Chapter 7

Local properties of plane algebraic curves

Throughout this chapter let K be an algebraically closed field of characteristic zero, and as usual let $\mathbb{A}^2(K)$ be embedded into $\mathbb{P}^2(K)$ by identifying the point $(a, b) \in \mathbb{A}^2(K)$ with the point $(a : b : 1) \in \mathbb{P}^2(K)$.

7.1 Singularities and tangents

Affine plane curves

An affine plane curve $\mathcal{C}$ over K is a hypersurface in $\mathbb{A}^2(K)$. Thus, it is an affine algebraic set defined by a non-constant polynomial f in $K[x, y]$. By Hilbert's Nullstellensatz the squarefree part of f defines the same curve $\mathcal{C}$, so we might as well require the defining polynomial to be squarefree.

Definition 7.1.1. An *affine plane algebraic curve* over K is defined as the set

$$\mathcal{C} = \{(a, b) \in \mathbb{A}^2(K) \mid f(a, b) = 0\}$$

for a non-constant squarefree polynomial $f(x, y) \in K[x, y]$.

We call f the *defining polynomial* of $\mathcal{C}$ (of course, a polynomial $g = c\,f$, for some nonzero $c \in K$, defines the same curve, so f is unique only up to multiplication by nonzero constants).

We will write f as

$$f(x, y) = f_d(x, y) + f_{d-1}(x, y) + \cdots + f_0(x, y)$$

where $f_k(x, y)$ is a homogeneous polynomial (form) of degree k, and $f_d(x, y)$ is nonzero. The polynomials f_k are called the *homogeneous components* of f, and d is called the *degree* of $\mathcal{C}$. Curves of degree one are called *lines*, of degree two *conics*, of degree three *cubics*, etc.

If $f = \prod_{i=1}^{n} f_i$, where f_i are the irreducible factors of f, we say that the affine curve defined by each polynomial f_i is a *component* of $\mathcal{C}$. Furthermore, the curve $\mathcal{C}$ is said to be *irreducible* if its defining polynomial is irreducible. □

Sometimes we need to consider curves with multiple components. This means that the given definition has to be extended to arbitrary polynomials $f = \prod_{i=1}^{n} f_i^{e_i}$, where f_i are the irreducible factors of f, and $e_i \in \mathbb{N}$ are their multiplicities. In this situation, the curve defined by f is the curve defined by its squarefree part, but the component generated by f_i carries multiplicity e_i. Whenever we will use this generalization we will always explicitly say so.

Definition 7.1.2. Let $\mathcal{C}$ be an affine plane curve over K defined by $f(x,y) \in K[x,y]$, and let $P = (a,b) \in \mathcal{C}$. The *multiplicity of* $\mathcal{C}$ *at* P (we denote it by $\mathrm{mult}_P(\mathcal{C})$) is defined as the order of the first non-vanishing term in the Taylor expansion of f at P, i.e.

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (x-a)^k (y-b)^{n-k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}}(a,b).$$

Clearly $P \notin \mathcal{C}$ if and only if $\mathrm{mult}_P(\mathcal{C}) = 0$. P is called a *simple point on* $\mathcal{C}$ iff $\mathrm{mult}_P(\mathcal{C}) = 1$. If $\mathrm{mult}_P(\mathcal{C}) = r > 1$, then we say that P is a *multiple* or *singular point* (or *singularity*) of *multiplicity* r on $\mathcal{C}$ or an *r-fold point*; if $r = 2$, then P is called a *double point*, if $r = 3$ a *triple point*, etc. We say that a curve is *non-singular* if it has no singular point. □

The singularities of the curve $\mathcal{C}$ defined by f are the solutions of the affine algebraic set $V(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Later we will see that this set is 0-dimensional, i.e. every curve has only finitely many singularities.

Let $P = (a,b) \in \mathbb{A}^2(K)$ be an r-fold point ($r \geq 1$) on the curve $\mathcal{C}$ defined by the polynomial f. Then the first nonvanishing term in the Taylor expansion of f at P is

$$T_r(x,y) = \sum_{i=0}^{r} \binom{r}{i} \frac{\partial^r f}{\partial x^i \partial y^{r-i}}(P)(x-a)^i (y-b)^{r-i}.$$

By a linear change of coordinates which moves P to the origin the polynomial T_r is transformed into a homogeneous bivariate polynomial of degree r. Hence, since the number of factors of a polynomial is invariant under linear changes of coordinates, we get that all irreducible factors of T_r are linear and they are the tangents to the curve at P.

