Chapter 9

Local Parametrization and Puiseux Expansion

Let us first give an example of what we want to do in this section.

Example 9.0.1. Consider the plane algebraic curve $\mathcal{C} \subset \mathbb{A}^2(\mathbb{C})$ defined by the equation

$$f(x,y) = y^5 - 4y^4 + 4y^3 + 2x^2y^2 - xy^2 + 2x^2y + 2xy + x^4 + x^3 = 0$$

(see Figure 9.1).

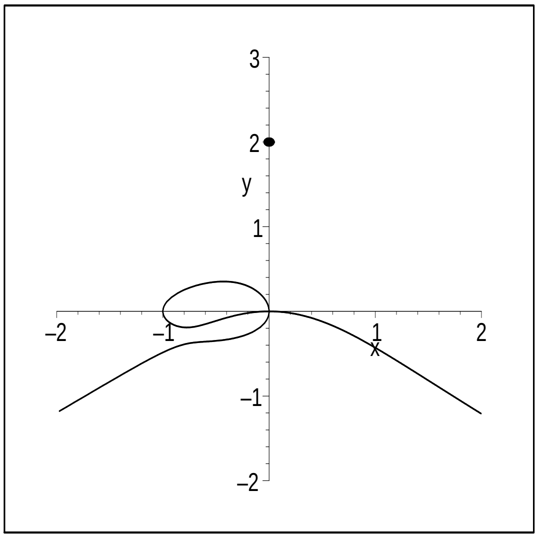

Figure 9.1: Real part of $\mathcal{C}$

Note that the affine point $(0,2)$ is an isolated singularity of $\mathcal{C}$.

Around the origin, $\mathcal{C}$ is parametrized by two different pairs of analytic functions (x_1, y_1) and (x_2, y_2) which have the following power series expansions:

$$\begin{aligned}
(A_1(t), B_1(t)) &= (t,\ -\tfrac{1}{2}t^2 + \tfrac{1}{8}t^4 - \tfrac{1}{8}t^5 + \tfrac{1}{16}t^6 + \tfrac{1}{16}t^7 + \ldots), \\
(A_2(t), B_2(t)) &= (-2t^2,\ t + \tfrac{1}{4}t^2 - \tfrac{27}{32}t^3 - \tfrac{7}{8}t^4 - \tfrac{4057}{2048}t^5 + \ldots).
\end{aligned}$$

In a neighborhood around the origin these power series actually converge to points of the curve $\mathcal{C}$. In fact, these two power series correspond to what we want to call the two branches of $\mathcal{C}$ through the origin. In Figure 9.2 one may check how $(A_i(t), B_i(t))$ approaches the curve $\mathcal{C}$ in a neigborhood of the origin.

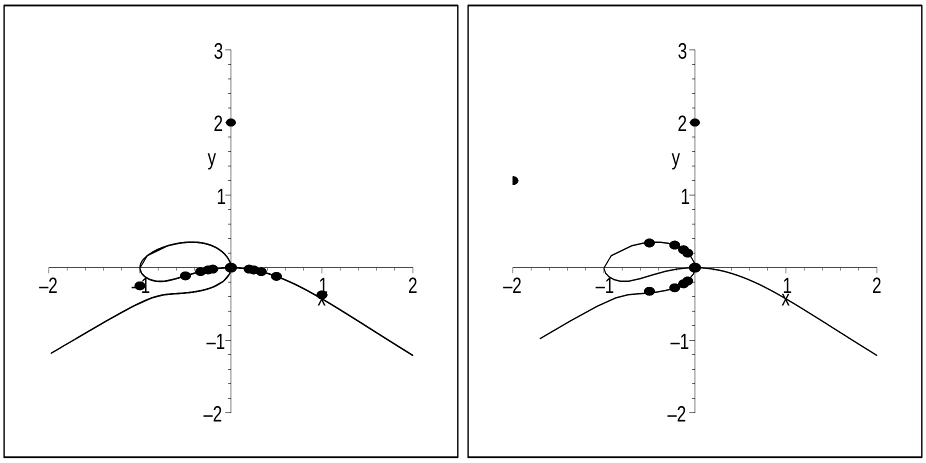

Figure 9.2: Real part of $\mathcal{C}$ and some points generated by $(A_1(t), B_1(t))$ (Left), Real part of $\mathcal{C}$ and some points generated by $(A_2(t), B_2(t))$ (Right)

We will be interested in determining such power series, i.e. in analyzing the topology of a curve in the neighborhood of some point. □

9.1 Power series, places, and branches

We denote by $K[[x]]$ the domain of *formal power series* in the indeterminate x with coefficients in the field K, i.e. the set of all sums of the form $\sum_{i=0}^{\infty} a_i x^i$, where $a_i \in K$.

