Chapter 6

Functions and mappings on varieties

6.1 Coordinate rings and polynomial functions

Throughout this chapter let K be a fixed algebraically closed field. We will consider algebraic sets in $\mathbb{A}^n = \mathbb{A}^n(K)$, for a fixed n. In particular, we will investigate varieties, i.e. irreducible algebraic sets.

All the rings and fields in this section will contain K as a subring. A homomorphism of such rings, $\varphi : R \longrightarrow S$, will always be a ring homomorphism which leaves K fixed, i.e. $\varphi(\lambda) = \lambda$ for all $\lambda \in K$.

Coordinate Rings and Polynomial Functions

Definition 6.1.1. Let V be a variety. W is a *subvariety* of V, iff W is a variety and $W \subset V$. □

If $V \subseteq \mathbb{A}^n$ is a variety, then $I(V)$ is prime, so $K[x_1, \dots, x_n]/I(V)$ is an integral domain.

Definition 6.1.2. Let $V \subseteq \mathbb{A}^n$ be a variety. The integral domain

$$\Gamma(V) = K[x_1, \dots, x_n]/I(V)$$

is called the *coordinate ring* of V. □

For a set $V \neq \emptyset$ let $\mathcal{J}(V, K)$ be the set of all functions from V to K. The set $\mathcal{J}(V, K)$ becomes a ring if we define

$$(f+g)(x) = f(x) + g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all $f, g \in \mathcal{J}(V, K)$, $x \in V$. The natural homomorphism from K into $\mathcal{J}(V, K)$, which maps a $\lambda \in K$ to the constant function $x \mapsto \lambda$, makes K a subring of $\mathcal{J}(V, K)$.

Definition 6.1.3. Let $V \subseteq \mathbb{A}^n$ be a variety. A function $\varphi \in \mathcal{J}(V, K)$ is called a *polynomial function* on V, iff there exists a polynomial $f \in K[x_1, \dots, x_n]$ with

$$\varphi(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

for all $(a_1, \dots, a_n) \in V$. In this case we say that f *represents* the function φ. □

The polynomial functions on a variety V form a subring of $\mathcal{J}(V, K)$ containing K (via the natural homomorphism). Two polynomials f, g represent the same function if and only if $(f-g)(P)=0$ for all $P \in V$, i.e. $f-g \in I(V)$. So we can identify the polynomial functions on V with the elements of the coordinate ring $\Gamma(V)$.

Lemma 6.1.1. *Let I be an ideal in the ring R, $\pi : R \longrightarrow R/I$ the natural homomorphism.*

(a) *Let J' be an ideal of R/I. Then $J = \pi^{-1}(J')$ is a superideal of I in R. This relation between ideals of R/I and superideals of I is 1-1.*

(b) *Let J and J' correspond as in* (a). *Then J is radical, prime, or maximal, respectively, if and only if J' is radical, prime, or maximal, respectively.*

Proof: (a) Obviously, if J' is an ideal in R/I, then $\pi^{-1}(J')$ is a superideal of I in R, and if J is a superideal of I in R, then $\pi(J)$ is an ideal in R/I.

Now let J_1, J_2 be two different superideals of I. W.l.o.g. let $f_1 \in J_1 \setminus J_2$. If $\pi(f_1)$ were in $\pi(J_2)$, then there would be an $f_2 \in J_2$ with $\pi(f_1) = \pi(f_2)$. So $f_1 - f_2 \in I$, and therefore $f_1 = (f_1 - f_2) + f_2 \in J_2$. Thus, $\pi(J_1) \neq \pi(J_2)$.

If J_1', J_2' are two different ideals in R/I, then obviously $\pi^{-1}(J_1) \neq \pi^{-1}(J_2)$.

(b) Let J be radical. If $\pi(a)^n = \pi(a^n) \in J'$, then $a^n \in J$, so $a \in J$, and therefore $\pi(a) \in J'$. Thus, J' is also radical. Conversely, let $J' = \pi(J)$ be radical in R/I. Let $f^n \in J$. Then $\pi(f^n) = \pi(f)^n \in J'$. So $\pi(f) \in J'$, and therefore $f \in \pi^{-1}(\pi(f)) \subseteq J$. Thus, J is also radical.