Definition 7.1.3. Let $\mathcal{C}$ be an affine plane curve with defining polynomial f, and $P = (a,b) \in \mathbb{A}^2(K)$ such that $\mathrm{mult}_P(\mathcal{C}) = r \geq 1$. Then the *tangents* to $\mathcal{C}$ at P are the irreducible factors of the polynomial

$$\sum_{i=0}^{r}\binom{r}{i}\frac{\partial^r f}{\partial x^i \partial y^{r-i}}(P)(x-a)^i(y-b)^{r-i}$$

and the *multiplicity* of a tangent is the multiplicity of the corresponding factor. □

Remark: We leave the proof of these remarks as an exercise. Let $\mathcal{C}$ be an affine plane curve defined by the polynomial $f \in K[x, y]$. Then the following hold:

(1) The notion of multiplicity is invariant under linear changes of coordinates.

(2) For every $P \in \mathbb{A}^2(K)$, $mult_P(\mathcal{C}) = r$ if and only if all the derivatives of f up to and including the $(r-1)$-st vanish at P but at least one r-th derivative does not vanish at P.

(3) The multiplicity of $\mathcal{C}$ at the origin of $\mathbb{A}^2(K)$ is the minimum of the degrees of the non-zero homogeneous components of f. I.e. the origin $(0, 0)$ is an m–fold point on $\mathcal{C}$ iff f is the sum of forms $f = f_m + f_{m+1} + \ldots + f_d$, with $\deg(f_i) = i$ for all i. The tangents to $\mathcal{C}$ at the origin are the linear factors of f_m, the form of lowest degree, with the corresponding multiplicities.

(4) Let $P = (a, b)$ be a point on $\mathcal{C}$, and let $T : (x, y) \mapsto (x + a, y + b)$ be the change of coordinates moving the origin to P. Let $f^T(x, y) = f(x + a, y + b)$. Then $\mathrm{mult}_P(f) = \mathrm{mult}_{(0,0)}(f^T)$. If $L(x, y)$ is a tangent to f^T at the origin, then $L(x - a, y - b)$ is a tangent to f at P with the same multiplicity. □

Hence, taking into account this remark, the multiplicity of P can also be determined by moving P to the origin by means of a linear change of coordinates and applying Remark (3).

Example 7.1.1. Let the curve $\mathcal{C}$ be defined by the equation $f(x, y) = 0$, $P = (a, b)$ a simple point on $\mathcal{C}$. Consider the line $\mathcal{L}$ through P (with slope μ/λ) defined by $\mu x - \lambda y = \mu a - \lambda b$, or parametrically by $x = a + \lambda t, y = b + \mu t$. We get the points of intersection of $\mathcal{C}$ and $\mathcal{L}$ as the solutions of $f(a + \lambda t, b + \mu t) = 0$.

An expansion of this equation in a Taylor series around P yields $(f(a, b) = 0)$:

$$(f_x\lambda + f_y\mu)t + \frac{1}{2!}(f_{xx}\lambda^2 + 2f_x f_y\lambda\mu + f_{yy}\mu^2)t^2 + \ldots = 0.$$

We get "higher order" intersection at P if some of these terms vanish, i.e. if λ, μ are such that $f_x\lambda + f_y\mu = 0, \ldots$. The line which yields this higher order intersection is what we want to call the tangent to $\mathcal{C}$ at P. □

For analyzing a singular point P on a curve $\mathcal{C}$ we need to know its multiplicity but also the multiplicities of the tangents. If all the r tangents at the r-fold point P are different, then this singularity is of well-behaved type. For instance, when we trace the curve through P we can simply follow the tangent and then approximate back onto the curve. This is not possible any more when some of the tangents are the same.

Definition 7.1.4. A singular point P of multiplicity r on an affine plane curve $\mathcal{C}$ is called *ordinary* iff the r tangents to $\mathcal{C}$ at P are distinct, and *non-ordinary* otherwise. non-ordinary. An ordinary double point is a *node*. □

Example 7.1.2. We consider the following curves in $\mathbb{A}^2(\mathbb{C})$. Pictures of such curves can be helpful, but we have to keep in mind that we can only plot the real components of such curves, i.e. the components in $\mathbb{A}^2(\mathbb{R})$. Interesting phenomena, such as singular points etc., might not be visible in the real plane.