The quotient field of $K[[x]]$ is called the field of *formal Laurent series* and is denoted by $K((x))$. As is well known, every non-zero formal Laurent series $A(x) \in K((x))$ can be written in the form

$$A(x) = x^k \cdot (a_0 + a_1 x + a_2 x^2 + \cdots), \text{ where } a_0 \neq 0 \text{ and } k \in \mathbb{Z}.$$

The exponent k of the first non-vanishing term of A is called the *order* of A, denoted by $\mathrm{ord}(A)$. We let the order of 0 be ∞.

The units in $K[[x]]$ are exactly the power series of order 0, i.e. those having a non-zero constant term. If $\mathrm{ord}(A) = 0$, then A^{-1} can be computed by an obvious recursive process, in which linear equations over K have to be solved. It is easy to check if a power series is a multiple of another one: $A \mid B$ if and only if $\mathrm{ord}(A) \leq \mathrm{ord}(B)$.

In the remainder of this chapter we will need power series with fractional exponents. So we will consider Laurent series $K((x^{1/n}))$ in $x^{1/n}$, $n \in \mathbb{N}$. In fact, the union of all these fields of Laurent series with denominator n, for $n \in \mathbb{N}$, is again a field.

Definition 9.1.1. The field $K \ll x \gg := \bigcup_{n=1}^{\infty} K((x^{1/n}))$ is called the field of *formal Puiseux series*. The *order* of a non-zero Puiseux series A is the smallest exponent of a term with non-vanishing coefficient in A. The *order* of 0 is ∞. $\square$

The Puiseux series are power series with fractional exponents. Every Puiseux series has a bound n for the denominators of exponents with non-vanishing coefficients.

The substitution of constants for the indeterminate x in a formal power series is usually meaningless. This operation only makes sense for convergent power series in a certain neighborhood of the origin. But we can always substitute 0 for the variable in a power series $A = a_0 + a_1 x + a_2 x^2 + \cdots$, getting the constant coefficient a_0.

It is useful to define the substitution of a power series into another. Let $A, B \in K[[x]]$, $A = a_0 + a_1 x + a_2 x^2 + \cdots$, $B = b_1 x + b_2 x^2 + \cdots$, i.e. $\mathrm{ord}(B) \geq 1$. Then the substitution $A(B)$ is defined as

$$\begin{aligned} A(B) &= a_0 + a_1 B + a_2 B^2 + a_3 B^3 + \cdots = \\ &= a_0 + a_1 b_1 x + (a_1 b_2 + a_2 b_1^2) x^2 + (a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3) x^3 + \cdots . \end{aligned}$$

In order to avoid the problem of substitution of constants we have to request that $\mathrm{ord}(B) \geq 1$.

The following properties of the substitution operation can be easly proved.

Lemma 9.1.1. *Let* $A, B, C \in K[[x]]$, $\mathrm{ord}(B), \mathrm{ord}(C) \geq 1$.

(a) $(A(B))(C) = A(B(C))$.

(b) *If* $\operatorname{ord}(B) = 1$ *then there exists a power series* B' *of order 1, such that* $A = (A(B))(B')$.

(c) *The mapping* $A \longrightarrow A(B)$ *is an endomorphism on* $K[[x]]$.

(d) *If* $\operatorname{ord}(B) = 1$ *then the mapping* $A \longrightarrow A(B)$ *is an automorphism of* $K[[x]]$ *over* K *which preserves the order of the elements.* □

A curve defined over the field K can be considered to have points over the bigger field $K((t))$ of Laurent series, i.e. in $\mathbb{P}^2(K((t)))$. Such a point is called a local parametrization of the curve. $\mathbb{P}^2(K)$ is naturally embedded in $\mathbb{P}^2(K((t)))$. $\mathbb{P}^2(K)$ corresponds to those points $(x : y : z) \in \mathbb{P}^2(K((t)))$, such that $u \cdot (x, y, z) \in K^3$ for some $u \in K((t))^*$.

Definition 9.1.2. Let $\mathcal{C} \subset \mathbb{P}^2(K)$ be a curve defined by the homogeneous polynomial $F(x, y, z) \in K[x, y, z]$. Let $A(t), B(t), C(t) \in K((t))$ such that

(i) $F(A, B, C) = 0$, and

(ii) there is no $D(t) \in K((t))$ such that $D \cdot (A, B, C) \in K^3$.

Then the point $\mathcal{P}(t) = (A : B : C) \in \mathbb{P}^2(K((t)))$ is called a *local parametrization* of $\mathcal{C}$. □

So, obviously, A, B, C are just one possible set of projective coordinates for the local parametrization $\mathcal{P}(t) = (A : B : C)$. For every $D \in K((t))^*$, $(DA : DB : DC)$ is another set of projective coordinates for $\mathcal{P}(t)$.

Lemma 9.1.2. *Every local parametrization of a projective curve* $\mathcal{C}$ *defined over* K *has coordinates* $(A_1 : A_2 : A_3)$ *with* $A_i \in K((t))$ *for* $i = 1, 2, 3$, *and* $\min(\operatorname{ord}(A_i)) = 0$.