Let J be maximal. If J' were not maximal, then there would be an H' with $J' \subset H' \subset R/I$ (all inclusions proper). So we would have $J \subset H \subset R$ (all inclusions proper), in contradiction to the maximality of J.

The rest of the proof is left as an exercise. □

Theorem 6.1.2. *Let $V \subseteq \mathbb{A}^n$ be a variety. There is a 1-1 correspondence between algebraic subsets, subvarieties, and points, respectively, of V and radical ideals, prime ideals, and maximal ideals, respectively, in $\Gamma(V)$.*

Proof: Let the mapping

$$\varphi : \{\text{alg. subsets of } V\} \longrightarrow \text{ideals of } \Gamma(V)$$

be defined as

$$\varphi = \varphi_2 \circ \varphi_1,$$

where

$$\begin{array}{rccc} \varphi_1 : & \{\text{alg. subs. of } V\} & \longrightarrow & \text{ideals in } K[x_1, \ldots, x_n] \\ & W & \mapsto & I(W) \end{array}$$

and

$$\varphi_2 : K[x_1, \ldots, x_n] \longrightarrow K[x_1, \ldots, x_n]/I(V) \quad \text{(natural homomorphism).}$$

Let W be an algebraic subset of V. Then

W algebraic
$\Longleftrightarrow$ (Thm. 4.2.4)
$\varphi_1(W)$ is radical
$\Longleftrightarrow$ (Lemma 6.1.1)
$\varphi(W) = \varphi_2(\varphi_1(W))$ is radical.

Furthermore,

W is a variety or a point, respectively,
$\Longleftrightarrow$ (Thm. 4.2.5)

$\varphi_1(W)$ is prime or maximal, respectively,
$\Longleftrightarrow$ (Lemma 6.1.1)
$\varphi(W) = \varphi_2(\varphi_1(W))$ is prime or maximal, respectively. $\square$

So for the algebraic subsets of V, $\Gamma(V)$ plays the same role as the polynomial ring $K[x_1, \ldots, x_n]$ plays for the algebraic sets in $\mathbb{A}^n$.

Remark.

(1) There is an effective method for computing in $\Gamma(V)$. Let G be a Gröbner basis for the prime ideal $I(V)$ Then

$$\Gamma(V) \simeq N_G = \{\ f \in K[x_1, \ldots, x_n] \mid f \text{ is in normal form w.r.t. } G\ \}.$$

So, if we have a Gröbner basis G for $I(V)$ w.r.t. any term ordering, then the irreducible terms w.r.t. G are representatives of the elements of $\Gamma(V)$. Addition in $\Gamma(V) = N_G$ is simply addition of the representatives, for multiplication we multiply the representatives and then reduce modulo the Gröbner basis G.

(2) If V is a hypersurface, then the ideal $I(V)$ is principal, and the defining polynomial of V is a Gröbner basis for $I(V)$. Hence, arithmetic in the coordinate ring $\Gamma(V)$ can be carried out by means of remainders. $\square$

Example 6.1.1.

(a) If $V = \mathbb{A}^n$, then $I(V) = \langle 0 \rangle$ and $\Gamma(V) = K[x_1, \ldots, x_n]$.

(b) Let $V \subseteq \mathbb{A}^n$ a variety. Then V is a point if and only if $\Gamma(V) = K$. To see this, let $P = (a_1, \ldots, a_n)$ and $V = \{P\}$. Then $I(V) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, so $\Gamma(V) = K[x_1, \ldots, x_n]/I(V) = K$. Conversely, let $\Gamma(V) = K$. The only subideals of $\Gamma(V)$ are K and $\langle 0 \rangle$. So, the only algebraic subsets of V are V and $\emptyset$. Thus, V must be a single point.

