(t)$, i.e.

$$I(\mathcal{C}) = \{h \in K[x,y] \,|\, h(\mathcal{P}(t)) = 0\}.$$

Indeed, if $h \in I(\mathcal{C})$ then $h(P) = 0$ for all $P \in \mathcal{C}$. In particular h vanishes on all points generated by the parametrization, and hence $h(\mathcal{P}(t)) = 0$. Conversely, let $h \in K[x, y]$ be such that $h(\mathcal{P}(t)) = 0$. Therefore, h vanishes on all points of the curve generated by $\mathcal{P}(t)$, i.e. on all points of $\mathcal{C}$ with finitely many exceptions. So, it vanishes on $\mathcal{C}$, i.e. $h \in \mathcal{I}(C)$.

Finally, in order to prove that $\mathcal{C}$ is irreducible, we prove that $I(\mathcal{C})$ is prime. Let $h_1 \cdot h_2 \in I(\mathcal{C})$. Then $h_1(\mathcal{P}(t)) \cdot h_2(\mathcal{P}(t)) = 0$. Thus, either $h_1(\mathcal{P}(t)) = 0$ or $h_2(\mathcal{P}(t)) = 0$. Therefore, either $h_1 \in I(\mathcal{C})$ or $h_2 \in I(\mathcal{C})$. □

We start our analysis by showing that the rationality of a curve does not depend on whether we embed it into an affine or projective plane. So, in the sequel, we can choose freely between projective and affine situations, whatever we find more convenient.

Lemma 8.1.2. *Let $\mathcal{C}$ be an irreducible affine curve and $\mathcal{C}^*$ its corresponding projective curve. Then $\mathcal{C}$ is rational if and only if $\mathcal{C}^*$ is rational. Furthermore, a parametrization of $\mathcal{C}$ can be computed from a parametrization of $\mathcal{C}^*$ and vice versa.*

Proof: Let

$$(\chi_1(t), \chi_2(t), \chi_3(t))$$

be a parametrization of $\mathcal{C}^*$. Observe that $\chi_3(t) \neq 0$, since the curve $\mathcal{C}^*$ can have only finitely many points at infinity. Hence,

$$\left(\frac{\chi_1(t)}{\chi_3(t)}, \frac{\chi_2(t)}{\chi_3(t)}\right)$$

is a parametrization of the affine curve $\mathcal{C}$.

Conversely, a rational parametrization of $\mathcal{C}$ can always be extended to a parametrization of $\mathcal{C}^*$ by setting the z–coordinate to 1. □

Definition 8.1.1. clearly implies that associated with any rational plane curve there exists a pair of univariate rational functions over K, not both simultaneously constant, which is a parametrization of the curve. The converse is also true. That is, associated with any pair of univariate rational functions over K, not both simultaneously constant, there is a rational plane curve $\mathcal{C}$ such that the image of the parametrization is dense in $\mathcal{C}$. The implicit equation of this curve $\mathcal{C}$ is directly related to a resultant.

In the following lemma we state this property. Afterwards, in Section 8.5, we give a geometric interpretation to the integer r that appears in Lemma 8.1.3., proving that it counts the number of times the curve is traced when one gives values to the parameter of the parametrization.

Note that in the second part of the statement of this lemma, we require that the parametrization should not have a constant component. This is not a loss of generality since this situation corresponds to the lines $x = \lambda$ or $y = \lambda$, for some $\lambda \in K$.

Lemma 8.1.3. *Let $\chi_1(t), \chi_2(t) \in K(t)$ be rational functions in reduced form, not both of them constant. Then,*

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$$

parametrizes an irreducible plane curve $\mathcal{C}$ over K. Moreover, if none of the two rational functions is constant and $f(x, y)$ is the defining polynomial of $\mathcal{C}$, there exists $r \in \mathbb{N}$ such that

$$\mathrm{res}_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y)) = (f(x, y))^r.$$

Proof: If one of the two rational functions is constant, then $\mathcal{P}(t)$ parametrizes a horizontal or vertical line.
Now, let us assume that none of the components of $\mathcal{P}(t)$ is constant. Let $\chi_i(t) = \frac{\chi_{i,1}(t)}{\chi_{i,2}(t)}$, and let

$$h(x, y) = \mathrm{res}_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y)).$$

First we observe that $H_1^{\mathcal{P}}$ and $H_2^{\mathcal{P}}$ are irreducible, because $\chi_1(t)$ and $\chi_2(t)$ are in reduced form. Hence $H_1^{\mathcal{P}}$ and $H_2^{\mathcal{P}}$ do not have common factors. Therefore, $h(x, y)$ is not the zero polynomial. Furthermore, h cannot be a constant polynomial either. Indeed: let $t_0 \in K$ be such that $\chi_{1\,2}(t_0)\chi_{2\,2}(t_0) \neq 0$. Then $H_1^{\mathcal{P}}(t_0, \mathcal{P}(t_0)) = H_2^{\mathcal{P}}(t_0, \mathcal{P}(t_0)) = 0$. So $h(\mathcal{P}(t_0)) = 0$, and since h is not the zero polynomial it cannot be constant.

Now, we consider the square–free part $h'(x, y)$ of $h(x, y)$ and the plane curve $\mathcal{C}$ defined by $h'(x, y)$ over K. Let us see that $\mathcal{P}(t)$ parametrizes $\mathcal{C}$. For this purpose, we check the conditions introduced in Definition 8.1.1.

1. Let $t_0 \in K$ be such that $\chi_{1\,2}(t_0)\chi_{2\,2}(t_0) \neq 0$. Reasoning as above, we see that $h(\mathcal{P}(t_0)) = 0$. So $h'(\mathcal{P}(t_0)) = 0$, and hence $\mathcal{P}(t_0)$ is on $\mathcal{C}$.

2. Let c_1, c_2 be the leading coefficients of $H_1^{\mathcal{P}}, H_2^{\mathcal{P}}$ w.r.t. t, respectively. Note that $c_1 \in K[x], c_2 \in K[y]$ are of degree at most 1. For every (x_0, y_0) on $\mathcal{C}$ such that $c_1(x_0) \neq 0$ or $c_2(y_0) \neq 0$ (note that there is at most one point in K^2 where c_1 and c_2 vanish simultaneously), we have $h(x_0, y_0) = 0$. Thus, since h is a resultant, there exists $t_0 \in K$ such that $H_1^{\mathcal{P}}(t_0, x_0) = H_2^{\mathcal{P}}(t_0, y_0) = 0$. Also, observe that $\chi_{1\,2}(t_0) \neq 0$ since otherwise the first component of the parametrization would not be in reduced form. Similarly, $\chi_{2\,2}(t_0) \neq 0$. Thus, $(x_0, y_0) = \mathcal{P}(t_0)$. Therefore, almost all points on $\mathcal{C}$ are generated by $\mathcal{P}(t)$.

Now by Theorem 8.1.1. it follows that h' is irreducible. Therefore, there exists $r \in \mathbb{N}$ such that $h(x, y) = (h'(x, y))^r$. □

Sometimes it is useful to apply equivalent characterizations of the concept of rationality. In Theorems 8.1.4, 8.1.6., 8.1.7., and 8.1.8. some such equivalent characterizations are established.

Theorem 8.1.4. *An irreducible curve $\mathcal{C}$, defined by $f(x, y)$, is rational if and only if there exist rational functions $\chi_1(t), \chi_2(t) \in K(t)$, not both constant, such that*

$f(\chi_1(t), \chi_2(t)) = 0$. *In this case,* $(\chi_1(t), \chi_2(t))$ *is a rational parametrization of* $\mathcal{C}$.

Proof: Let $\mathcal{C}$ be rational. So there exist rational functions $\chi_1, \chi_2 \in K(t)$ satisfying conditions (1) and (2) in Definition 8.1.1. Obviously not both rational functions χ_i are constant, and clearly $f(\chi_1(t), \chi_2(t)) = 0$.

Conversely, let $\chi_1, \chi_2 \in K(t)$, not both constant, be such that $f(\chi_1(t), \chi_2(t))$ is identically zero. Let $\mathcal{D}$ be the irreducible plane curve defined by $(\chi_1(t), \chi_2(t))$ (see Lemma 8.1.3.). Then $\mathcal{C}$ and $\mathcal{D}$ are both irreducible, because of Theorem 8.1.1., and have infinitely many points in common. Thus, by Bézout's theorem one concludes that $\mathcal{C} = \mathcal{D}$. Hence, $(\chi_1(t), \chi_2(t))$ is a parametrization of $\mathcal{C}$. □

An alternative characterization of rationality in terms of field theory is given in Theorem 8.1.6. This theorem can be seen as the geometric version of Lüroth's Theorem. Lüroth's Theorem appears in basic text books on algebra such as [Wae70]. Here we do not give a proof of this result.

Theorem 8.1.5. (Lüroth's Theorem) *Let* $\mathbb{L}$ *be a field (not necessarily algebraically closed). Then every subfield* $\mathbb{K}$ *of* $\mathbb{L}(t)$, *where* t *is a transcendental element over* $\mathbb{L}$, *such that* $\mathbb{K}$ *strictly contains* $\mathbb{L}$, *is* $\mathbb{L}$*-isomorphic to* $\mathbb{L}(t)$.