Proof: Let $(\tilde{A}_1 : \tilde{A}_2 : \tilde{A}_3)$ be a local parametrization of $\mathcal{C}$. Let $h = -\min(\operatorname{ord}(\tilde{A}_i))$. We set $A_i := t^h \cdot \tilde{A}_i$. Then $(A_1 : A_2 : A_3)$ satisfies the conditions of the lemma. □

Definition 9.1.3. Let $\mathcal{P} = (A : B : C)$ be a local parametrization of $\mathcal{C}$ with $\min\{\operatorname{ord}(A), \operatorname{ord}(B), \operatorname{ord}(C)\} = 0$. Let a, b, c be the constant coefficients of A, B, C, respectively. Then the point $(a : b : c) \in \mathbb{P}^2(K)$ is called the *center* of the local parametrization $\mathcal{P}$. □

Since local parametrizations are just points in the projective space over a bigger field, we can also introduce the notion of affine local parametrization in the obvious way. Namely, let $\mathcal{C}$ be an affine curve, and $\mathcal{C}^*$ the corresponding projective curve. Let $(A^* : B^* : C^*)$ be a local parametrization of $\mathcal{C}^*$. Setting $A := A^*/C^*$ and $B := B^*/C^*$ we get

(i) $f(A, B) = F(A, B, 1) = 0$, and

(ii) not both A and B are in K.

Then the pair of Laurent series (A, B) is called an *affine local parametrization* of the affine curve $\mathcal{C}$.

Let $(A^*(t) : B^*(t) : C^*(t))$ be a (projective) local parametrization of the projective curve $\mathcal{C}^*$ corresponding to the affine curve $\mathcal{C}$, such that $A^*, B^*, C^* \in K[[t]]$ and $\min\{\mathrm{ord}(A^*), \mathrm{ord}(B^*), \mathrm{ord}(C^*)) = 0$. If the constant coefficient c of C^* is non-zero, then $\mathrm{ord}(C^{*-1}) = 0$, so if we set $A := A^*/C^*, B := B^*/C^*$, then (A, B) is an affine local parametrization of $\mathcal{C}$ with $A, B \in K[[t]]$, i.e. with center at a finite affine point. Conversely, every affine local parametrization with center at a finite affine point has coordinates in $K[[t]]$.

Substituting a non-zero power series of positive order into the coordinates of a local parametrization yields a parametrization with the same center.

Definition 9.1.4. Two (affine or projective) local parametrizations $\mathcal{P}_1(t), \mathcal{P}_2(t)$ of an algebraic curve $\mathcal{C}$ are called *equivalent* iff there exists $C \in K[[t]]$ with $\mathrm{ord}(C) = 1$ such that $\mathcal{P}_1(t) = \mathcal{P}_2(C)$. □

This equivalence of local parametrizations is actually an equivalence relation, because of Lemma 9.1.1.

Theorem 9.1.3. *In a suitable affine coordinate system any given local parametrization is equivalent to one of the type*

$$(t^n, \ a_1 t^{n_1} + a_2 t^{n_2} + a_3 t^{n_3} + \cdots),$$

where $0 < n$, *and* $0 < n_1 < n_2 < n_3 < \cdots$.

Proof: We choose the origin of the affine coordinate system to be the center of the parametrization. This means the parametrization will have the form (B, C), with

$$\begin{aligned} B(t) = & \ t^n(b_0 + b_1 t + b_2 t^2 + \cdots), \quad n > 0, \\ C(t) = & \ t^m(c_0 + c_1 t + c_2 t^2 + \cdots), \quad m > 0. \end{aligned}$$

At least one of b_0, c_0 is not 0; w.l.o.g. (perhaps after interchanging the axes) we may assume $b_0 \neq 0$. So now we have to find a power series $D(t)$ of order 1 such that $B(D) = t^n$. This can be done by making an indetermined ansatz for $D(t)$, and solving the linear equations derived from $B(D) = t^n$. The condition $b_0 \neq 0$ guarantees that these linear equations are solvable. □

Definition 9.1.5. If a local parametrization $\mathcal{P}$, or one equivalent to it, has coordinates in $K((t^n))$, for some natural number $n > 1$, i.e. $\mathcal{P}(t) = \mathcal{P}'(t^n)$ for some parametrization $\mathcal{P}'$, then $\mathcal{P}$ is said to be *reducible*. Otherwise, $\mathcal{P}$ is said to be *irreducible*. □

The following criterion for irreducibility is proved in [Wal50].