$\mathcal{A} : a(x,y) = y - x^2 = 0$ $\quad$ $\mathcal{B} : b(x,y) = y^2 - x^3 + x = 0$ $\quad$ $\mathcal{C} : c(x,y) = y^2 - x^3$

$\mathcal{D} : d(x,y) = y^2 - x^3 - x^2 = 0$ $\quad$ $\mathcal{E} : e(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$ $\quad$ $\mathcal{F} : f(x,y) = (x^2+y^2)^3 - 4x^2y^2$

The curves $\mathcal{A}, \mathcal{B}$ are non-singular. The only singular point on $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ is the origin $(0,0)$.

The linear forms in the equations for $\mathcal{A}$ and $\mathcal{B}$ define the tangents to these curves at the origin.

The forms of lowest degree in $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are $y^2, y^2 - x^2 = (y-x)(y+x), 3x^2y - y^3 = y(\sqrt{3}x - y)(\sqrt{3}x + y)$ and $-4x^2y^2$. The factors of these forms of lowest degree determine the tangents to these curves at the origin.

Intersecting $\mathcal{F}$ by the line $x = y$ we get the point $P = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$\frac{\partial f}{\partial x}(P) = (6x^5 + 12x^3y^2 + 6xy^4 - 8xy^2)(P) = \sqrt{2}, \quad \frac{\partial f}{\partial y}(P) = \sqrt{2}.$$

So the tangent to $\mathcal{F}$ at P is defined by

$$\sqrt{2}(x - \frac{1}{\sqrt{2}}) + \sqrt{2}(y - \frac{1}{\sqrt{2}}) = 0 \quad \sim \quad x + y - \sqrt{2} = 0.$$

$\mathcal{D}$ has a node at the origin. $\mathcal{E}$ has an ordinary 3–fold point at the origin. $\mathcal{C}$ and $\mathcal{F}$ both have a non–ordinary singular point at the origin. □

Lemma 7.1.1. *Let $\mathcal{C}$ be an affine plane curve defined by the squarefree polynomial $f = \prod_{i=1}^{n} f_i$ where all the components f_i are irreducible. Let P be a point in $\mathbb{A}^2(K)$. Then the following holds:*

(1) $\text{mult}_P(\mathcal{C}) = \sum_{i=1}^{n} \text{mult}_P(\mathcal{C}_i)$, *where $\mathcal{C}_i$ is the component of $\mathcal{C}$ defined by f_i.*

(2) If L is a tangent to $\mathcal{C}_i$ at P with multiplicity s_i, then L is a tangent to $\mathcal{C}$ at P with multiplicity $\sum_{i=1}^{n} s_i$.

Proof.
(1) By the remark above we may assume that P is the origin. Let

$$f_i(x, y) = \sum_{j=r_i}^{n_i} g_{i,j}(x, y) \quad \text{for} \quad i = 1, \ldots, n,$$

where n_i is the degree of $\mathcal{C}_i$, $r_i = \text{mult}_P(\mathcal{C}_i)$, and $g_{i,j}$ is the homogeneous component of f_i of degree j. Then the lowest degree homogeneous component of f is $\prod_{i=1}^{n} g_{i,r_i}$. Hence, (1) follows from Remark (3).
(2) follows directly from Remark (4) and from the expression of the lowest degree homogeneous component of f deduced in the proof of statement (1). □

Example 7.1.3. Determine the singular points and the tangents at these singular points to the following curves:

(a) $f_1 = y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy$,

(b) $f_2 = x^4 + y^4 - x^2y^2$,

(c) $f_3 = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$,

(d) $f_4 = y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25)$.

Plot the real components (i.e. in $\mathbb{A}^2(\mathbb{R})$) of these curves. □

Theorem 7.1.2. *An affine plane curve has only finitely many singular points.*

Proof: Let $\mathcal{C}$ be an affine plane curve with defining polynomial f, let $f = f_1 \cdots f_r$ be the irreducible factorization of f, and let $\mathcal{C}_i$ be the component generated by f_i (note that f is squarefree, so the f_i's are pairwise relatively prime). Then, applying Lemma 7.1.1. one deduces that the singular points of $\mathcal{C}$ are the singular points of each component $\mathcal{C}_i$ and the intersection points of all pairs of components (i.e. the points in the affine algebraic sets $V(f_i, f_j)$, $i \neq j$). Hence the set W of singular points of $\mathcal{C}$ is:

$$W = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \bigcup_{i=1}^{r} V\left(f_i, \frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}\right) \cup \bigcup_{i \neq j} V(f_i, f_j).$$