Theorem 8.1.6. *An irreducible affine curve* $\mathcal{C}$ *is rational if and only if the field of rational functions on* $\mathcal{C}$, *i.e.* $K(\mathcal{C})$, *is isomorphic to* $K(t)$ *(*t *a transcendental element).*

Proof: Let $f(x, y)$ be the defining polynomial of $\mathcal{C}$, and let $\mathcal{P}(t)$ be a parametrization of $\mathcal{C}$. We consider the map

$$\begin{array}{rccc} \varphi_{\mathcal{P}}: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

First we observe that $\varphi_{\mathcal{P}}$ is well-defined. Let $\frac{p_1}{q_1}, \frac{p_2}{q_2}$, where $p_i, q_i \in K[x, y]$, be two different expressions of the same element in $K(\mathcal{C})$. Then f divides $p_1q_2 - q_1p_2$. Thus, by Theorem 8.1.4, $f(\mathcal{P}(t))$ is identically equal to zero, and therefore $p_1(\mathcal{P}(t))q_2(\mathcal{P}(t)) - q_1(\mathcal{P}(t))p_2(\mathcal{P}(t))$ is also identicall zero. Furthermore, since $q_1 \neq 0$ in $K(\mathcal{C})$, we have $q_1(\mathcal{P}(t)) \neq 0$. Similarly $q_2(\mathcal{P}(t)) \neq 0$. Therefore, $\varphi_{\mathcal{P}}(\frac{p_1}{q_1}) = \varphi_{\mathcal{P}}(\frac{p_2}{q_2})$.

Now, since $\varphi_{\mathcal{P}}$ is not the zero homomorphism, and $\varphi_{\mathcal{P}}$ is injective [1] one has that $\varphi_{\mathcal{P}}$ defines an isomorphism of $K(\mathcal{C})$ onto a subfield of $K(t)$ that properly contains K. Thus, by Lüroth's Theorem, this subfield, and $K(\mathcal{C})$ itself, must be isomorphic to $K(t)$.

Conversely, let $\psi : K(\mathcal{C}) \to K(t)$ be an isomorphism and $\chi_1(t) = \psi(x), \chi_2(t) = \psi(y)$. Clearly, since the image of ψ is $K(t)$, χ_1 and χ_2 cannot both be constant. Furthermore

$$f(\chi_1(t), \chi_2(t)) = f(\psi(x), \psi(y)) = \psi(f(x, y)) = 0.$$

Hence, by Theorem 8.1.4, $(\chi_1(t), \chi_2(t))$ is a rational parametrization of $\mathcal{C}$. □

[1] $\frac{p_1}{q_1}(\mathcal{P}) = \frac{p_2}{q_2}(\mathcal{P})$ implies $p_1q_2 - p_2q_1 = 0$ on infinitely many points, so it is identically 0

Remark. From the proof of Theorem 8.1.6 we see that every parametrization $\mathcal{P}(t)$ induces a monomorphism $\varphi_{\mathcal{P}}$ from $K(\mathcal{C})$ to $K(t)$. We will refer to $\varphi_{\mathcal{P}}$ as the *monomorphism induced* by $\mathcal{P}(t)$. □

Rationality can also be established by means of rational maps. The next characterization shows that Definitions 8.1.1 and 6.2.5 (for plane curves) are equivalent. Furthermore, it implies that the notions of rationality and unirationality are equivalent for plane curves.

Theorem 8.1.7. *An affine algebraic curve $\mathcal{C}$ is rational if and only if it is birationally equivalent to K (i.e. the affine line $\mathbb{A}^1(K)$).*

Proof: By Theorem 6.2.3. one has that $\mathcal{C}$ is birationally equivalent to K if and only if $K(\mathcal{C})$ is isomorphic to $K(t)$. Thus, by Theorem 8.1.6 we get the desired result. □

The following theorem states that rational curves are precisely those with genus zero. In fact, all irreducible conics are rational, and an irreducible cubic is rational if and only if it has a double point. We get this theorem by using the fact (which we have not proved) that the genus is invariant under birational maps.

Theorem 8.1.8. *An algebraic curve $\mathcal{C}$ is rational if and only if* $\mathrm{genus}(\mathcal{C}) = 0$.

8.2 Proper Parametrizations

Although the implicit representation for a plane curve is unique, up to a constant, there exist infinitely many different parametrizations of the same rational curve. For instance, for every $i \in \mathbb{N}$, (t^i, t^{2i}) parametrizes the parabola $y = x^2$. Obviously (t, t^2) is the parametrization of lowest degree in this family. Such parametrizations are called proper parametrizations.

The parametrization algorithms presented in this book always output proper parametrizations. Furthermore, there are algorithms for determining whether a given parametrization of a plane curve is proper, and if that is not the case, for transforming it to a proper one. In Section 6.1 we will describe these methods.

In this section, we introduce the notion of proper parametrization and we study some of the main properties. For this purpose, in the following we assume that $\mathcal{C}$ is an affine rational plane curve, and $\mathcal{P}(t)$ is a rational affine parametrization of $\mathcal{C}$.

Definition 8.2.1. An affine parametrization $\mathcal{P}(t)$ of a rational curve $\mathcal{C}$ is *proper* if the map

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{A}^1(K) & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t) \end{array}$$

is birational, or equivalently, if almost every point on $\mathcal{C}$ is generated by exactly one value of the parameter t.
We define the *inversion* of a proper parametrization $\mathcal{P}(t)$ as the inverse rational mapping of $\mathcal{P}$, and we denote it by $\mathcal{P}^{-1}$. □

Lemma 8.2.1. *Every rational curve can be properly parametrized.*

Proof: From Theorem 8.1.7. one deduces that any rational curve $\mathcal{C}$ is birationally equivalent to $\mathbb{A}^1(K)$. Therefore, any rational curve can be properly parametrized. □

The notion of properness can also be stated algebraically in terms of fields of rational functions. From Theorem 6.2.3 we deduce that a rational parametrization $\mathcal{P}(t)$ is proper if and only if the induced monomorphism $\varphi_{\mathcal{P}}$ (see Remark to Theorem 8.1.6)

$$\begin{array}{rccc} \varphi_{\mathcal{P}}: & K(\mathcal{C}) & \longrightarrow & K(t) \\ & R(x,y) & \longmapsto & R(\mathcal{P}(t)). \end{array}$$

is an isomorphism. Therefore, $\mathcal{P}(t)$ is proper if and only if the mapping $\varphi_{\mathcal{P}}$ is surjective, that is, if and only if $\varphi_{\mathcal{P}}(K(\mathcal{C})) = K(\mathcal{P}(t)) = K(t)$. More precisely, we have the following theorem.

Theorem 8.2.2. *Let $\mathcal{P}(t)$ be a rational parametrization of a plane curve $\mathcal{C}$. Then, the following statements are equivalent:*

(1) $\mathcal{P}(t)$ is proper.

(2) The monomorphism $\varphi_{\mathcal{P}}$ induced by $\mathcal{P}$ is an isomorphism.

(3) $K(\mathcal{P}(t)) = K(t)$.

Remark. We have introduced the notion of properness for affine parametrizations. For projective parametrizations the notion can be extended by asking the rational map, obtained by homogenizing the projective parametrization, from $\mathbb{P}^1(K)$ onto the curve to be birational. Moreover, if $\mathcal{C}$ is an irreducible affine curve and $\mathcal{C}^\star$ is its projective closure, then $K(\mathcal{C}) = K(\mathcal{C}^\star)$. Thus, taking into account Theorem 8.2.2. one has that the properness of affine and projective parametrizations are equivalent. □

Now, we characterize proper parametrizations by means of the degree of the corresponding rational curve. To state this result, we first introduce the notion of degree of a parametrization.

Definition 8.2.2. Let $\chi(t) \in K(t)$ be a non-zero rational function in reduced form. If $\chi(t)$ is not zero, the *degree* of $\chi(t)$ is the maximum of the degrees of the numerator and denominator of $\chi(t)$. If $\chi(t)$ is zero, we define its degree to be -1. We denote the degree of $\chi(t)$ as $\deg(\chi(t))$.
Rational functions of degree 1 are called *linear*. □

Obviously the degree is multiplicative with respect to the composition of rational functions. Furthermore, invertible rational functions are exactly the linear rational functions.

Definition 8.2.3. We define the *degree* of a rational affine parametrization $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ as the maximum of the degrees of its rational components; i.e.

$$\deg(\mathcal{P}(t)) = \max\{\deg(\chi_1(t)), \deg(\chi_2(t))\}. \qquad \square$$

We start this study with a lemma that shows how proper and improper parametrizations of a affine plane curve are related.

Lemma 8.2.3. *Let $\mathcal{P}(t)$ be a proper parametrization of a rational affine plane curve $\mathcal{C}$, and let $\mathcal{P}'(t)$ be any other rational parametrization of $\mathcal{C}$. Then*

(1) there exists a rational function $R(t) \in K(t) \setminus K$ such that $\mathcal{P}'(t) = \mathcal{P}(R(t))$;

(2) $\mathcal{P}'(t)$ is proper if and only if there exists a linear rational function $L(t) \in K(t)$ such that $\mathcal{P}'(t) = \mathcal{P}(L(t))$.

Proof: (1) We consider the following diagram

$$\begin{tikzcd}
\mathbb{A}^1(K) \arrow[r, "\mathcal{P}"] & \mathcal{C} \subset \mathbb{A}^2(K) \\
& \mathbb{A}^1(K) \arrow[u, "\mathcal{P}'"'] \arrow[ul, "\mathcal{P}^{-1} \circ \mathcal{P}'"]
\end{tikzcd}$$

Then, since $\mathcal{P}$ is a birational mapping, it is clear that $R(t) = \mathcal{P}^{-1}(\mathcal{P}'(t)) \in K(t)$.
(2) If $\mathcal{P}'(t)$ is proper, then from the diagram above we see that $\varphi = \mathcal{P}^{-1} \circ \mathcal{P}'$ is a birational mapping from $\mathbb{A}^1(K)$ onto $\mathbb{A}^1(K)$. Hence, by Theorem 6.2.3 one has that φ induces an automorphism $\tilde{\varphi}$ of $K(t)$ defined as:

$$\begin{array}{rccc} \tilde{\varphi}: & K(t) & \longrightarrow & K(t) \\ & t & \longmapsto & \varphi(t). \end{array}$$

Therefore, since K-automorphisms of $K(t)$ are the invertible rational functions (see e.g. [Wae70]), we see that $\tilde{\varphi}$ is our linear rational function.