Theorem 9.1.4. *The local parametrization*

$$(\ t^n, \ a_1 t^{n_1} + a_2 t^{n_2} + a_3 t^{n_3} + \cdots \ ),$$

where $0 < n$, $0 < n_1 < n_2 < n_3 < \cdots$ *and* $a_i \neq 0$, *is reducible if and only if the integers* $n, n_1, n_2, n_3, \ldots$ *have a common factor greater than 1.*

Definition 9.1.6. An equivalence class of irreducible local parametrizations of the algebraic curve $\mathcal{C}$ is called a *place* of $\mathcal{C}$. The common center of the local parametrizations is the *center* of the place. □

This notion of a place on a curve $\mathcal{C}$ can be motivated by looking at the case $K = \mathbb{C}$. Let us assume that $\mathcal{C}$ is defined by $f \in \mathbb{C}[x, y]$ and the origin O of the affine coordinate system is a point on $\mathcal{C}$. We want to study the parametrizations of $\mathcal{C}$ around O. If O is a regular point, we may assume w.l.o.g. that $\frac{\partial f}{\partial y}(0,0) \neq 0$. Then, by the Implicit Function Theorem, there exists a function $y(x)$, analytic in some neighborhood of $x = 0$, such that

- $y(0) = 0$,
- $f(x, y(x)) = 0$, and
- for all (x_0, y_0) in some neighborhood of $(0, 0)$ we have $y_0 = y(x_0)$.

This means that the pair of analytic functions $(x, y(x))$ parametrizes $\mathcal{C}$ around the origin. The analytic function $y(x)$ defined by $f(x, y(x)) = 0$ can be expanded into a Taylor series $\sum_{i=0}^{\infty} c_i t^i$ for some $c_i \in \mathbb{C}$, convergent in a certain neighborhood of the origin. If we set $X(t) = t, Y(t) = \sum_{i=0}^{\infty} c_i t^i$, then $f(X(t), Y(t)) = 0$ and X and Y are convergent around $t = 0$. Hence, $(X(t), Y(t))$ is a parametrization of $\mathcal{C}$ with center at the origin.

If the origin is a singular point of $\mathcal{C}$ then, by Newton's Theorem there exist finitely many pairs of functions $(x(t), y(t))$, analytic in some neighborhood of $t = 0$, such that

- $x(0) = 0, y(0) = 0$,
- $f(x(t), y(t)) = 0$, and
- for every point $(x_0, y_0) \neq (0, 0)$ on $\mathcal{C}$ in a suitable neighborhood of $(0, 0)$ there is exactly one of the pairs of functions $(x(t), y(t))$ for which there exists a unique t_0 such that $x(t_0) = x_0$ and $y(t_0) = y_0$.

Again the pairs of analytic functions can be expanded into power series $(X(t), Y(t))$, resulting in parametrizations of $\mathcal{C}$. These parametrizations are irreducible because of the claim of uniqueness of t_0.

It is important to note that the pairs of analytic functions parametrizing $\mathcal{C}$ are not unique. However, any such collection of parametrizations gives the same set of points in a suitable neighborhood. Let $(x'(t), y'(t))$ be a pair of analytic functions different from $(x(t), y(t))$ but giving the same set of points in a suitable neighborhood of $t = 0$. Then there exists a non-constant analytic function $v(t)$ with $v(0) = 0$, such that $(x(t), y(t)) = (x'(v(t)), y'(v(t)))$. So the two parametrizations are equivalent.

All parametrizations in an equivalence class determine the same set of points as t varies in a certain neighborhood of 0. So all these parametrizations determine a *branch* of $\mathcal{C}$, a branch being a set of all points $(x(t), y(t))$ obtained by allowing t to vary within some neighborhood of 0 within which $x(t)$ and $y(t)$ are analytic. A place on $\mathcal{C}$ is an algebraic counterpart of a branch of $\mathcal{C}$.

It is not hard to see that the center of a parametrization of $\mathcal{C}$ is a point on $\mathcal{C}$, and a proof is left for the exercises. The converse, namely that every point on $\mathcal{C}$ is the center of a least one place of $\mathcal{C}$, follows from the fundamental theorem of Puiseux about the algebraic closure of the field of Puiseux series.

9.2 Puiseux's Theorem and the Newton Polygon Method

Let us view $f \in K[x, y]$ as a polynomial in y with coefficients in $K \ll x \gg$. Computing a power series expansion for y can be seen as solving a polynomial equation in one variable over the field of Puiseux series. Puiseux's Theorem states that a root can always be found.

Theorem 9.2.1 (Puiseux's Theorem) *The field $K \ll x \gg$ is algebraically closed.*

A proof of Puiseux's Theorem can be given constructively by the Newton Polygon Method. We describe the Newton Polygon Method here, and point out how it solves the construction of solutions of univariate polynomial equations over $K \ll x \gg$. Details of the proof can be found in [Wal50].