Now, observe that $\gcd(f_i, f_j) = 1$ for $i \neq j$. Thus, applying Theorem 3.3.5., we get that, for $i \neq j$, $V(f_i, f_j)$ is finite. Similarly, since f_i is irreducible and $\deg(\frac{\partial f_i}{\partial x}), \deg(\frac{\partial f_i}{\partial y}) < \deg(f)$, one deduces that $\gcd(f_i, \frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}) = 1$. Therefore, again by Theorem 3.3.5, we conclude that W is finite. □

Projective plane curves

A projective plane curve is a hypersurface in the projective plane.

Definition 7.1.5. A *projective plane algebraic curve* over K is defined as the set

$$\mathcal{C} = \{(a : b : c) \in \mathbb{P}^2(K) \mid F(a, b, c) = 0\}$$

for a non-constant squarefree homogeneous polynomial $F(x, y, z) \in K[x, y, z]$.

We call F the *defining polynomial* of $\mathcal{C}$ (of course, a polynomial $G = c\,F$, for some nonzero $c \in K$ defines the same curve, so F is unique only up to multiplication by nonzero constants).

Similarly as in Definition 7.1.1. the concepts of degree, components and irreducibility are introduced. □

Also, as in the case of affine curves, we will sometimes need to refer to multiple components of a projective plane curve. Again, one introduces this notion by extending the concept of curve to arbitrary forms. We will also always explicitly indicate when we make use of this generalization.

As we saw in the previous chapter, there is a close relationship between affine and projective algebraic sets. Thus, associated to every affine curve there is a projective curve (its projective closure).

Definition 7.1.6. The *projective plane curve* $\mathcal{C}^*$ *corresponding* to an affine plane curve $\mathcal{C}$ over K is the projective closure of $\mathcal{C}$ in $\mathbb{P}^2(K)$. □

If the affine curve $\mathcal{C}$ is defined by the polynomial $f(x,y)$, then from Section 5.3 we immediately get that the corresponding projective curve $\mathcal{C}^*$ is defined by the homogenization $F(x,y,z)$ of $f(x,y)$. Therefore, if $f(x,y) = f_d(x,y)+f_{d-1}(x,y)+\cdots+f_0(x,y)$ is the decomposition of f into forms, then $F(x,y,z) = f_d(x,y) + f_{d-1}(x,y)z + \cdots + f_0(x,y)z^d$, and

$$\mathcal{C}^* = \{(a:b:c) \in \mathbb{P}^2(K) \mid F(a,b,c) = 0\}.$$

Every point (a,b) on $\mathcal{C}$ corresponds to a point on $(a:b:1)$ on $\mathcal{C}^*$, and every additional point on $\mathcal{C}^*$ is a point at infinity. In other words, the first two coordinates of the additional points are the nontrivial solutions of $f_d(x,y)$. Thus, the curve $\mathcal{C}^*$ has only finitely many points at infinity. Of course, a projective curve, not associated to an affine curve, could have $z = 0$ as a component and therefore have infinitely many points at infinity.

On the other hand, associated to every projective curve there are infinitely many affine curves. We may take any line in $\mathbb{P}^2(K)$ as the line at infinity, by a linear change of coordinates move it to $z = 0$, and then dehomogenize. But in practice we mostly use dehomogenizations provided by taking the axes as lines at infinity. More precisely, if $\mathcal{C}$ is the projective curve defined by the form $F(x,y,z)$, we denote by $\mathcal{C}_{*,z}$ the affine plane curve defined by $F(x,y,1)$. Similarly, we consider $\mathcal{C}_{*,y}$, and $\mathcal{C}_{*,x}$.

So, any point P on a projective curve $\mathcal{C}$ corresponds to a point on a suitable affine version of $\mathcal{C}$. The notions of multiplicity of a point and tangents at a point are local properties. So for determining the multiplicity of P at $\mathcal{C}$ and the tangents to $\mathcal{C}$ at P we choose a suitable affine plane (by dehomogenizing w.r.t to one of the projective variables) containing P, determine the multiplicity and tangents there, and afterwards homogenize the tangents to move them back to the projective plane. This process does not depend on the particular dehomogenizing variable (compare, for instance, [Ful69] Chap. 5).