Conversely, let ψ be the birational mapping from $\mathbb{A}^1(K)$ onto $\mathbb{A}^1(K)$ defined by the linear rational function $L(t) \in K(t)$. Then, it is clear that $\mathcal{P}' = \mathcal{P} \circ \psi : \mathbb{A}^1(K) \to \mathcal{C}$ is a birational mapping, and therefore $\mathcal{P}'(t)$ is proper. $\square$

Lemma 8.2.3. seems to suggest that a parametrization of prime degree is proper. But in fact, this is not true, as can easily be seen from the parametrization (t^2, t^2) of a line.

Before we can characterize the properness of a parametrization via the degree of the curve, we first derive the following technical property.

Lemma 8.2.4. *Let $p(x), q(x) \in K[x]^\star$ be relatively prime such that at least one of them is non–constant. There exist only finitely many values $a \in K$ such that the polynomial $p(x) - aq(x)$ has multiple roots.*

Proof: Let us consider the polynomial $f(x, y) = p(x) - yq(x) \in K[x, y]$. Since $\gcd(p, q) = 1$ and p, q are non-zero, one has that f is irreducible. Now we study the existence of roots of the discriminant of f w.r.t. y. Let $g(x, y) = \frac{\partial f}{\partial x}$. Note that g is non-zero, since at least one of the two polynomials p and q is not constant. If g is a constant it follows that there exists no $a \in K$ such that the polynomial $p(x) - aq(x)$ has multiple roots. On the other hand, if g is not constant, applying Bézout's Theorem one concludes that the curves defined by f and g have finitely many intersection points. Hence the result follows immediately. $\square$

Taking into account how intersection points of two curves are computed and the proof of the previous lemma, one has the following corollary.

Corollary. *Let $p(x), q(x) \in K[x]^\star$ be relatively prime such that at least one of them is non-constant, and let $R(y)$ be the resultant*

$$R(y) = \mathrm{res}_x(p(x) - yq(x), p'(x) - yq'(x)).$$

Then, for all $b \in K$ such that $R(b) \neq 0$, the polynomial $p(x) - bq(x)$ is squarefree.

Remark. If $\deg(p) > \deg(q)$ then the roots of the resultant are exactly the values of b for which $p(x) - bq(x)$ has multiple roots. However, if $\deg(p) \leq \deg(q)$ the leading coefficients, w.r.t. x, of the polynomials involved in the resultant may have a common root, and this root may generate extraneous factors in the resultant. For instance, take $p(x) = x^2, q(x) = 2x^2 + 1$. Then

$$p(x) - yq(x) = (1 - 2y)x^2 - y, \;\; p'(x) - yq'(x) = 2(1 - 2y)x$$

and, $R(y) = -4y(2y - 1)^2$. But taking $b = \frac{1}{2}$, one has that $p(x) - \frac{1}{2}q(x) = -\frac{1}{2}$ is squarefree. □

The next theorem characterizes the properness of a parametrization by means of the degree of the implicit equation of the curve.

Theorem 8.2.5. *Let $\mathcal{C}$ be a rational affine curve defined over K with defining polynomial $f(x, y) \in K[x, y]$, and let $\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$ be a parametrization of $\mathcal{C}$. Then $\mathcal{P}(t)$ is proper if and only if*

$$\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}.$$

Furthermore, if $\mathcal{P}(t)$ is proper, then $\deg(\chi_1(t)) = \deg_y(f)$, and $\deg(\chi_2(t)) = \deg_x(f)$.

Proof: First we prove the result for the special case of parametrizations having a constant component; i.e. for horizontal or vertical lines. Afterwards, we consider the general case. Let $\mathcal{P}(t)$ be a parametrization such that one of its two components is constant, say $\mathcal{P}(t) = (\chi_1(t), \lambda)$ for some $\lambda \in K$. Then the curve $\mathcal{C}$ is the line of equation $y = \lambda$. Hence, by Lemma 8.2.3 (2) and because (t, λ) parametrizes $\mathcal{C}$ properly, we get that all proper parametrizations of $\mathcal{C}$ are of the form $(\frac{at+b}{ct+d}, \lambda)$, where $a, b, c, d, \in K$ and $ad - bc \neq 0$. Therefore, $\deg(\chi_1) = 1$, and the theorem clearly holds.

In order to prove the general case, let $\mathcal{P}(t)$ be proper, in reduced form, such that none of its components is constant. Then we prove that $\deg(\chi_2(t)) = \deg_x(f)$, and analogously one can prove that $\deg(\chi_1(t)) = \deg_y(f)$. From these relations, we inmediately get that $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$. For this purpose, let $\chi_2(t) = \chi_{2,1}(t)/\chi_{2,2}(t)$ be in reduced form. We define $\mathcal{S}$ as the subset of K containing:

(a) all the second coordinates of those points on $\mathcal{C}$ that are not generated by $\mathcal{P}(t)$,

(b) those $b \in K$ such that the polynomial $\chi_{2\,1}(t) - b\chi_{2\,2}(t)$ has multiple roots,

(c) $lc(\chi_{2\,1})/lc(\chi_{2\,2})$, where " lc" denotes the leading coefficient,

(d) those $b \in K$ such that the polynomial $f(x, b)$ has multiple roots,

(e) the roots of the leading coefficient, with respect to x, of $f(x, y)$.

We claim that $\mathcal{S}$ is finite. Indeed: $\mathcal{P}(t)$ is a parametrization, so only finitely many points on the curve are not generated by $\mathcal{P}(t)$, and therefore only finitely many field elements satisfy (a). According to Lemma 8.2.4. there are only infinitely many field elements satisfying (b). The argument for (c) is trivial. An element $b \in K$ satisfies (d) if and only if b is the second coordinate of a singular point of $\mathcal{C}$ or the line $y = b$ is tangent to the curve at some simple point. $\mathcal{C}$ has only finitely many singular points, and $y = b$ is tangent to $\mathcal{C}$ at some point (a, b) if (a, b) is a solution of the system $\{f = 0, \frac{\partial f}{\partial x} = 0\}$. However, by Bézout's Theorem, this system has only finitely many solutions; note that f is not a line. So only finitely many field elements satisfy (d). Since the leading coefficient, with respect to x, of $f(x, y)$ is a non-zero univariate polynomial (note that, since $\mathcal{C}$ is not a line, f is a non-linear irreducible bivariate polynomial), only finitely many field elements satisfy (e). Therefore, $\mathcal{S}$ is finite.

Now we take an element $b \in K \setminus \mathcal{S}$ and we consider the intersection of $\mathcal{C}$ and the line of equation $y = b$. Since $b \notin \mathcal{S}$, by (e), one has that the degree of $f(x, b)$ is exactly $\deg_x(f(x, y))$, say $m := \deg_x(f(x, y))$. Furthermore, by (d), $f(x, b)$ has m different roots, say $\{r_1, \ldots, r_m\}$. So, there are m different points on $\mathcal{C}$ having b as a second coordinate, say $\{(r_i, b)\}_{i=1,\ldots,m}$, and they can be generated by $\mathcal{P}(t)$, because of (a).

On the other hand, we consider the polynomial $M(t) = \chi_{2\,1}(t) - b\chi_{2\,2}(t)$. We note that, since every point (r_i, b) is generated by some value of the parameter t, $\deg_t(M) \geq m$. But, since $\mathcal{P}(t)$ is proper and M cannot have multiple roots, we get that $\deg_t(M) = m = \deg_x(f(x, y))$. Now, since b is not the quotient of the leading coefficients of $\chi_{2\,1}$ and $\chi_{2\,2}$, we finally see that $\deg_x(f(x, y)) = \deg(M) = \max\{\deg(\chi_{2\,1}), \deg(\chi_{2\,2})\}$.

Conversely, let $\mathcal{P}(t)$ be a parametrization of $\mathcal{C}$ such that $\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$, and let $\mathcal{P}'(t)$ be any proper parametrization of $\mathcal{C}$. Then, by Lemma 8.2.3(1), there exists $R(t) \in K(t)$ such that $\mathcal{P}'(R(t)) = \mathcal{P}(t)$. Now, since $\mathcal{P}'(t)$ is proper, one deduces that $\deg(\mathcal{P}'(t)) = \max\{\deg_x(f), \deg_y(f)\} = \deg(\mathcal{P}(t))$. Therefore, since the degree is multiplicative with respect to composition, $R(t)$ must be of degree 1, and hence invertible. Thus, by Lemma 8.2.3(2), $\mathcal{P}(t)$ is proper. □

The next corollary follows from Theorem 8.2.5 and Lemma 8.2.3.

Corollary. *Let $\mathcal{C}$ be a rational affine plane curve defined by $f(x, y) \in K[x, y]$. Then the degree of any rational parametrization of $\mathcal{C}$ is a multiple of* $\max\{\deg_x(f), \deg_y(f)\}$.