(c) Let V be defined by $x \cdot y = 1$ in $\mathbb{A}^2$. Then

$$\Gamma(V) = \{\ f(x) + g(y) \mid f \in K[x],\ g \in K[y]\ \}.$$

$\square$

Theorem 6.1.3. *Let V be a variety. Then $\Gamma(V)$ is a Noetherian ring.*

Proof: By Lemma 6.1.1 there is a 1-1 relation between the ideals J' in $\Gamma(V)$ and the superideals J of $I(V)$ in $K[x_1, \ldots, x_n]$. The theorem follows from the fact that $K[x_1, \ldots, x_n]$ is a Noetherian ring (satisfying the ascending chain condition for ideals). $\square$

Polynomial Mappings

Definition 6.1.4 Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be varieties (over the same field K). A function $\varphi : V \to W$ is called a *polynomial* or *regular mapping* iff there are polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ such that $\varphi(P) = (f_1(P), \dots, f(P))$ for all $P \in V$. □

Theorem 6.1.4. *Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be varieties. There is a natural 1-1 correspondence between the polynomial mappings $\varphi : V \to W$ and the homomorphisms $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$.*

Proof: Let $\varphi : V \to W$ be regular. With φ we associate the homomorphism

$$\begin{array}{rccc} \tilde{\varphi}: & \Gamma(W) & \to & \Gamma(V) \\ & f & \mapsto & f \circ \varphi \end{array} .$$

The map $\tilde{}: \varphi \to \tilde{\varphi}$ is injective:

let $\varphi = (f_1, \dots, f_m), \varphi' = (f_1', \dots, f_m')$ be two regular mappings from V to W, where $f_i, f_i' \in K[x_1, \dots, x_n]$ for $1 \leq i \leq m$. $\tilde{\varphi} = \tilde{\varphi}'$ means that $\tilde{\varphi}(f) = \tilde{\varphi}'(f)$ for all $f \in \Gamma(W)$ (as functions on V). So, in particular,

$$f_1 = x_1 \circ \varphi = \tilde{\varphi}(x_1) = \tilde{\varphi}'(x_1) = x_1 \circ \varphi' = f_1'$$

as functions on V. Thus, as functions on V, $f_i = f_i'$ for $1 \leq i \leq m$, i.e. $\varphi = \varphi'$.

The map $\tilde{}: \varphi \to \tilde{\varphi}$ is surjective, i.e. every element of $\mathrm{Hom}(\Gamma(W), \Gamma(V))$ is reached by $\tilde{}$:

let $\lambda : \Gamma(W) \to \Gamma(V)$ be an arbitrary element of $\mathrm{Hom}(\Gamma(W), \Gamma(V))$, and $f_i \in K[x_1, \dots, x_n]$ such that $\lambda(x_i) \equiv_{I(V)} f_i$, for $1 \leq i \leq m$. We define

$$\begin{array}{rccc} \mu: & \mathbb{A}^n & \to & \mathbb{A}^m \\ & (a_1, \dots, a_n) & \mapsto & (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)). \end{array}$$

For $g \in I(W)$ we have (as functions on V):

$$g(f_1, \dots, f_m) \equiv_{I(V)} g(\lambda(x_1), \dots, \lambda(x_m)) \underset{\substack{\uparrow \\ \lambda \text{ homom.} \\ g \text{ polynomial}}}{=} \lambda(g) \underset{\substack{\uparrow \\ g \in I(W)}}{\equiv_{I(V)}} 0.$$

Therefore, for $P = (a_1, \dots, a_n) \in V$:

$$g(\mu(P)) = g(f_1(P), \dots, f_m(P)) = 0.$$

So, every element $g \in I(W)$ vanishes on $\mu(P)$, i.e. $\mu(P) \in W$. Thus, $\mu_{|V}$, the restriction of μ on V, is a regular mapping from V to W.

λ and $\tilde{\mu}$ agree on $x_1, \dots, x_m$, and therefore on arbitrary functions in $\Gamma(W)$. □

Definition 6.1.5. A regular mapping $\varphi : V \to W$ is a *regular isomorphism* iff there is a regular mapping $\psi : W \to V$, such that

$$\varphi \circ \psi = \mathrm{id}_W \qquad \text{and} \qquad \psi \circ \varphi = \mathrm{id}_V.$$

In this case the varieties V and W are *regularly isomorphic* (it via φ, ψ). □

Theorem 6.1.5. *V and W are regularly isomorphic via φ if and only if $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$ is an isomorphism of K–algebras.*