Theorem 7.1.3. *$P \in \mathbb{P}^2(K)$ is a singularity of the projective plane curve $\mathcal{C}$ defined by the homogeneous polynomial $F(x,y,z)$ if and only if $\frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial z}(P) = 0$.*

Proof. Let $d = \deg(F)$. We may assume w.l.o.g. that P is not on the line at infinity $z = 0$, i.e. $P = (a:b:1)$. Let $\mathcal{C}_{*,z}$ be the affine curve defined by $f(x,y) = F(x,y,1)$ and let $P_* = (a,b)$ be the image of P in this affine version of the plane.

P is a singular point of $\mathcal{C}$ if and only if P_* is a singular point of $\mathcal{C}_{*,z}$, i.e. if and only if

$$f(P_*) = \frac{\partial f}{\partial x}(P_*) = \frac{\partial f}{\partial y}(P_*) = 0.$$

But

$$\frac{\partial f}{\partial x}(P_*) = \frac{\partial F}{\partial x}(P), \quad \frac{\partial f}{\partial y}(P_*) = \frac{\partial F}{\partial y}(P).$$

Furthermore, by Euler's Formula for homogeneous polynomials [1] we have

$$x \cdot \frac{\partial F}{\partial x}(P) + y \cdot \frac{\partial F}{\partial y}(P) + z \cdot \frac{\partial F}{\partial z}(P) = d \cdot F(P).$$

The theorem now follows at once. $\square$

By an inductive argument this theorem can be extended to higher multiplicities. We leave the proof as an exercise.

Theorem 7.1.4. *$P \in \mathbb{P}^2(K)$ is a point of multiplicity at least r on the projective plane curve $\mathcal{C}$ defined by the homogeneous polynomial $F(x, y, z)$ of degree d (where $r \leq d$) if and only if all the $(r-1)$-th partial derivatives of F vanish at P.*

We finish this section with an example where all these notions are illustrated.

Example 7.1.4. Let $\mathcal{C}$ be the projective plane curve over $\mathbb{C}$ defined by the homogeneous polynomial:

$$\begin{aligned} F(x,y,z) \;\; = \;\; & -4\,y^4\,z^3\,x^2 + 2\,y^7\,x^2 + y^9 + 3\,y^7\,z^2 - 9\,y^6\,x^2\,z - 2\,x^8\,z + 2\,x^8\,y + 3\,x^4\,y^5 \\ & -y^6\,z^3 + 4\,x^6\,y^3 - 7\,x^6\,y^2\,z + 5\,x^6\,y\,z^2 + 10\,x^2\,y^5\,z^2 - 11\,x^4\,y^4\,z \\ & +9\,x^4\,y^3\,z^2 - 4\,x^4\,y^2\,z^3 + y^3\,z^4\,x^2 - 3\,y^8\,z \end{aligned}$$

The degree of the curve is 9. In Figure 7.1. the real part of $\mathcal{C}_{\star,z}$ is plotted.

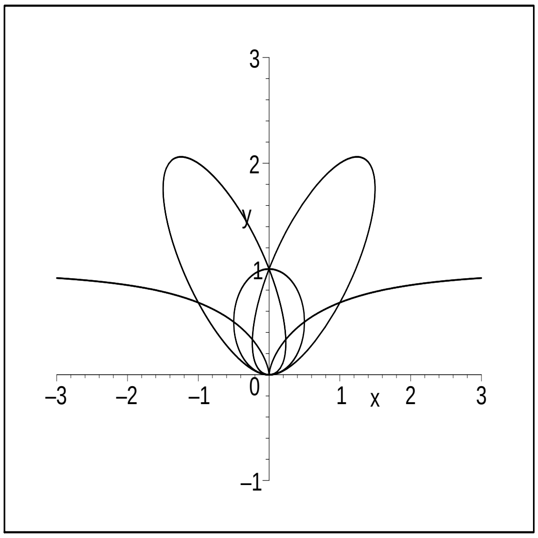

Figure 7.1: Real part of $\mathcal{C}_{\star,z}$

[1] Euler's formula for homogeneous polynomials $F(x_1, x_2, x_3)$: $\sum_{i=1}^{3} x_i \cdot \frac{\partial F}{\partial x_i} = n \cdot F$, where $n = \deg(F)$

First, we compute the finitely many points at infinity of the curve. We observe that

$$F(x, y, 0) = y(2x^4 + y^4)(y^2 + x^2)^2$$

does not vanish identically, so the line $z = 0$ is not a component of $\mathcal{C}$. In fact, the points at infinity are $(1 : 0 : 0), (1 : \alpha : 0)$ where $\alpha^4 + 2 = 0$, and the cyclic points $(1 : \pm i : 0)$. Hence, the line at infinity intersects $\mathcal{C}$ at 7 points (compare to Bézout's Theorem, in this chapter).