Example 8.2.1. We consider the rational quintic $\mathcal{C}$ defined by the polynomial $f(x, y) = y^5 + x^2y^3 - 3\,x^2y^2 + 3\,x^2y - x^2$. Theorem 8.2.5 ensures that any rational proper parametrization of $\mathcal{C}$ must have a first component of degree 5, and a second component of degree 2. It is easy to check that

$$\mathcal{P}(t) = \left(\frac{t^5}{t^2+1}, \frac{t^2}{t^2+1} \right)$$

parametrizes properly $\mathcal{C}$. Note that $f(\mathcal{P}(t)) = 0$. $\square$

8.3 Parametrization by Lines

In this section we treat some straight-forward cases in which we can easily parametrize implicitly given algebraic curves. This approach will be generalized in the section. The basic idea consists in using a pencil of lines through a suitable point on the curve such that computing the intersection point of a generic element of the pencil with the curve one determines a parametrization of the curve.

We start with the simple case of rational conics (i.e. irreducible conics). Let $\mathcal{C}$ be an irreducible conic defined by the quadratic polynomial

$$f(x,y) = f_2(x,y) + f_1(x,y) + f_0(x,y)$$

where $f_i(x,y)$ is the homogeneous component of degree i. Let us first assume w.l.o.g. that $\mathcal{C}$ passes through the origin, so $f_0(x,y) = 0$. Let $\mathcal{H}(t)$ be the linear system of lines through the origin, the elements of $\mathcal{H}(t)$ being parametrized by their slope t. So the defining polynomial of $\mathcal{H}(t)$ is

$$h(x,y,t) = y - tx.$$

Now, we compute the intersection points of a generic element of $\mathcal{H}(t)$ and $\mathcal{C}$. That is, we solve in the variables x, y the system

$$\begin{cases} y = tx \\ f(x,y) = 0 \end{cases}$$

whose solutions are

$$P = (0,0) \quad \text{and} \quad Q = \left(-\frac{f_1(1,t)}{f_2(1,t)}, -\frac{t \cdot f_1(1,t)}{f_2(1,t)}\right).$$

Note that $f_1(x,y)$ is not identically zero, since $\mathcal{C}$ is an irreducible curve. Therefore, Q depends on the parameter t. Furthermore, the affine point Q is not reachable by at most two particular values of t, namely the roots of the quadratic form $f_2(x,y)$. Thus, for all but finitely many values of $t \in K$, $\mathcal{H}(t)$ and $\mathcal{C}$ intersects exactly at two different affine points (see Figure 8.2). The intersection point Q depends rationally on the parameter of t of $\mathcal{H}(t)$, and it yields the desired parametrization of the conic. So we have proved the following theorem.

Theorem 8.3.1. *The irreducible projective conic $\mathcal{C}$ defined by the polynomial $F(x,y,z) = f_2(x,y) + f_1(x,y)z$ (f_i a form of degree i, resp.), has the rational parametrization*

$$\mathcal{P}(t) = (-f_1(1,t), -tf_1(1,t), f_2(1,t)).$$

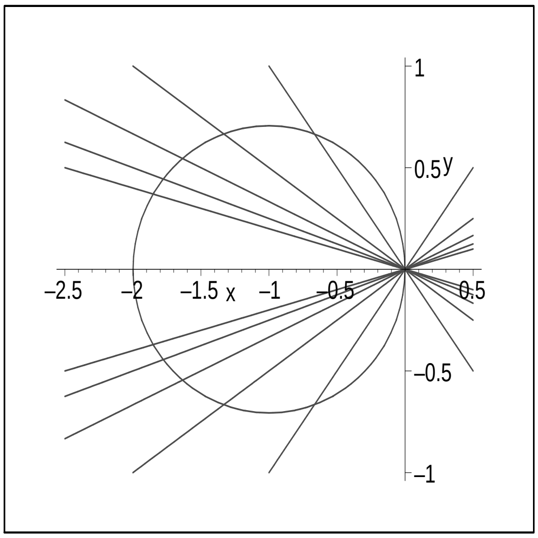

Figure 8.2: Ellipse $x^2 + 2x + 2y^2 = 0$ and pencil $\mathcal{H}(t)$

In this situation, using the previous theorem and making a suitable change of coordinates, one may derive the following parametrization algorithm for conics.

Algorithm CONIC-PARAMETRIZATION.
Given the defining polynomial $F(x, y, z)$ of an irreducible projective conic $\mathcal{C}$, the algorithm computes a rational parametrization.

1. Compute the homogeneus components f_2, f_1, f_0 of $F(x, y, 1)$.
2. If $(0 : 0 : 1) \in \mathcal{C}$ then return $\mathcal{P}(t) = (-f_1(1, t), -tf_1(1, t), f_2(1, t))$.
3. Compute a point $(a : b : 1) \in \mathcal{C}$.
4. $g(x, y) = F(x + a, y + b, 1)$. Let $g_2(x, y)$ and $g_1(x, y)$ be the homogeneous components of $g(x, y)$ of degree 2 and 1, respectively.
5. Return $\mathcal{P}(t) = (-g_1(1, t) + ag_2(1, t), -tg_1(1, t) + bg_2(1, t), g_2(1, t))$.

Remark. Note that, because of the geometric construction, the output parametrization of algorithm CONIC-PARAMETRIZATION is proper. Moreover, if $\mathcal{P}_{\star,z}(t)$ is the affine parametrization of $\mathcal{C}_{\star,z}$ derived from $\mathcal{P}(t)$, then its inverse can be expressed as

$$\mathcal{P}_{\star,z}^{-1}(x,y) = \frac{y-b}{x-a}.$$

Similary for $\mathcal{C}_{\star,y}$ and $\mathcal{C}_{\star,z}$ □

Example 8.3.1.: Let $\mathcal{C}$ be the ellipse defined by

$$f(x,y) = x^2 + 2y^2 - z^2.$$

We apply algorithm CONIC–PARAMETRIZATION. It is clear that $\mathcal{C}$ does not pass through $(0:0:1)$. Thus, we take a point on $\mathcal{C}$, for instance $(1:0:1)$ (Step 3). Then, performing Step 4. one gets $g(x,y) = x^2 + 2x + y^2$ (see Figure 8.2). Then, a parametrization of $\mathcal{C}$ is

$$\mathcal{P}(t) = (-1+2\,t^2, -2\,t, 1+2\,t^2).$$ □

Obviously, this approach can be immediately generalized to the situation where we have an irreducible projective curve $\mathcal{C}$ of degree d with a $(d-1)$–fold point P. W.l.o.g. we consider that $P = (0:0:1)$, so the defining polynomial of $\mathcal{C}$ is of the form

$$F(x,y,z) = f_d(x,y) + f_{d-1}(x,y)z$$

(f_i a form of degree i, resp.). Of course, there can be no other singularity of $\mathcal{C}$, since otherwise the line passing through the two singularities would intersect $\mathcal{C}$ in more than d times.

As above, we consider the linear system of lines $\mathcal{H}(t)$ through $(0:0:1)$. Intersecting $\mathcal{C}$ with an element of $\mathcal{H}$ we get the origin as an intersection point of multiplicity at least $d-1$. Reasoning as above, one has that since $\mathcal{C}$ is irreducible for all but finitely many values of t, P is an intersection point of multiplicity at most $d-1$. Thus, by Bézout's Theorem, we must get exactly one more intersection point Q depending rationally on the value of t. So the coordinates of Q are polynomials in t, in fact

$$Q = (-f_{d-1}(1,t) : -t \cdot f_{d-1}(1,t) : f_d(1,t)).$$

This is a rational parametrization of the curve $\mathcal{C}$. We summarize this in the following theorem.

Theorem 8.3.2. *Let $\mathcal{C}$ be an irreducible projective curve of degree d defined by the polynomial $F(x,y,z) = f_d(x,y) + f_{d-1}(x,y)z$ (f_i a form of degree i, resp.), i.e. having a $(d-1)$–fold point at $(0:0:1)$. Then $\mathcal{C}$ is rational and a rational parametrization is*

$$\mathcal{P}(t) = (-f_{d-1}(1,t), -tf_{d-1}(1,t), f_d(1,t)).$$

Applying the previous theorems, one may derive an algorithm for parametrizing by lines. For this purpose, one just has to move the base point of the pencil of lines to the origin. More precisely, one has the following algorithm.

Algorithm PARAMETRIZATION-BY-LINES.
Given the defining polynomial $F(x, y, z)$ of an irreducible projective curve $\mathcal{C}$ of degree $d > 1$, having a $(d-1)$–fold point, the algorithm computes a rational parametrization of $\mathcal{C}$.

1. If $d = 2$ apply algorithm CONIC-PARAMETRIZATION.
2. Compute the $(d-1)$–fold point P of $\mathcal{C}$.
3. If P is an affine point of $\mathcal{C}$ (say $P = (a : b : 1)$), then compute $g(x, y) = F(x + a, y + b, 1)$ (i.e. apply the linear change of coordinates which moves the origin $(0 : 0 : 1)$ to P and then dehomogenize w.r.t. z) and return
$$\mathcal{P}(t) = (-g_{d-1}(1, t) + ag_d(1, t), -tg_{d-1}(1, t) + bg_d(1, t), g_d(1, t)),$$
where $g_d(x, y)$ and $g_{d-1}(x, y)$ are the homogeneous components of $g(x, y)$ of degree d and $d - 1$, respectively. (so we have applied Theorem 8.3.2 and then reversed the change of coordinates.)
4. If P is at infinity (say $P = (a : b : 0)$), then
   - 4.1. If $a \neq 0$ then compute $g(x, y) = F(a, y + b, x)$ and return
   $$\mathcal{P}(t) = (-ag_d(1, t), tg_{d-1}(1, t) - bg_d(1, t), g_{d-1}(1, t)),$$
   where $g_d(x, y)$ and $g_{d-1}(x, y)$ are the homogeneous components of $g(x, y)$ of degree d and $d - 1$, respectively.
   - 4.2. If $a = 0$, then compute $g(x, y) = F(x, 1, y)$ and return
   $$\mathcal{P}(t) = (g_{d-1}(1, t), -g_d(1, t), tg_{d-1}(1, t)),$$
   where $g_d(x, y)$ and $g_{d-1}(x, y)$ are the homogeneous components of $g(x, y)$ of degree d and $d - 1$, respectively.