Proof: By Theorem 6.1.4 $\tilde{\varphi}$ is a homomorphism. Let $\psi : W \to V$ be such that $\varphi \circ \psi = \mathrm{id}_W$, $\psi \circ \varphi = \mathrm{id}_V$. $\tilde{\psi}$ is a homomorphism from $\Gamma(V)$ to $\Gamma(W)$. $\tilde{\varphi} \circ \tilde{\psi} : \Gamma(V) \to \Gamma(V)$ is the identity on $\Gamma(V)$, since

$$\tilde{\varphi} \circ \tilde{\psi}(f) = \tilde{\varphi}(f \circ \psi) = f \circ \psi \circ \varphi = f.$$

Analogously we get that $\tilde{\psi} \circ \tilde{\varphi} = \mathrm{id}_{\Gamma(W)}$. Thus, $\tilde{\varphi}$ is an isomorphism.

Conversely, if λ is an isomorphism from $\Gamma(W)$ to $\Gamma(V)$, then the corresponding φ is an isomorphism from V to W. $\square$

Example 6.1.2. (a) Let the (generalized) parabola $V \subset \mathbb{A}^2(\mathbb{C})$ be defined by $y = x^k$. V and $\mathbb{A}^1(\mathbb{C})$ are isomorphic via

$$\begin{array}{cccc} \varphi : & V & \to & \mathbb{A}^1 \\ & (x,y) & \mapsto & x \end{array} \quad , \quad \begin{array}{cccc} \psi : & \mathbb{A}^1 & \to & V \\ & t & \mapsto & (t, t^k) \end{array} .$$

(b) The projection $\varphi(x,y) = x$ of the hyperbola $xy = 1$ to the x–axis ($\mathbb{A}^1$) is <u>not</u> an isomorphism. There is no point (x,y) on the hyperbola such that $\varphi(x,y) = 0$.
(c) Let $V \subset \mathbb{A}^2(\mathbb{C})$ be defined by $y^2 = x^3$.
Then $\Gamma(V) \cong \{p(x) + q(x)y | p, q \in \mathbb{C}[x]\}$.
The mapping $\varphi : t \mapsto (t^2, t^3)$ from $\mathbb{A}^1$ to V is 1-1, but not an isomorphism. Otherwise we would have that $\tilde{\varphi} : \Gamma(V) \to \Gamma(\mathbb{A}^1) = \mathbb{C}[t]$ is an isomorphism. But for arbitrary $p, q \in \mathbb{C}[x]$ we have $\tilde{\varphi}(p(x) + q(x)y) = p(t^2) + q(t^2)t^3 \neq t$. $\square$

Theorem 6.1.6. *Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be varieties. Let $\varphi : V \to W$ be a regular mapping, X an algebraic subset of W, φ surjective on X.*
(a) *$\varphi^{-1}(X)$ is an algebraic subset of V.*
(b) *If $\varphi^{-1}(X)$ is irreducible, then also X is irreducible.*

Proof: (a) Let $\varphi = (\varphi_1, \ldots, \varphi_m)$, $f_1 = \ldots = f_r = 0$ be the defining equations for X, $g_i = f_i(\varphi_1, \ldots, \varphi_m) \in K[x_1, \ldots, x_n]$ for $1 \leq i \leq r$. Let $P = (a_1, \ldots, a_n)$ be an arbitrary point in V. Then

$$P \in \varphi^{-1}(X) \Longleftrightarrow \varphi(P) \in X \Longleftrightarrow g_1(P) = \cdots = g_r(P) = 0.$$

(b) If $X = X_1 \cup X_2$, then $\varphi^{-1}(X) = \varphi^{-1}(X_1) \cup \varphi^{-1}(X_2)$. Suppose $X_1 \not\subset X_2$. Because of surjectivity, $\varphi^{-1}(P) \neq \emptyset$. Therefore, $\emptyset \neq \varphi^{-1}(P) \subseteq \varphi^{-1}(X_1) \not\subset \varphi^{-1}(X_2)$ and thus $\varphi^{-1}(X_1) \not\subset \varphi^{-1}(X_2)$. So if X is reducible, then so is $\varphi^{-1}(X)$. $\square$

Example 6.1.3. We show that $V = V(y - x^2, z - x^3) \subset \mathbb{A}^3$ is a variety. The regular mapping

$$\begin{array}{cccc} \varphi : & \mathbb{A}^1 & \to & V \\ & t & \mapsto & (t, t^2, t^3) \end{array}$$

is surjective and $\mathbb{A}^1$ is irreducible. So by Theorem 6.1.6 also V is irreducible. $\square$

There are some kinds of very frequently used and important regular mappings. One such kind of mappings are the *projections*

$$\begin{array}{rccc} \mathrm{pr}: & \mathbb{A}^n & \to & \mathbb{A}^r \\ & (a_1,\ldots,a_n) & \mapsto & (a_1,\ldots,a_r), \end{array}$$

for $n \geq r$.