We are given a polynomial $f \in K \ll x \gg [y]$ of degree $n > 0$, i.e.

$$f(x, y) = A_0(x) + A_1(x)y + \cdots + A_n(x)y^n, \qquad \text{with } A_n \neq 0.$$

If $A_0 = 0$, then obviously $y = 0$ is a solution. So now let us assume that $A_0 \neq 0$. Let $\alpha_i := \operatorname{ord}(A_i)$ and a_i the coefficient of x^{α_i} in A_i, i.e.

$$A_i(x) = a_i x^{\alpha_i} + \text{terms of higher order}.$$

We will recursively construct a solution $Y(x)$, a Puiseux series in x, of the equation $f(x, y) = 0$. $Y(x)$ must have the form

$$Y(x) = c_1 x^{\gamma_1} + \underbrace{c_2 x^{\gamma_2} + c_3 x^{\gamma_3} + \cdots}_{Y_1(x)},$$

with $c_j \neq 0, \gamma_j \in \mathbb{Q}, \gamma_j < \gamma_{j+1}$ for all j. There might be finitely or infinitely many such j, but there is at least one.

In order to get necessary conditions for c_1 and γ_1, we substitute $Y(x) = c_1 x^{\gamma_1} + Y_1(x)$ for y in $f(x, y)$, getting

$$f(x, Y(x)) = A_0(x) + A_1(x) \cdot (c_1 x^{\gamma_1} + Y_1(x)) + \cdots + A_n(x) \cdot (c_1 x^{\gamma_1} + Y_1(x))^n = 0.$$

The terms of lowest order must cancel. Therefore there must exist at least two indices j, k with $j \neq k$ and $0 \leq j, k \leq n$ such that

$$c_1^j A_j(x) x^{j\gamma_1} = c_1^j a_j x^{\alpha_j + j\gamma_1} + \cdots \quad \text{and} \quad c_1^k A_k(x) x^{k\gamma_1} = c_1^k a_k x^{\alpha_k + k\gamma_1} + \cdots$$

have the same order and this order is minimal. So if we think of the pairs (i, α_i), for $0 \leq i \leq n$, as points in the affine plane over $\mathbb{Q}$ (if $A_i(x) = 0$ then $\alpha_i = \infty$ and this point is not contained in the affine plane) then this condition means that all the points

(i, α_i) are on or above the line connecting (j, α_j) and (k, α_k). If we set $\beta_1 := \alpha_j + j\gamma_1$, then the points (u, v) on this line L satisfy $v = -\beta_1 - u\gamma_1$, i.e. γ_1 is the negative slope of L.

A convenient way of determining the possible values for γ_1 is to consider the so-called Newton polytope of f. This is the smallest convex polytope in the affine plane over $\mathbb{Q}$, which is open in the right and upper directions and contains all the points (i, α_i). The faces in the left and lower corner of the Newton polytope are the lines which have possible values for γ_1 as their negative slopes.

In this way we can determine the possible values for γ_1. There can be at most n possible values for γ_1. Having determined a value for γ_1, we now take all the points (i, α_i) on the line L. They correspond to the terms of lowest order in $f(x, Y(x))$. So we have to determine a c_1 such that

$$\sum_{\alpha_i + i\gamma_1 = \beta_1} a_i c_1^i \ = \ 0.$$

Since K is algebraically closed, this equation will always have non-zero solutions in K. The possible values for c_1 are the non-zero roots of this equation.

So after γ_1 and c_1 have been determined, the same process is performed on $Y_1(x)$, which must be a root of the equation

$$f_1(x, y_1) = f(x, c_1 x^{\gamma_1} + y_1) = 0. \quad (f \in K \ll x \gg [y_1])$$

Again the Newton polygon may be used to derive necessary conditions on c_2 and γ_2. However, this time only those lines are considered whose corresponding negative slope γ_2 is greater than γ_1.

This recursive process in the Newton Polygon Method can be iterated until the desired number of terms is computed, or no further splitting of solutions is possible.

A detailled proof of the fact, that the Newton Polygon Method can be performed on any polynomial f and that it actually yields Puiseux series (with bounded denominators of exponents) is given in [Wal50].

Now we are ready to see that every point P on an affine curve $\mathcal{C}$ has a correponding place with center at P. For a proof of the following theorem we refer to [Wal50]. Theorem 4.1 in Chap.4.

Theorem 9.2.2. *Let $f(x, y)$ be a polynomial in $K[x, y]$, and let $\mathcal{C}$ be the curve defined by f. To each root $Y(x) \in K \ll x \gg$ of $f(x, y) = 0$ with $\mathrm{ord}(Y) > 0$ there corresponds a unique place of $\mathcal{C}$ with center at the origin. Conversely, to each place $(X(t), Y(t))$ of $\mathcal{C}$ with center at the origin there correspond $\mathrm{ord}(X)$ roots of $f(x, y) = 0$, each of order greater than zero.*

If $Y(x)$ is a Puiseux series solving $f(x, y) = 0$, $\mathrm{ord}(Y) > 0$, and n is the least integer for which $Y(x) \in K((x^{\frac{1}{n}}))$, then we put $x^{\frac{1}{n}} = t$, and (t^n, Y) is a local parametrization with center at the origin.