Remark. Note that, because of the geometric construction, the parametrization computed by algorithm CONIC-PARAMETRIZATION is proper. Furthermore, if $\mathcal{P}_{\star,z}(t)$ is the affine parametrization of $\mathcal{C}_{\star,z}$ derived from $\mathcal{P}(t)$, then its inverse can be computed as follows. If $\mathcal{C}$ is a conic see the remark to algorithm CONIC–PARAMETRIZATION. If $\mathcal{C}$ is not a conic, let $P = (a : b : c)$ be the singularity of the curve, then:

1. If $c \neq 0$ then $\mathcal{P}_{\star,z}^{-1}(x, y) = \frac{cy-b}{cx-a}$.
2. If $c = 0$ and $a \neq 0$ then $\mathcal{P}_{\star,z}^{-1}(x, y) = y - \frac{b}{a}x$.
3. If $c = a = 0$ then $\mathcal{P}_{\star,z}^{-1}(x, y) = \frac{1}{x}$.

Similarly for $\mathcal{P}_{\star,y}$ and $\mathcal{P}_{\star,z}$. □

In the following we illustrate the algorithm with two different examples. The first one has an affine singularity while the second has the singularity at infinity.

Example 8.3.2.: Let $\mathcal{C}$ be the affine quartic curve defined by (see Figure 8.3).

$$f(x, y) = 1 + x - 15\,x^2 - 29\,y^2 + 30\,y^3 - 25\,xy^2 + x^3y + 35\,xy + x^4 - 6\,y^4 + 6\,x^2y$$

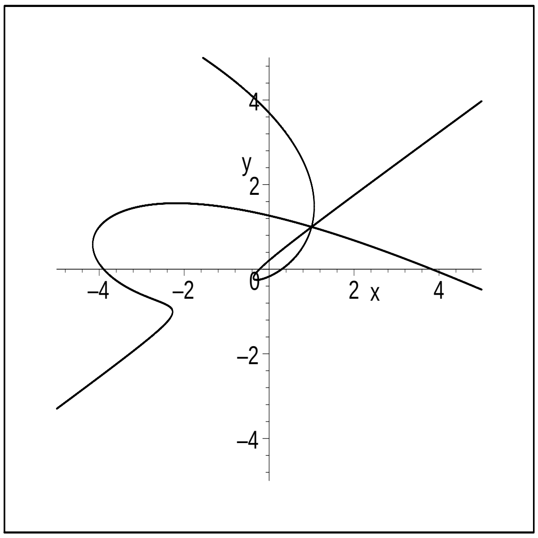

Figure 8.3: Quartic $\mathcal{C}$

$\mathcal{C}$ has an affine triple point at $(1, 1)$. We apply algorithm PARAMETRIZATION-BY-LINES to parametrize $\mathcal{C}$. In Step 3.1. we compute the polynomial

$$g(x, y) = 5\,x^3 + 6\,y^3 - 25\,xy^2 + x^3y + x^4 - 6\,y^4 + 9\,x^2y$$

And, determining the homogeneous forms of $g(x,y)$, in Step 4.2., we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(\frac{4 + 6\,t^3 - 25\,t^2 + 8\,t + 6\,t^4}{-1 + 6\,t^4 - t}, \frac{4\,t + 12\,t^4 - 25\,t^3 + 9\,t^2 - 1}{-1 + 6\,t^4 - t} \right).$$

Furthermore, taking into account the remark to the algorithm we have that

$$\mathcal{P}^{-1}(x,y) = \frac{y-1}{x-1}. \qquad \square$$

Example 8.3.3.: Let $\mathcal{C}$ be the affine quintic curve defined by

$$f = -\frac{75}{8}\,x^2y^2 + \frac{125}{8}\,x^3y - \frac{1875}{256}\,x^4 + x + y^4 + \frac{625}{16}\,x^3y^2 - \frac{9375}{256}\,x^4y - \frac{125}{8}\,x^2y^3 + \frac{3125}{256}\,x^5 + y^5.$$

$\mathcal{C}$ has a quadruple point at $(4:5:0)$. We apply algorithm PARAMETRIZATION-BY-LINES to parametrize $\mathcal{C}$. In Step 4.1. we compute the polynomial

$$g(x,y) = 6400y^4 + 1024x^4 + 256y^5 + 256xy^4 + 5120xy^3$$

And, determining the homogeneous forms of $g(x,y)$, in Step 4.1.2., we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(-4\,\frac{t^4\,(t+1)}{25\,t^4 + 20\,t^3 + 4}, \frac{t\,(20\,t^4 + 15\,t^3 + 4)}{25\,t^4 + 20\,t^3 + 4} \right).$$

Furthermore, taking into account the remark to the algorithm we have that

$$\mathcal{P}^{-1}(x,y) = y - \frac{5}{4}x. \qquad \square$$

A natural question is whether only the rational curves considered previously are those parametrizable by lines. In the last part of this section we characterize those curves that can be parametrized by lines. For this purpose, first of all, we must be more precise and give a formal definition of what we mean by a curve parametrizable by lines.

Definition 8.3.1. The irreducible projective curve $\mathcal{C}$ is *parametrizable by lines* if there exists a linear system of curves $\mathcal{H}$ of degree 1 (i.e. a pencil of lines) such that

(1) $\dim(\mathcal{H}) = 1$,

(2) the intersection of a generic element in $\mathcal{H}$ and $\mathcal{C}$ contains a non–constant point whose coordinates depend rationally on the free parameter of $\mathcal{H}$.

We say that an irreducible affine curve is *parametrizable by lines* if its projective closure is parametrizable by lines. □

Remark.

1. Note that in Definition 8.3.1 we have not required that the base point of $\mathcal{H}$ is on the curve. Later, we will see that in fact the base point must lie on $\mathcal{C}$, unless $\mathcal{C}$ is a line.

2. Any line is parametrizable by lines (Exercise).

3. Note that an affine curve parametrizable by lines is in fact rational. Moreover, the implicit equation of $\mathcal{C}$ vanishes on the generic intersection point depending rationally on the parameter (see Theorem 4.1.4). So this generic point is a rational parametrization of $\mathcal{C}$. Furthermore, if the irreducibility condition in Definition 8.3.1 is not imposed, then it follows that the curve has a rational component.

4. Let $\mathcal{C}$ be an affine curve such that its associated projective curve $\mathcal{C}^\star$ is parametrizable by the pencil of lines $\mathcal{H}(t)$ of equation $L_1(x, y, z) - tL_2(x, y, z)$. Then, the affine parametrization of $\mathcal{C}$, generated by $\mathcal{H}(t)$, is proper and $\frac{L_1(x,y,1)}{L_2(x,y,1)}$ is its inverse (Exercise).

Theorem 8.3.3. *Let $\mathcal{C}$ be an irreducible projective plane curve of degree $d > 1$. The following statements are equivalent:*

(1) $\mathcal{C}$ is parametrizable by a pencil of lines $\mathcal{H}(t)$.

(2) $\mathcal{C}$ has a point of multiplicity $d - 1$ which is the base point of $\mathcal{H}(t)$.

Proof: That (2) implies (1) follows from Definition 8.3.1 and Theorems 8.3.1 and 8.3.2.

Conversely, if $\mathcal{C}$ is parametrizable by lines, then $\mathcal{C}$ is rational. Let $\mathcal{P}(t)$ be the parametrization generated by $\mathcal{H}(t)$. Taking into account Remark 4 to Definition 8.3.1, one has that $\mathcal{P}(t)$ is injective for almost all $t \in K$. Let Q be the base point of $\mathcal{H}(t)$. Since $d > 1$, for almost all $t_0 \in K$, $\mathcal{H}(t_0)$ intersects $\mathcal{C}$ in at least two points, and one of them is $\mathcal{P}(t_0)$. Moreover, since $\mathcal{P}(t)$ is a parametrization, it generates all but finitely many points on the curve. Thus, we may assume w.l.o.g. that for almost all $t_0 \in K$, all intersection points of $\mathcal{H}(t_0)$ and $\mathcal{C}$ are reachable by the parametrization $\mathcal{P}(t)$.

Let $P \in (\mathcal{H}(t_0) \cap \mathcal{C}) \setminus \mathcal{P}(t_0)$. Then, $P \in \mathcal{H}(t_0)$, and there exists $t_1 \in K$, $t_1 \neq t_0$, such that $\mathcal{P}(t_1) = P$. This implies that $P \in \mathcal{H}(t_1)$. Therefore, P is on two different lines of the pencil, and thus $P = Q$. Then, for almost all $t_0 \in K$ it holds that $\mathcal{H}(t_0) \cap \mathcal{C} = \{\mathcal{P}(t_0), Q\}$. On the other hand, since $\mathcal{C}$ is rational, it has finitely many singularities. Thus, for almost all $t_0 \in K$, $\mathrm{mult}_{\mathcal{P}(t_0)}(\mathcal{C}, \mathcal{H}(t_0)) = 1$. This implies, by Bézout's Theorem that for almost all $t_0 \in K$, $\mathrm{mult}_Q(\mathcal{C}, \mathcal{H}(t_0)) = d - 1$. Thus, $d - 1 = \mathrm{mult}_Q(\mathcal{C})$. Therefore, the base point of $\mathcal{H}(t)$ is a point on $\mathcal{C}$ of multiplicity $d - 1$. Thus, (1) implies (2). □

We have seen that the inverse of an affine parametrization generated by the algorithm PARAMETRIZATION-BY-LINES is linear. In the next theorem we see that this phenomenon also characterizes the curves parametrizable by lines.