Let $V \subseteq \mathbb{A}^n$ be a variety, $f \in \Gamma(V)$. Let

$$G(f) = \{(a_1,\ldots,a_{n+1}) \mid (a_1,\ldots,a_n) \in V, a_{n+1} = f(a_1,\ldots,a_n)\} \subseteq \mathbb{A}^{n+1}$$

be the *graph* of f. $G(f)$ is an affine variety, and

$$\begin{array}{rccc} \varphi: & V & \to & G(f) \\ & (a_1,\ldots,a_n) & \mapsto & (a_1,\ldots,a_n,f(a_1,\ldots,a_n)) \end{array}$$

is an isomorphism between V and $G(f)$. The projection from $\mathbb{A}^{n+1}$ to $\mathbb{A}^n$ is the inverse of φ.

Another important kind of regular mappings are changes of coordinates.

Definition 6.1.6. (Compare Def. 5.2.6) An *affine change of coordinates* in $\mathbb{A}^n$ is a bijective linear polynomial mapping, i.e. a bijective mapping of the form

$$\begin{array}{rccc} T: & \mathbb{A}^n & \to & \mathbb{A}^n \\ & (a_1,\ldots,a_n) & \mapsto & (T_1(a_1,\ldots,a_n),\ldots,T_n(a_1,\ldots,a_n)), \end{array}$$

where $\deg(T_i) = 1$ for $1 \leq i \leq n$.

If $V = V(f_1,\ldots,f_m)$ is an algebraic set in $\mathbb{A}^n$, then by V^T we denote the image of V under T, i.e.

$$V^T = V(f_1^T,\ldots,f_m^T),$$

where $f^T(x_1,\ldots,x_n) = f(T_1(x_1,\ldots,x_n),\ldots,T_n(x_1,\ldots,x_n))$, for any polynomial f. □

Using column notation for the coordinates of points, every linear polynomial mapping from $\mathbb{A}^n$ into itself can be written as

$$T(\overline{x}) = A \cdot \overline{x} + b$$

for some matrix A and vector b. T is an affine change of coordinates, if and only if A is an invertible matrix.

Affine geometry is the geometry of properties which are invariant under affine changes of coordinates.

6.2 Rational functions and local rings

The coordinate ring $\Gamma(V)$ of a variety $V \subseteq \mathbb{A}^n$ is an integral domain. So it can be embedded into its quotient field.

Definition 6.2.1. The *field of rational functions* $K(V)$ of a variety $V \subseteq \mathbb{A}^n(K)$ is the quotient field of $\Gamma(V)$. So

$$K(V) \;\simeq\; \{\frac{f}{g} \mid f, g \in K[x_1, \ldots, x_n], g \notin I(V)\}/\sim\;,$$

where $\frac{f}{g} \sim \frac{f'}{g'} \iff fg' - f'g \in I(V)$. □

Definition 6.2.2. A rational function $\varphi \in K(V)$ is *defined* or *regular* at $P \in V$ iff φ can be written as $\varphi = f/g$ with $g(P) \neq 0$. In this case $f(P)/g(P)$ is the *value* of φ at P. The set of points in V at which a rational function φ is defined is called the *domain of definition* of φ. A point $P \in V$ at which the function φ is not defined is a *pole* of φ.
For $P \in V$ the *local ring* of V at P is defined as $\mathcal{O}_P(V) = \{\varphi \in K(V) \mid \varphi \text{ regular at } P\}$.
□