Now, we proceed to determine and analyze the singularities. We apply Theorem 7.1.3. Solving the system

$$\{\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0\}$$

we find that the singular points of $\mathcal{C}$ are

$$(1 : \pm i : 0), (0 : 0 : 1), (\pm\frac{1}{3\sqrt{3}} : \frac{1}{3} : 1), (\pm\frac{1}{2}, \frac{1}{2} : 1), (0 : 1 : 1), \text{ and } (\pm 1 : \alpha : 1)$$

where $\alpha^3 + \alpha - 1 = 0$. So $\mathcal{C}$ has 11 singular points.

Now we compute the multiplicities and the tangents to $\mathcal{C}$ at each singular point. For this purpose, we determine the first non-vanishing term in the corresponding Taylor expansion. The result of this computation is shown in Table 7.1. In this table we denote the singularities of $\mathcal{C}$ as follows:

$$P_1^{\pm} := (1 : \pm i : 0),\ P_2 := (0 : 0 : 1),\ P_3^{\pm} := (\pm\frac{1}{3\sqrt{3}} : \frac{1}{3} : 1),$$

$$P_4^{\pm} := (\pm\frac{1}{2}, \frac{1}{2} : 1),\ P_5 := (0 : 1 : 1),\ P_{\alpha}^{\pm} := (\pm 1 : \alpha : 1)$$

All these singular points are ordinary, except the affine origin $(0 : 0 : 1)$. Factoring F over $\mathbb{C}$ we get

$$F(x, y, z) = (x^2 + y^2 - yz)(y^3 + yz^2 - zx^2)(y^4 - 2y^3z + y^2z^2 - 3yzx^2 + 2x^4).$$

Therefore, $\mathcal{C}$ decomposes into a union of a conic, a cubic, and a quartic. Furthermore, $(0 : 0 : 1)$ is a double point on the quartic, a double point on the cubic, and a simple point on the conic. Thus, applying Lemma 7.1.1, the multiplicity of $\mathcal{C}$ at $(0 : 0 : 1)$ is 5. $(0 : 1 : 1)$ is a double point on the quartic and a simple point on the conic. Hence, the multiplicity of $\mathcal{C}$ at $(0 : 1 : 1)$ is 3. $(\pm\frac{1}{2} : \frac{1}{2} : 1)$ are simple points on the conic and the cubic. $(\pm\frac{1}{3\sqrt{2}} : \frac{1}{3} : 1)$ are simple points on the quartic and the cubic. Similarly, the points $(\pm 1 : \alpha : 1)$ are also simple points on the quartic and the cubic (two of them are real, and four of them complex). Finally, the cyclic points are simple points on the cubic and the conic. Hence, all these singular points are double points on $\mathcal{C}$.

point	tangents	multiplicity
P_1^+	$(2y-z-2ix)(2y+z-2ix)$	2
P_1^-	$(2y+z+2ix)(2y-z+2ix)$	2
P_2	y^3x^2	5
P_3^+	$(8x+\sqrt{2}z-7\sqrt{2}y)(28x-5\sqrt{2}z+\sqrt{2}y)$	2
P_3^-	$(8x-\sqrt{2}z+7\sqrt{2}y)(28x+5\sqrt{2}z-\sqrt{2}y)$	2
P_4^+	$(2x-z)(2x-4y+z)$	2
P_4^-	$(2x+z)(2x+4y-z)$	2
P_5	$(y-z)(3x^2+2yz-y^2-z^2)$	3
P_α^+	$((1+\frac{1}{2}\alpha+\frac{1}{2}\alpha^2)z+x+(-\frac{5}{2}-\frac{1}{2}\alpha-2\alpha^2)y)$ $((-\frac{35}{146}-\frac{23}{73}\alpha+\frac{1}{73}\alpha^2)z+x+(-\frac{65}{146}-\frac{1}{73}\alpha-\frac{111}{146}\alpha^2)y)$	2
P_α^-	$((-1-\frac{1}{2}\alpha-\frac{1}{2}\alpha^2)z+x+(\frac{5}{2}+\frac{1}{2}\alpha+2\alpha^2)y)$ $((\frac{35}{146}+\frac{23}{73}\alpha-\frac{1}{73}\alpha^2)z+x+(\frac{65}{146}+\frac{1}{73}\alpha+\frac{111}{146}\alpha^2)y)$	2

Table 7.1.