Theorem 8.3.4. *Let $\mathcal{C}$ be an irreducible affine plane curve. The following statements are equivalent:*

(1) The associated projective curve $\mathcal{C}^\star$ is parametrizable by lines.

(2) There exists a proper affine parametrization of $\mathcal{C}$ with a linear inverse.

(3) The inversion of any proper affine parametrization of $\mathcal{C}$ is linear.

Proof: Let d be the degree of $\mathcal{C}$. If $d = 1$ the result is trivial.

So now let us assume that $d > 1$. If (1) holds, by Theorem 8.3.3. we know that $\mathcal{C}^\star$ has a $(d-1)$–fold point. Therefore, applying algorithm PARAMETRIZATION-BY-LINES one gets a proper affine parametrization of $\mathcal{C}$ with linear inverse. Thus, (2) holds.

We prove now that (2) implies (3). Let $\mathcal{P}(t)$ be a proper affine parametrization with linear inverse, and let $\mathcal{P}'(t)$ be any other proper affine parametrization of $\mathcal{C}$. From Lemma 8.2.3 (2) one has that there exists a linear rational function $L(t)$ such that $\mathcal{P}'(t) = \mathcal{P}(L(t))$. Therefore, ${\mathcal{P}'}^{-1} = L^{-1} \circ \mathcal{P}^{-1}$, which is also linear.

Finally, we prove that (3) implies (1). Let $\mathcal{P}(t)$ be a proper affine parametrization of $\mathcal{C}$ whose rational inverse can be expressed as $(ax + by + c)/(a'x + b'y + c')$, and let $\mathcal{P}^\star(t)$ be the projective parametrization generated by $\mathcal{P}(t)$. Then, we consider the pencil of lines $\mathcal{H}(t)$ defined by $H(x, y, z, t) = (ax + by + cz) - (a'x + b'y + c'z)t$. Clearly, $H(\mathcal{P}^\star(t), t) = 0$. Thus, $\mathcal{P}^\star(t) \in \mathcal{H}(t) \cap \mathcal{C}^\star$. Therefore, $\mathcal{C}^\star$ is parametrizable by lines. □

8.4 Parametrization by Adjoint Curves

In Theorem 8.3.3 we have seen that, in general, rational curves cannot be parametrized by lines. In fact, we have proved that a rational curve $\mathcal{C}$ of degree d is parametrizable by lines if and only if it has a $(d-1)$–fold point. In order to treat the general case, we develop here a method based on the notion of adjoint curves. The method described in this section follows basically the approach in [SeWi91]. There are alternative parametrization methods, e.g. based on the computation of the anticanonical divisor.

Throughout this section, $\mathcal{C}$ will be an irreducible projective curve of degree $d > 2$. Note that this is not a loss of generality, because we have seen in the previous section that lines and irreducible conics can be parametrizable by lines.

Before showing how adjoints are defined and how they can be used to solve the parametrization problem, we give a generalization of the notion of parametrization by lines.

Definition 8.4.1. We say that a linear system of curves $\mathcal{H}$ *parametrizes* $\mathcal{C}$ if it holds that

(1) $\dim(\mathcal{H}) = 1$,

(2) the intersection of a generic element in $\mathcal{H}$ and $\mathcal{C}$ contains a non–constant point whose coordinates depend rationally on the free parameter in $\mathcal{H}$,

(3) $\mathcal{C}$ is not a common component of any curves in $\mathcal{H}$.

Lemma 8.4.1. *Let $\mathcal{H}(t)$ be a linear system of curves parametrizing $\mathcal{C}$, then there exists only one non–constant intersection point $\mathcal{P}(t)$ of $\mathcal{H}(t)$ and $\mathcal{C}$ depending on t, and it is a proper parametrization of $\mathcal{C}$.*

Proof. Since $\mathcal{P}(t)$ is an intersection point, it is clear that the defining polynomial of $\mathcal{C}$ vanishes at it. Thus, $\mathcal{P}(t)$ is a parametrization of $\mathcal{C}$. In order to see that it is proper, we find the inverse of the affine parametrization $\mathcal{P}_{\star,z}(t)$ of $\mathcal{C}_{\star,z}$ generated by $\mathcal{P}(t)$. Let $H(t,x,y,z) = H_0(x,y,z) - tH_1(x,y,z)$ be the defining polynomial of $\mathcal{H}(t)$. Then, $H(t,\mathcal{P}(t)) = 0$. Moreover, $H_1(t,\mathcal{P}(t)) \neq 0$, because otherwise one gets that $H_0(t,\mathcal{P}(t)) = 0$, and this is impossible because of condition (3) in Definition 8.4.1. Therefore, $M = H_0/H_1$ is defined at $\mathcal{P}(t)$ and $M(\mathcal{P}(t)) = t$. Thus, $M(x,y,1)$ is the inverse of $\mathcal{P}_{\star,z}(t)$.

Finally, let us see that $\mathcal{P}(t)$ is unique. If there exists another non–constant intersection point $\mathcal{Q}(t)$ depending on t, reasoning as above one deduces that M is also the inverse of $\mathcal{Q}_{\star,z}$. Therefore, $\mathcal{P}_{\star,z}(t) = \mathcal{Q}_{\star,z}(t)$ and then $\mathcal{P}(t) = \mathcal{Q}(t)$. □

The following result shows how to actually compute a parametrization from a parametrizing linear system of curves. For this purpose, for a polynomial G in $K[t][x,y,z]$ we use the notation $\mathrm{pp}_t(G)$ to denote the primitive part of G w.r.t. t.

Theorem 8.4.2. *Let $F(x,y,z)$ be the defining polynomial of $\mathcal{C}$, and let $H(t,x,y,z)$ be the defining polynomial of a linear system $\mathcal{H}(t)$ parametrizing $\mathcal{C}$. Then, the proper parametrization $\mathcal{P}(t)$ generated by $\mathcal{H}(t)$ is the solution in $\mathbb{P}^2(K(t))$ of the system of algebraic equations*

$$\left.\begin{array}{r}\mathrm{pp}_t(\mathrm{res}_y(F,H))=0\\ \mathrm{pp}_t(\mathrm{res}_x(F,H))=0\end{array}\right\}.$$

Proof. Let $\{P_1,\ldots,P_s,\mathcal{P}(t)\}$ be the intersection points of $\mathcal{H}(t)$ and $\mathcal{C}$. By Lemma 8.4.1. we know that $P_i\in\mathbb{P}^2(K)$ and $\mathcal{P}(t)\in\mathbb{P}^2(K(t))$. Let $P_i=(a_i:b_i:c_i)$ and $\mathcal{P}(t)=(\chi_1(t),\chi_2(t),\chi_3(t))$. From condition (3) in Definition 4.7.1. one has that $\mathrm{res}_y(F,H)$ and $\mathrm{res}_x(F,H)$ are not identically zero. Furthermore, from Section 7.2, one has that

$$\mathrm{res}_y(F,H)=(\chi_3(t)x-\chi_1(t)z)^{\beta}\prod_{i=1}^{s}(c_ix-a_iz)^{\alpha_i}$$

$$\mathrm{res}_x(F,H)=(\chi_3(t)y-\chi_2(t)z)^{\beta'}\prod_{i=1}^{s}(c_iy-b_iz)^{\alpha'_i}$$

for some $\alpha_i,\alpha'_i,\beta,\beta'\in\mathbb{N}$. Now, the result follows taking primitive parts w.r.t. t. □

The following theorem gives sufficient conditions on a linear system of curves to be a parametrizing system.

Theorem 8.4.3. *Let $\mathcal{H}$ be a linear systems of curves of degree k such that*

(1) $\dim(\mathcal{H})=1$.

(2) If $\mathcal{B}$ is the set of base points of $\mathcal{H}$, then for almost all curves $\mathcal{C}'\in\mathcal{H}$ it holds that

$$\sum_{P\in\mathcal{B}}\mathrm{mult}_P(\mathcal{C},\mathcal{C}')=dk-1.$$

(3) $\mathcal{C}$ is not a common component of any curve in $\mathcal{H}$.

Then, $\mathcal{H}$ parametrizes $\mathcal{C}$.

Proof. We just have to prove that condition (2) in the theorem statement implies condition (2) in Definition 8.4.1. By condition (3) we know that $\mathcal{C}$ is not a component of a generic element of $\mathcal{H}$. Thus, by Bézout's Theorem and condition (2) in the thoerem one has that for almost all $\mathcal{C}'\in\mathcal{H}$, $(\mathcal{C}'\cap\mathcal{C})\setminus\mathcal{B}$ consists of a point. Therefore, this point should depend rationally on the parameter defining $\mathcal{H}$. □

Now, the natural question is how to determine parametrizing linear systems of curves. We will show that adjoints provide an answer to this question. Adjoint curves can be defined for reducible curves. However, since our final goal is to work with rational curves we only consider irreducible curves.