The notion of value of a rational function at a point on a variety is well defined. The local ring $\mathcal{O}_P(V)$ is indeed a local ring in the sense of having a unique maximal ideal. This maximal ideal is the subset of $\mathcal{O}_P(V)$ containing those rational functions which vanish on P. One easily verifies that $\mathcal{O}_P(V)$ is a subring of $K(V)$ containing $\Gamma(V)$. So we have the following increasing chain of rings:

$$K \subseteq \Gamma(V) \subseteq \mathcal{O}_P(V) \subseteq K(V).$$

Example 6.2.1. Let V be the unit circle in $\mathbb{A}^2(\mathbb{C})$ defined by

$$x^2 + y^2 = 1.$$

The rational function

$$\varphi(x, y) = \frac{1-y}{x}$$

is obviously regular in all points of V except $(0, \pm 1)$. But φ is also regular in $(0, 1)$, which can be seen by the following transformation

$$\frac{1-y}{x} = \frac{x(1-y)}{x^2} = \frac{x(1-y)}{1-y^2} = \frac{x}{1+y}.$$

□

Theorem 6.2.1. *The set of poles of a rational function φ on a variety $V \subseteq \mathbb{A}^n$ is an algebraic set.*

Proof: Consider $J_\varphi = \{g \in K[x_1, \ldots, x_n] \mid g\varphi \in \Gamma(V)\}$. J_φ is an ideal in $K[x_1, \ldots, x_n]$ containing $I(V)$. The points of $V(J_\varphi)$ are exactly the poles of φ: if $P \in V(J_\varphi)$, then for every representation $\varphi = f/g$ we have $g \in J_\varphi$, so $g(P) = 0$, and therefore P is a pole. On the other hand, if $P \notin V(J_\varphi)$, then for some $g \in J_\varphi$ we have $g(P) \neq 0$. So there is an $r \in \Gamma(V)$ such that $\varphi = r/g$ and $g(P) \neq 0$, i.e. P is not a pole. $\square$

Theorem 6.2.2. *A rational function $\varphi \in K(V)$, which is regular on every point of the variety V, is a regular function on V. So*

$$\Gamma(V) = \bigcap_{P \in V} \mathcal{O}_P(V).$$

Proof: If φ is regular on every point of V, then $V(J_\varphi) = \emptyset$ (proof of Theorem 6.2.1). So, by Hilbert's Nullstellensatz we have $1 \in J_\varphi$, i.e. $1 \cdot \varphi = \varphi \in \Gamma(V)$. $\square$

Example 6.2.1. continued: $\varphi = (1-y)/x$ cannot be regular on the whole variety V, because otherwise, by Theorem 6.2.2, there should be a polynomial $p(x, y) \in \mathbb{C}[x, y]$ such that

$$\varphi(x, y) = \frac{1-y}{x} = p(x, y).$$

This would mean $1 - y - x \cdot p(x, y) \in I(V) = \langle x^2 + y^2 - 1 \rangle$, or, equivalently, $1 - y \in \langle x^2 + y^2 - 1, x \rangle = \langle y^2 - 1, x \rangle$. This, however, is impossible, as can be seen from the theory of Gröbner bases. $\square$

As we have extended the notion of a regular function to that of a regular mapping, we will now extend the notion of a rational function on a variety to that of a rational mapping on the variety.

Definition 6.2.3. Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be varieties (over the algebraically closed field K). An m-tuple φ of rational functions $\varphi_1, \ldots, \varphi_m \in K(V)$, with the property that for an arbitrary point $P \in V$, at which all the φ_i are regular, we have $(\varphi_1(P), \ldots, \varphi_m(P)) \in W$, is called a *rational mapping* from V to W, $\varphi : V \to W$. φ is *regular* at $P \in V$ iff all the φ_i are regular at P. $\square$

So, a rational mapping is not a mapping of the whole variety V into W, but of a certain non-empty open (in the Zariski topology) subset $U \subseteq V$ into W.