The solutions of $f(x,y)$ of order 0 correspond to places with center on the y-axis (but different from the origin), and the solutions of negative order correpond to places at infinity.

Example 9.2.1. We consider the curve of Example 9.0.1. So the defining polynomial for $\mathcal{C}$ is Figure 9.3 shows the Newton polygon of f. There are three segments on the

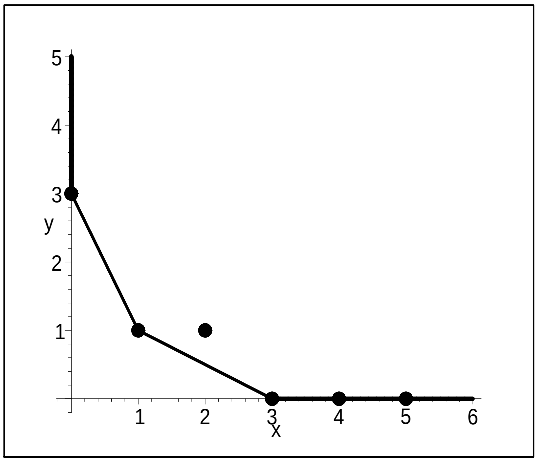

Figure 9.3: Newton polygon of f

lower left boundary of the Newton polygon of f. These three segments give three possible choices for the first exponent γ_1 in the Puiseux series expansion of a solution, namely

$$\gamma_1 \in \{2, \frac{1}{2}, 0\}.$$

In all three cases the corresponding equation has non-zero roots. In the case of $\gamma_1 = 2$, there are two points on the segment of the Newton polygon, and the corresponding equation

$$1 + 2c_1 = 0$$

has the solution $c_1 = -\frac{1}{2}$. For $\gamma_1 = \frac{1}{2}$ the equation is $4c_1^3 + 2c_1 = 0$, the non-zero solutions are $\pm\frac{1}{\sqrt{-2}}$. Finally, for $\gamma_1 = 0$ the equation is $c_1^5 - 4c_1^4 + 4c_1^3 = c_1^3(c_1-2)^2 = 0$, the non-zero solution is 2.

So we get 4 possible smallest terms of Puiseux series solving $f(x,y) = 0$. Since the field of Puiseux series is algebraically closed, and f is a squarefree polynomial of degree 5, there must be 2 solutions starting with the same term. This is the case for the series starting with the term $2x^0$. Continuing the process with this series, we would see that in the next step it splits into the two different solutions

$$2 + \frac{1+\sqrt{-95}}{8}x + \cdots \quad \text{and} \quad 2 + \frac{1-\sqrt{-95}}{8}x + \cdots.$$

We continue to expand the series starting with $-\frac{1}{2}x^2$. For determining the next highest exponent γ_2 and non-zero coefficient c_2, we make the ansatz $Y(x) = -\frac{1}{2}x^2 + Y_1(x)$. Now $Y_1(x)$ must solve the modified equation

$$\begin{aligned} f_1(x, y_1) &= f(x, -\tfrac{1}{2}x^2 + y_1) \\ &= y_1^5 - (\tfrac{5}{2}x^2 - 4)y_1^4 + (\tfrac{5}{2}x^4 + 8x^2 + 4)y_1^3 - (\tfrac{5}{4}x^6 + 6x^4 + 4x^2 + x)y_1^2 \\ &\quad +(\tfrac{5}{16}x^8 + 2x^6 + x^4 + x^3 + 2x^2 + 2x)y_1 - \tfrac{1}{32}x^{10} - \tfrac{1}{4}x^8 - \tfrac{1}{4}x^5. \end{aligned}$$

The Newton polygon of f_1 has only one segment with negative slope greater than $\gamma_1 = 2$. So we get $\gamma_2 = 4$ and $c_2 = \frac{1}{8}$.

Repeating this process sufficiently often, we finally get the following series expansions for the solutions of $f(x, y) = 0$:

$$\begin{aligned} Y_1(x) &= -\tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 - \tfrac{1}{8}x^5 + \tfrac{1}{16}x^6 + \tfrac{1}{16}x^7 + \cdots\ , \\ Y_2(x) &= \tfrac{\sqrt{-2}}{2}x^{\frac{1}{2}} - \tfrac{1}{8}x + \tfrac{27\sqrt{-2}}{128}x^{\frac{3}{2}} - \tfrac{7}{32}x^2 - \tfrac{4057\sqrt{-2}}{16384}x^{\frac{5}{2}} + \cdots\ , \\ Y_3(x) &= -\tfrac{\sqrt{-2}}{2}x^{\frac{1}{2}} - \tfrac{1}{8}x - \tfrac{27\sqrt{-2}}{128}x^{\frac{3}{2}} - \tfrac{7}{32}x^2 + \tfrac{4057\sqrt{-2}}{16384}x^{\frac{5}{2}} + \cdots\ , \\ Y_4(x) &= 2 + \tfrac{1+\sqrt{-95}}{8}x + \tfrac{1425-47\sqrt{-95}}{3040}x^2 + \cdots\ , \\ Y_5(x) &= 2 + \tfrac{1-\sqrt{-95}}{8}x + \tfrac{1425+47\sqrt{-95}}{3040}x^2 + \cdots\ . \end{aligned}$$