□

7.2 Intersection of Curves

In this section we analyze the intersection of two plane curves. Since curves are algebraic sets, and since the intersection of two algebraic sets is again an algebraic set, we see that the intersection of two plane curves is an algebraic set in the plane consisting of 0-dimensional and 1-dimensional components. The ground field K is algebraically closed, so the intersection of two curves is non-empty. In fact, the intersection of two curves contains a 1-dimensional component, i.e. a curve, if and only if the gcd of the corresponding defining polynomials is not constant, or equivalently if both curves have a common component. Hence, in this case the intersection is given by the gcd.

Therefore, the problem of analyzing the intersection of curves is reduced to the case of two curves without common components. There are two questions which we need to answer. First, we want to compute the finitely many intersection points of the two curves. This means solving a zero dimensional system of two bivariate polynomials. Second, we also want to analyze the number of intersection points of the curves when the two curves do not have common components. This counting of intersections points with proper multiplicities is achieved by Bézout's Theorem (in this section). For this purpose, one introduces the notion of multiplicity of intersection.

We start with the problem of computing the intersection points. Let $\mathcal{C}$ and $\mathcal{D}$ be two projective plane curves defined by $F(x, y, z)$ and $G(x, y, z)$, respectively, such that $\gcd(F, G) = 1$. We want to compute the finitely many points in $V(F, G)$. Since we are working in the plane, the solutions of this system of algebraic equations can be determined by resultants.

First, we observe that if both polynomials F and G are bivariate forms in the same variables, say $F, G \in K[x, y]$ (similarly if $F, G \in K[x, z]$ or $F, G \in K[y, z]$), then each curve is a finite union of lines passing through $(0 : 0 : 1)$. Hence, since the curves do not have common components, one has that $\mathcal{C}$ and $\mathcal{D}$ intersect only in $(0 : 0 : 1)$.

So now let us assume that at least one of the defining polynomials is not a bivariate form in x and y, say $F \notin K[x, y]$. Then, we consider the resultant $R(x, y)$ of F and G with respect to z. Since $\mathcal{C}$ and $\mathcal{D}$ do not have common components, $R(x, y)$ is not identically zero. Furthermore, since $\deg_z(F) \geq 1$ and G is not constant, the resultant R is a non-constant bivariate homogeneous polynomial. Hence it factors as

$$R(x, y) = \prod_{i=1}^{r} (b_i x - a_i y)^{n_i}$$

for some $a_i, b_i \in K$, and $r_i \in \mathbb{N}$. For every $(a, b) \in K^2$ such that $R(a, b) = 0$ there exists $c \in K$ such that $F(a, b, c) = G(a, b, c) = 0$, and conversely. Therefore, the solutions of R provide the intersection points. Since $(0 : 0 : 0)$ is not a point in $\mathbb{P}^2(K)$, but it might be the formal result of extending the solution $(0, 0)$ of R, we check whether $(0 : 0 : 1)$ is an intersection point. The remaining intersection points are given by $(a_i : b_i : c_{i,j})$, where a_i, b_i are not simultaneously zero, and $c_{i,j}$ are the roots in K of $\gcd(F(a_i, b_i, z), G(a_i, b_i, z))$.

Example 7.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be the projective plane curves defined by $F(x,y,z) = x^2 + y^2 - yz$ and $G(x,y,z) = y^3 + x^2y - x^2z$, respectively. Observe that these curves are the conic and cubic that appear in Example 7.1.4. Since $\gcd(F,G) = 1$, $\mathcal{C}$ and $\mathcal{D}$ do not have common components. Obviously $(0:0:1)$ is an intersection point. For determining the other intersection points, we compute

$$R(x,y) = \text{res}_z(F,G) = x^4 - y^4 = (x-y)(x+y)(x^2+y^2).$$

For extending the solutions of R to the third coordinate we compute $\gcd(F(1,1,z), G(1,1,z)) = z - 2$, $\gcd(F(1,-1,z), G(1,-1,z)) = z + 2$, $\gcd(F(1,\pm i,z), G(1,\pm i,z)) = z$. So the intersection points of $\mathcal{C}$ and $\mathcal{D}$ are

$$(0:0:1), (\frac{1}{2} : \frac{1}{2} : 1), (\frac{1}{2} : -\frac{1}{2} : 1), (1 : i : 0), (1 : -i : 0)$$

□

We now proceed to the problem of analyzing the number of intersections of two projective curves without common components. For this purpose, we first study upper bounds, and then we see how these upper bounds can always be reached by a suitable definition of the notion of intersection multiplicity.