Example 6.2.2. Let $\mathcal{C}$ be the curve in $\mathbb{A}^2(\mathbb{C})$ defined by

$$f(x, y) = y^2 - x^3 - x^2 = 0.$$

The tuple of rational (in fact polynomial) functions

$$\varphi_1(t) = t^2 - 1, \quad \varphi_2(t) = t(t^2 - 1)$$

determines a rational mapping φ from $\mathbb{A}^1$ to $\mathcal{C}$. This rational mapping has a rational inverse, i.e. a rational mapping from $\mathcal{C}$ to $\mathbb{A}^1$:

$$\chi(x, y) = y/x.$$

We check that χ really is the inverse of φ:

$$\begin{aligned}
\chi(\varphi_1(t), \varphi_2(t)) &= \frac{t(t^2-1)}{t^2-1} = t, \\
\varphi_1(\chi(x,y)) &= \frac{y^2}{x^2} - 1 = \frac{y^2 - x^2}{x^2} = \frac{x^3}{x^2} = x, \\
\varphi_2(\chi(x,y)) &= \frac{y}{x} \cdot \left(\frac{y^2}{x^2} - 1\right) = \frac{y(y^2-x^2)}{x^3} = y.
\end{aligned}$$

So, up to finitely many exceptions, the points in $\mathbb{A}^1$ and $\mathcal{C}$ correspond uniquely to each other. Later (see Chapter 8) we will call $(\varphi_1(t), \varphi_2(t))$ a rational parametrization of the curve $\mathcal{C}$. □

Definition 6.2.4. Let the rational mapping $\varphi : V \to W$ have a rational inverse, i.e. a rational mapping $\psi : W \to V$ such that $\psi \circ \varphi = \mathrm{id}_V$, $\varphi \circ \psi = \mathrm{id}_W$ (wherever the composition of these mappings is defined), and $\varphi(V), \psi(W)$ are dense in W, V, respectively. In this case φ is called a *birational isomorphism* from V to W (and ψ a birational isomorphism from W to V), and that V and W are *birationally isomorphic* or *birationally equivalent*. □

Definition 6.2.5. We say that a variety is *rational* if it is birationally isomorphic to an affine space $\mathbb{A}^m$. A variety W is called *unirational* if, for some $\mathbb{A}^m$, there exists a rational mapping $\varphi : \mathbb{A}^m \to W$, such that $\varphi(\mathbb{A}^m) \subset W$ is dense. □

In the previous example we have seen that the curve defined by $y^2 - x^3 - x^2$ is birationally isomorphic to the affine line, so it is a rational curve. In Chapter 8 rational plane curves over algebraically closed fields of characteristic zero are analyzed in detail. In fact, the notions of rationality and unirationality for plane curves are equivalent. This is a consequence of Lüroth's Theorem. Furthermore, for surfaces over an algebraically closed field the two concepts are also the same (Castelnuovo's theorem). However, in general the equivalence is not true. For further details see [Sch72].

Isomorphism of varieties is reflected in the function fields of these varieties.

Theorem 6.2.3. *The varieties V and W are birationally isomorphic if and only if the corresponding function fields $K(V)$ and $K(W)$ are isomorphic.* □

Proof: Let $V \subseteq \mathbb{A}^n(K)$ and $W \subseteq \mathbb{A}^m(K)$.

Let $\varphi : V \to W$ be a birational isomorphism from V to W and let $\psi : W \to V$ be its inverse. Consider the following homomorphisms between the function fields:

$$\begin{array}{rccc} \tilde{\varphi}: & K(W) & \to & K(V) \\ & r & \mapsto & r \circ \varphi \end{array} \qquad \begin{array}{rccc} \tilde{\psi}: & K(V) & \to & K(W) \\ & s & \mapsto & s \circ \psi \end{array}$$

(Actually by $r \circ \varphi$ we mean the rational function on $K(V)$ whose restriction to the dense subset $\varphi(W)$ of W is $r \circ \varphi$, and analogously for $s \circ \psi$.) These homomorphisms $\tilde{\varphi}$ and $\tilde{\psi}$ are inverses of each other, so we have an isomorphism of the function fields.

On the other hand, let α be an isomorphism from $K(V)$ to $K(W)$, and β its inverse, i.e.

$$\alpha : K(V) \to K(W), \qquad \beta : K(W) \to K(V).$$

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be the coordinate functions of V and W, respectively. Then

$$\tilde{\beta} = (\beta(y_1)(x_1, \ldots, x_n),\ \ldots, \beta(y_m)(x_1, \ldots, x_n))$$

is a birational isomorphism from V to W and

$$\tilde{\alpha} = (\alpha(x_1)(y_1, \ldots, y_m),\ \ldots, \alpha(x_n)(y_1, \ldots, y_m))$$

is its inverse from W to V. $\square$