Y_1, Y_2, Y_3 have order greater than 0, so they correspond to places of $\mathcal{C}$ centered at the origin. Y_1 corresponds to the parametrization

$$(A_1(t), B_1(t)) = (t,\ -\frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{8}t^5 + \frac{1}{16}t^6 + \frac{1}{16}t^7 + \cdots),$$

and Y_2, Y_3 both correspond to the parametrization

$$(A_2(t), B_2(t)) = (-2t^2,\ t + \frac{1}{4}t^2 - \frac{27}{32}t^3 - \frac{7}{8}t^4 - \frac{4057}{2048}t^5 + \cdots).$$

Y_4, Y_5 have order 0, and they correspond to parametrizations centered at $(0, 2)$. The branches corresponding to these parametrizations cannot be seen in Fig. 9.3, since they are complex, except for the point $(0, 2)$. □

Example 9.2.2. Let us consider the curve $\mathcal{C}$ in $\mathbb{A}^2(\mathbb{C})$ defined by

$$f(x, y) = y^2 - x^3 + 2x^2 - x.$$

A plot of $\mathcal{C}$ around the origin is given in Figure 9.4. Let us determine the local parametrizations of $\mathcal{C}$ centered at the origin. $\mathcal{C}$ has only one branch at $(0, 0)$, so there should be exactly one place at the origin.

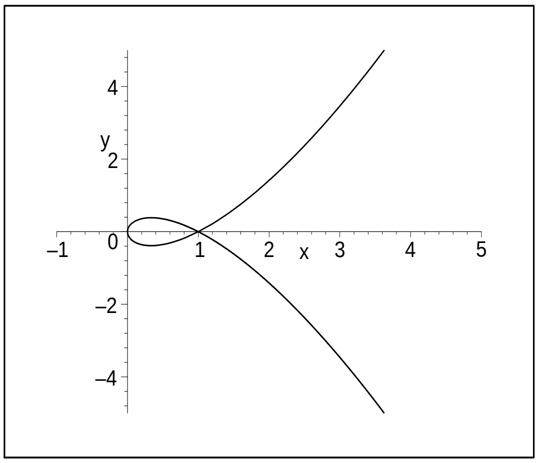

Figure 9.4: Real part of $\mathcal{C}$

We have

$$\begin{array}{lll} A_0(x) = -x + 2x^2 - x^3, & \alpha_0 = 1, & a_0 = -1, \\ A_1(x) = 0, & \alpha_0 = \infty, & \\ A_2(x) = 1, & \alpha_0 = 0, & a_2 = 1. \end{array}$$

The Newton polygon of f is given in Figure 9.5 (left).

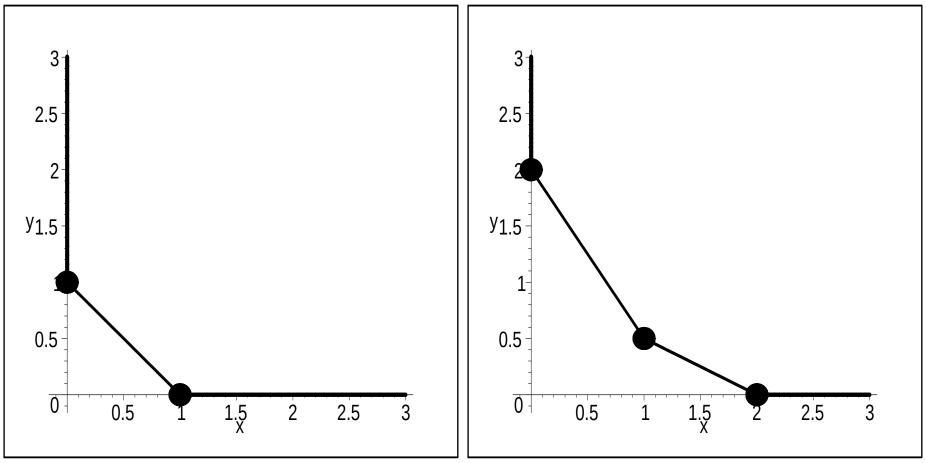

Figure 9.5: Newton polygon of f (Left), Newton polygon of f_1 (Right)

So $\gamma_1 = \frac{1}{2}$, and c_1 is the solution of the equation $-1 + c_1^2 = 0$, i.e. $c_1 = \pm 1$. We get two different Puiseux series solutions of $f(x, y) = 0$, starting with

$$Y_1(x) = x^{\frac{1}{2}} + \cdots, \quad \text{and} \quad Y_2(x) = -x^{\frac{1}{2}} + \cdots.$$