When we defined the genus of a curve, we did this only for curves having only ordinary singularities. A curve with a non-ordinary singularity P can be "blown-up" at P to exhibit so-called neighboring singularities. They can be treated similarly to singular points of the curve. We do not go into details on this, but refer to [Wal50] or [SeWi91]. But the following definition depends also on these neighboring singularities. For curves with only ordinary singularities, there are no such neighboring singularities to be considered.

With the notion of neighboring singularities the genus of an irreducible curve $\mathcal{C}$ is

$$\text{genus}(\mathcal{C}) = \frac{1}{2} \cdot [(n-1)(n-2) - \sum r_i(r_i - 1)], \qquad (*)$$

where the notation is as in Def. 7.3.4 and the sum runs over all the singular and neighboring singular points of $\mathcal{C}$.

Definition 8.4.2. We say that a projective curve $\mathcal{C}'$ is an *adjoint curve* of the irreducible $\mathcal{C}$ if and only if the following holds:

(1) if P is a singular point of $\mathcal{C}$, then $\text{mult}_P(\mathcal{C}') \geq \text{mult}_P(\mathcal{C}) - 1$,

(2) if P is a neighboring singular point of $\mathcal{C}$, then $\text{mult}_P(\mathcal{C}') \geq \text{mult}_P(\mathcal{C}) - 1$.

We say that $\mathcal{C}'$ is an *adjoint curve of degree* k of $\mathcal{C}$, if $\mathcal{C}'$ is an adjoint of $\mathcal{C}$ and $\deg(\mathcal{C}') = k$. □

All algebraic conditions required in the definition of adjoint curve are linear. Therefore if one fixes the degree, the set of all adjoint curves of $\mathcal{C}$ is a linear system of curves. In fact, if $\mathcal{C}$ has only ordinary singularities, then the set of adjoint curves of degree k of $\mathcal{C}$ is the linear system generated by the effective divisor

$$\sum_{P \in \text{Sing}(\mathcal{C})} (\text{mult}_P(\mathcal{C}) - 1)P.$$

This remark motivates the following definition.

Definition 8.4.3. The set of all adjoints of $\mathcal{C}$ of degree k, $k \in \mathbb{N}$, is called the *system of adjoints* of $\mathcal{C}$ of degree k. We denote this system by $\mathcal{A}_k(\mathcal{C})$. □

Theorem 8.4.4. *If $\mathcal{C}$ is rational and $k \geq d-2$ then $\mathcal{A}_k(\mathcal{C}) \neq \emptyset$.*

Proof. The full linear system of curves of degree k has dimension

$$\frac{(k+1)(k+2)}{2} - 1 \quad = \quad \frac{k(k+3)}{2}.$$

Furthermore, the number of linear conditions required by $\mathcal{A}_k(\mathcal{C})$ is

$$\sum_{P \in \mathrm{Ngr}(\mathcal{C})} \frac{\mathrm{mult}_P(\mathcal{C})(\mathrm{mult}_P(\mathcal{C})-1)}{2} = \frac{(d-1)(d-2)}{2}.$$

The last equality holds because $\mathcal{C}$ is rational (see $(*)$). Therefore,

$$\dim(\mathcal{A}_k(\mathcal{C})) \geq \frac{k(k+3)}{2} - \frac{(d-1)(d-2)}{2}\ .$$

Now, if $k \geq d-2$, then $\dim(\mathcal{A}_k(\mathcal{C})) \geq d-2 > 0$ and hence $\mathcal{A}_k(\mathcal{C}) \neq \emptyset$. □

Theorem 8.4.5. *Let $k \geq d-2$, then*

$$\dim(\mathcal{A}_k(\mathcal{C})) = \frac{k(k+3)}{2} - \frac{(d-1)(d-2)}{2}.$$

Proof. Let $\ell = k(k+3)/2 - (d-1)(d-2)/2$. We have already seen in the proof of Theorem 8.4.4. that

$$\dim(\mathcal{A}_k(\mathcal{C})) \geq \ell$$

Now, let us suppose that $\dim(\mathcal{A}_k(\mathcal{C})) > \ell$. Then, we choose a set $\mathcal{S} \subset \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ such that $\mathrm{card}(\mathcal{S}) = kd - (d-1)(d-2) - 1$, and we consider the linear subsystem $\mathcal{H}$ of $\mathcal{A}_k(\mathcal{C})$ by imposing that the adjoint curves pass through the points in $\mathcal{S}$. That is

$$\mathcal{H} = \mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}} P).$$

If k is expressed as $d+s$, where $s \geq -2$, one has that

$$\dim(\mathcal{H}) \geq \dim(\mathcal{A}_k(\mathcal{C})) - [kd - (d-1)(d-2) - 1] > \ell - [kd - (d-1)(d-2) - 1] =$$

$$= \frac{(d+s)(d+s+3)}{2} - \frac{(d-1)(d-2)}{2} - [(d+s)d - (d-1)(d-2) - 1] =$$

$$= \frac{(s+2)(s+1)}{2} + 1$$

Now, we distinguish two cases depending on whether s is negative or not.

(1) Let $s < 0$ (i.e. if $k = d-2$ or $k = d-1$) then $\dim(\mathcal{H}) \geq 2$. Then, we take two different points $Q_1, Q_2 \in \mathcal{C} \setminus (\mathrm{Sing}(\mathcal{C}) \cup \mathcal{S})$, and we consider the linear subsystem

$$\mathcal{H}' = \mathcal{H} \cap \mathcal{H}(k, Q_1 + Q_2).$$

Observe that $\dim(\mathcal{H}') \geq 0$. Thus, $\mathcal{H}' \neq \emptyset$ (note that $\mathcal{H}'$ is a projective linear variety). Let $\mathcal{C}' \in \mathcal{H}'$. Since $\deg(\mathcal{C}') < \deg(\mathcal{C})$ and $\mathcal{C}$ is irreducible, we know that $\mathcal{C}'$ and $\mathcal{C}$ do not have common components. Therefere, by Theorem 7.6. in [Walker] Chapter III, one has that

$$kd \geq \sum_{P \in \mathrm{Ngr}(\mathcal{C})} \mathrm{mult}_P(\mathcal{C})\,\mathrm{mult}_P(\mathcal{T}_p(\mathcal{C}')) + \sum_{P \in \mathcal{S} \cup \{Q_1, Q_2\}} \mathrm{mult}_P(\mathcal{C})\,\mathrm{mult}_P(\mathcal{C}') \geq$$

$$\geq \sum_{P \in \mathrm{Ngr}(\mathcal{C})} \mathrm{mult}_P(\mathcal{C})\,(\mathrm{mult}_P(\mathcal{C}) - 1) + \sum_{P \in \mathcal{S} \cup \{Q_1, Q_2\}} \mathrm{mult}_P(\mathcal{C})\,\mathrm{mult}_P(\mathcal{C}') =$$

$$= (d-1)(d-2) + \sum_{P \in \mathcal{S} \cup \{Q_1, Q_2\}} \mathrm{mult}_P(\mathcal{C})\,\mathrm{mult}_P(\mathcal{C}') \geq$$

$$\geq (d-1)(d-2) + [kd - (d-1)(d-2) - 1] + 2 = kd + 1,$$

which is impossible.

(2) Let $s \geq 0$. Then,

$$\dim(\mathcal{H}) \geq \frac{(s+2)(s+1)}{2} + 2.$$

In this situation, we take two different points $Q_1, Q_2 \in \mathcal{C} \setminus (\mathrm{Sing}(\mathcal{C}) \cup \mathcal{S})$, and one point $Q_3 \in \mathbb{P}(K) \setminus \mathcal{C}$, and we consider the linear subsystem

$$\mathcal{H}' = \mathcal{H} \cap \mathcal{H}(k, Q_1 + Q_2 + (s+1)Q_3).$$

Note that the number of linear conditions introduced by the effective divisor is precisely the lower bound of $\dim(\mathcal{H})$. Thus, $\dim(\mathcal{H}') \geq 0$, and therefore $\mathcal{H}' \neq \emptyset$. Let $\mathcal{C}' \in \mathcal{H}'$. We prove that $\mathcal{C}$ and $\mathcal{C}'$ do not have a common component. Indeed: if they have a common component, since $\deg(\mathcal{C}) = d \leq k = \deg(\mathcal{C}')$, then $\mathcal{C}'$ decomposes as $\mathcal{C} \cup \mathcal{C}''$, where $\deg(\mathcal{C}'') = k - d = s$. But Q_3 in on $\mathcal{C}''$ with multiplicity at least $s+1$, which is impossible since otherwise for any line $\mathcal{L}$ not being a component of $\mathcal{C}''$ and passing through Q_3 one would have, by Bézout's Theorem, that $s \geq \mathrm{mult}_{Q_3}(\mathcal{C}'', \mathcal{L}) \geq s+1$. From here, the proof ends as in case (1). □

Now, we proceed to show how adjoint curves may be used to generate parametrizing linear systems. We start with the following theorem.

Theorem 8.4.6. *Let $k \in \{d-1, d-2\}$ and let $\mathcal{S} \subset \mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})$ be such that $\mathrm{card}(\mathcal{S}) = kd - (d-1)(d-2) - 1$. Then*

$$\mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}} P)$$

parametrizes $\mathcal{C}$.

Proof. Let $\mathcal{H} = \mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}} P)$. We see that conditions in Theorem 8.4.3. are satisfied. Note that condition (3) trivially holds because $\mathcal{C}$ is irreducible and $k < d$.