We continue to expand Y_1. For determining the next term in Y_1, we get the equation

$$f_1(x, y_1) = f(x, x^{\frac{1}{2}} + y_1) = y_1^2 + 2x^{\frac{1}{2}} y_1 + 2x^2 - x^3.$$

The Newton polygon of f_1 is given in Figure 9.5 (right). There is only one segment with negative slope greater than $\frac{1}{2}$, namely $\gamma_2 = \frac{3}{2}$. The corresponding equation $2 + 2c_2 = 0$ yields $c_2 = -1$. So now we have

$$Y_1(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}} + \cdots \ .$$

For determining the next term, we consider the equation

$$f_2(x, y_2) = f(x, x^{\frac{1}{2}} - x^{\frac{3}{2}} + y_2) = y_2(y_2 + 2x^{\frac{1}{2}} - 2x^{\frac{3}{2}}).$$

Since y_2 divides $f_2(x, y_2)$, $y_2 = 0$ is a solution. Thus,

$$Y_1(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}.$$

In the same way we could expand Y_2 further, and we would get

$$Y_2(x) = -x^{\frac{1}{2}} + x^{\frac{3}{2}}.$$

So, by setting $x^{\frac{1}{2}} = \pm t$ in Y_1, Y_2, respectively, we get the local parametrization

$$\mathcal{P} = (t^2, t - t^3).$$

The series in this local parametrization converge for every t, and in fact $\mathcal{P}$ is a global parametrization of $\mathcal{C}$. □

9.3 Rational Newton Polygon Method

As we have seen above, several different Puiseux series may correspond to the same parametrization. So we have the problem of identifying equivalent Puiseux series. Moreover, these series may have coefficients in a more complicated algebraic extension field than the corresponding parametrizations. Compare, for instance, the Puiseux series

$$\begin{aligned} Y_2(x) &= \tfrac{\sqrt{-2}}{2}x^{\frac{1}{2}} - \tfrac{1}{8}x + \tfrac{27\sqrt{-2}}{128}x^{\frac{3}{2}} - \tfrac{7}{32}x^2 - \tfrac{4057\sqrt{-2}}{16384}x^{\frac{5}{2}} + \cdots \ , \\ Y_3(x) &= -\tfrac{\sqrt{-2}}{2}x^{\frac{1}{2}} - \tfrac{1}{8}x - \tfrac{27\sqrt{-2}}{128}x^{\frac{3}{2}} - \tfrac{7}{32}x^2 + \tfrac{4057\sqrt{-2}}{16384}x^{\frac{5}{2}} + \cdots \ , \end{aligned}$$

of Example 9.2.1, corresponding to the parametrization

$$(A_2(t), B_2(t)) = (-2t^2,\ t + \frac{1}{4}t^2 - \frac{27}{32}t^3 - \frac{7}{8}t^4 - \frac{4057}{2048}t^5 + \cdots).$$

The parametrization is obtained from $(x, Y_2(x))$ by setting $t = \frac{\sqrt{-2}}{2}x^{1/2}$ and from $(x, Y_3(x))$ by setting $t = -\frac{\sqrt{-2}}{2}x^{1/2}$.

D. Duval has developed a refinement of the classical Newton polygon method, the so-called rational Newton polygon method, which allows to detect this kind of parameter substitutions at the stage where the corresponding equation of a segment of the Newton polygon has to be solved. If the exponents in the equation

$$\sum_{\alpha_i + i\gamma_1 = \beta_1} a_i c_1^i \ = \ 0$$

have a greatest common divisor $q > 1$, then

$$\sum_{\alpha_i + i\gamma_1 = \beta_1} a_i c_1^i \ = \ \sum_{\alpha_i + i\gamma_1 = \beta_1} a_i \tilde{c}_1^{i/q} \quad \text{for } \tilde{c}_1 = {c_1}^q.$$

It suffices to solve for $\tilde{c}_1$. In terms of parametrizations, this amounts to setting $X(t) = t^{mq}$, if $X(t) = t^m$ has been computed so far, starting with $X(t) = t$. Additionally it is sometimes possible to avoid algebraic extension for the coefficients if one allows $X(t) = \lambda t^m$.

Applying this idea to the computation in Example 9.2.1, we see that the segment corresponding to $\gamma = \frac{1}{2}$ has the equation $4c^2 + 2 = 0$, which can be simplified to $4\tilde{c} + 2 = 0$. This equation has the root $\tilde{c} = -\frac{1}{2}$. For determining the corresponding parametrization, it is sufficient to set $X(t) = -2t^2$ and $Y(t) = t$. In this way the algebraic extension of the coefficient field $\mathbb{Q}$ by $\sqrt{-2}$ can be avoided.