(1) $\dim(\mathcal{H}) \geq \dim(\mathcal{A}_k(\mathcal{C})) - [kd - (d-1)(d-2) - 1]$, and by Theorem 8.4.5. we know that $\dim(\mathcal{H}) \geq 1$. Furthermore, if $\dim(\mathcal{H}) > 1$ reasoning as in step (1) of the proof of Theorem 8.4.5. one arrives at a contradiction.

(2) The set of base points $\mathcal{B}$ of $\mathcal{H}$ includes $\mathrm{Sing}(\mathcal{C}) \cup \mathcal{S}$. Moreover, if $\mathcal{B} \neq \mathrm{Sing}(\mathcal{C}) \cup \mathcal{S}$, then there exists $Q \in \mathcal{B} \setminus (\mathrm{Sing}(\mathcal{C}) \cup \mathcal{S})$. Then, choose a curve $\mathcal{C}' \in \mathcal{H}$ passing through a point $Q' \in \mathcal{C} \setminus \mathcal{B}$. This is possible because $\dim(\mathcal{H}) = 1$. Then, since $\mathcal{C}$ and $\mathcal{C}'$ do not have common components, by Theorem 7.6. in [Walker] Chapter III, one has that

$$kd \geq \sum_{P \in \mathrm{Ngr}(\mathcal{C})} \mathrm{mult}_P(\mathcal{C}) \, \mathrm{mult}_P(\mathcal{T}_P(\mathcal{C}')) + \sum_{P \in \mathcal{S} \cup \{Q, Q'\}} \mathrm{mult}_P(\mathcal{C}) \, \mathrm{mult}_P(\mathcal{C}') \geq$$

$$\geq \sum_{P \in \mathrm{Ngr}(\mathcal{C})} \mathrm{mult}_P(\mathcal{C}) \, (\mathrm{mult}_P(\mathcal{C}) - 1) + \sum_{P \in \mathcal{S} \cup \{Q, Q'\}} \mathrm{mult}_P(\mathcal{C}) \, \mathrm{mult}_P(\mathcal{C}') =$$

$$= (d-1)(d-2) + \sum_{P \in \mathcal{S} \cup \{Q, Q'\}} \mathrm{mult}_P(\mathcal{C}) \, \mathrm{mult}_P(\mathcal{C}') \geq$$

$$\geq (d-1)(d-2) + [kd - (d-1)(d-2) - 1] + 1 + 1 = kd + 1,$$

which is impossible. Thus, $\mathcal{B} = \mathrm{Sing}(\mathcal{C}) \cup \mathcal{S}$.

Now, take any curve $\mathcal{C}' \in \mathcal{H}$ such that the tangents of $\mathcal{C}'$ and $\mathcal{C}$ at the base points are transversal [2]. Note that this only excludes finitely many curves of the linear system. Then, the above inequalities are equalities. This concludes the proof. □

Theorem 8.4.7. *Let $k \geq d$, let $Q \notin \mathcal{C}$, and let $\mathcal{S} \subset (\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C}))$ be such that $\mathrm{card}(\mathcal{S}) = kd - (d-1)(d-2) - 1$. Then*

$$\mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}} P + (d-k+1)Q)$$

parametrizes $\mathcal{C}$.

Proof. The proof goes similarly to the proof of Theorem 8.4.6 and we leave it as an exercise. The only main different argument in the proof is to guarantee condition (3) in Theorem 8.4.3, but this is proved in Step (2) of the proof of Theorem 4.7.5. □

The results proved till here provide a family of algorithms to parametrize any rational curve by means of adjoints.

[2] $\sum_{\mathrm{Sing}(\mathcal{C}) \cup S} \mathrm{mult}(\mathcal{C}, \mathcal{C}') = (d-1)(d-2) + kd - (d-1)(d-2) - 1 = kd - 1$

Algorithm PARAMETRIZATION-BY-ADJOINTS.
Given the defining polynomial $F(x, y, z)$ of a rational irreducible projective curve $\mathcal{C}$ of degree d the algorithm computes a rational parametrization of $\mathcal{C}$.

1. If $d \leq 3$ or $\mathrm{Sing}(\mathcal{C})$ contains only one point of multiplicity $d-1$ apply algorithm PARAMETRIZATION–BY–LINES.
2. Choose $k \geq d-2$ and compute the defining polynomial of $\mathcal{A}_k(\mathcal{C})$.
3. Choose a set $\mathcal{S} \subset (\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C}))$ such that $\mathrm{card}(\mathcal{S}) = kd - (d-1)(d-2) - 1$.
4. Compute the defining polynomial H of $\mathcal{H} = \mathcal{A}_k(\mathcal{C}) \cap \mathcal{H}(k, \sum_{P \in \mathcal{S}} P)$.
5. If $k \geq d$ then choose $Q \notin \mathcal{C}$ and replace H by the defining polynomial of $\mathcal{H} \cap \mathcal{H}(k, (d-k+1)Q)$.
6. Return the solution in $\mathbb{P}^2(K(t))$ of $\{\mathrm{pp}_t(\mathrm{res}_y(F, H)) = 0, \mathrm{pp}_t(\mathrm{res}_x(F, H)) = 0\}$.

From the point of view of time efficiency one must choose $k = d-2$ in Step 2, since then degrees of polynomials are the smallest. Nevertheless, the selection of $k = d$ can be also interesting in the sense that at most one algebraic number of degree d has to be introduced (see Theorem 4.8.5), and therefore it is a first fast approach to algebraic optimality in the output (see Chapter 5). In the next section, we briefly develop these ideas. But first, we illustrate the algorithm by an example.

Example 8.4.1: Let $\mathcal{C}$ be the affine curve defined by

$$f(x, y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2).$$

Figure 8.4

$\mathcal{C}$ has a double point at the origin $(0,0)$ as the only affine singularity. But if we move to the associated projective curve $\mathcal{C}^*$ defined by the homogeneous polynomial

$$F(x,y,z) = (x^2+4yz+y^2)^2 - 16(x^2+y^2)z^2,$$

we see that the singularities of $\mathcal{C}^*$ are

$$O = (0:0:1), \quad P_{1,2} = (1:\pm i:0).$$

$P_{1,2}$ is a family of conjugate algebraic points on $\mathcal{C}^*$. All of these singularities have multiplicity 2, so the genus of $\mathcal{C}^*$ is 0, i.e. it can be parametrized. So also the affine curve $\mathcal{C}$ is parametrizable.

In order to achieve a parametrization, we need a simple point on $\mathcal{C}^*$. Intersecting $\mathcal{C}^*$ by the line $x = 0$, we get of course the origin as a multiple intersection point. The other intersection point is

$$Q = (0:-8:1).$$

So now we construct the system L^2 of curves of degree 2, having $O, P_{1,2}$ and Q as base points of multiplicity 1. The full system of curves of degree 2 is of the form

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$$

for arbitrary coefficients $a_1, \ldots, a_6$. Requiring that O be a base point leads to the linear equation

$$a_3 = 0.$$

Requiring that $P_{1,2}$ should be base points of L_2 leads to the equations

$$\begin{aligned} a_4 &= 0, \\ a_1 - a_2 &= 0. \end{aligned}$$

Finally, to make Q a base point we have to satisfy

$$64a_2 + a_3 - 8a_6 = 0.$$

This leaves exactly 2 parameters unspecified, say a_1 and a_5. Since curves are defined uniquely by polynomials only up to a nonzero constant factor, we can set one of these parameters to 1. Thus, the system L^2 depends on 1 free parameter $a_1 = t$, and its defining equation is

$$H(x,y,z,t) = tx^2 + ty^2 + xz + 8tyz.$$

The affine version L^2_a of L^2 is defined by

$$h(x,y,t) = tx^2 + ty^2 + x + 8ty.$$

Now we determine the free intersection point of L_a^2 and $\mathcal{C}$. The non–constant factors of $\mathrm{res}_x(f(x,y), h(x,y,t))$ are

$$\begin{aligned}&y^2,\\&y+8,\\&(256t^4+32t^2+1)y+(2048t^4-128t^2).\end{aligned}$$

The first two factors correspond to the affine base points of the linear system L^2, and the third one determines the y–coordinate of the free intersection point depending rationally on t.

Similarly, the non–constant factors of $\mathrm{res}_y(f(x,y), h(x,y,t))$ are

$$\begin{aligned}&x^3,\\&(256t^4+32t^2+1)x+1024t^3.\end{aligned}$$

The first factor corresponds to the affine base points of the linear system L^2, and the second one determines the x–coordinate of the free intersection point depending rationally on t.

So we have found a rational parametrization of $\mathcal{C}$, namely

$$x(t) = \frac{-1024t^3}{256t^4+32t^2+1}\ , \quad y(t) = \frac{-2048t^4+128t^2}{256t^4+32t^2+1}\ .$$

□

In the previous example we were lucky enough to find a rational simple point on the curve, allowing us to determine a rational parametrization over the field of definition $\mathbb{Q}$. In fact, there are methods for determining whether a curve of genus 0 has rational simple points, and if so find one. We cannot go into more details here, but we refer the reader to [SeWi97].

From the work of Noether Hilbert, Hurwitz we know that it is possible to parametrize any curve $\mathcal{C}$ of genus 0 over the field of definition K, if $\deg(\mathcal{C})$ is odd, and over some quadratic extension of K, if $\deg(\mathcal{C})$ is even. An algorithm which actually achieves this optimal field of parametrization is presented in [SeWi97]. Moreover, if the field of definition is $\mathbb{Q}$, we can also decide if the curve can be parametrized over $\mathbb{R}$, and if so, compute a parametrization over $\mathbb{R}$.

Space curves can be handled by projecting them to a plane along a suitable axis, parametrizing the plane curve, and inverting the projection.