Theorem 7.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two projective plane curves without common components and degrees n and m, respectively. Then the number of intersection points of $\mathcal{C}$ and $\mathcal{D}$ is at most $n \cdot m$.*

Proof. First observe that the number of intersection points is invariant under linear changes of coordinates. Let k be the number of intersection points of $\mathcal{C}$ and $\mathcal{D}$. W.l.o.g we assume that (perhaps after a suitable linear change of coordinates) $P = (0:0:1)$ is not a point on $\mathcal{C}$ or $\mathcal{D}$ and also not on a line connecting any pair of intersection points of $\mathcal{C}$ and $\mathcal{D}$. Let $F(x,y,z)$ and $G(x,y,z)$ be the defining polynomials of $\mathcal{C}$ and $\mathcal{D}$, respectively. Note that since $(0:0:1)$ is not on the curves, we have

$$\begin{aligned} F(x,y,z) &= A_0 z^n + A_1 z^{n-1} + \cdots + A_n, \\ G(x,y,z) &= B_0 z^m + B_1 z^{m-1} + \cdots + B_m, \end{aligned}$$

where A_0, B_0 are non-zero constants and A_i, B_j are homogeneous polynomials of degree i, j, respectively. Let $R(x,y)$ be the resultant of F and G with respect to z. Since the curves do not have common components, R is a non-zero homogeneous polynomial in $K[x,y]$ of degree $n \cdot m$ (compare [Wal50], Theorem I.10.9 on p.30). Furthermore, as we have already seen, each linear factor of R generates a set of intersection points. Thus, if we can prove that each linear factor generates exactly one intersection point, we have shown that $k \leq \deg(R) = n \cdot m$. Let us assume that $(bx - ay)$ is a linear factor generating at least two different intersection points P_1 and P_2. But this implies that the line $bx - ay$ passes through P_1, P_2 and $(0:0:1)$, which we have excluded. □

Given two projective curves of degrees n and m, respectively, and without common components, one gets exactly $n \cdot m$ intersection points if the intersection points are counted *properly*. This leads to the definition of multiplicity of intersection. First we present the notion for curves such that the point $(0 : 0 : 1)$ is not on any of the two curves, nor on any line connecting two of their intersection points. Afterwards, we observe that the concept can be extended to the general case by means of linear changes of coordinates.

Definition 7.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be projective plane curves, without common components, such that $(0 : 0 : 1)$ is not on $\mathcal{C}$ or $\mathcal{D}$ and such that it is not on any line connecting two intersection points of $\mathcal{C}$ and $\mathcal{D}$. Let $P \in \mathcal{C} \cap \mathcal{D}$, and let $F(x, y, z)$ and $G(x, y, z)$ be the defining polynomials of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then, the *multiplicity of intersection of* $\mathcal{C}$ *and* $\mathcal{D}$ *at* P (we denote it by $\text{mult}_P(\mathcal{C}, \mathcal{D})$) is defined as the multiplicity of the factor $bx - ay$ in the resultant of F and G with respect to z.

If $P \notin \mathcal{C} \cap \mathcal{D}$ then we define the multiplicity of intersection at P as 0. □

We observe that the condition on $(0 : 0 : 1)$, required in Definition 7.2.1., can be avoided by means of linear changes of coordinates. Moreover, the extension of the definition to the general case does not depend on the particular linear change of coordinates, as remarked in [Wal50], Sect. IV.5. Indeed, since $\mathcal{C}$ and $\mathcal{D}$ do not have common components the number of intersection points is finite. Therefore, there always exist linear changes of coordinates satisfying the required conditions in the definition. Furthermore, as we have remarked in the last part of the proof of Theorem 7.2.1., if T is any linear change of coordinates satisfying the conditions of the definition, each factor of the corresponding resultant is generated by exactly one intersection point. Therefore, the multiplicity of the factors in the resultant is preserved by this type of linear changes of coordinates.

Theorem 7.2.2. (Bézout's Theorem) *Let $\mathcal{C}$ and $\mathcal{D}$ be two projective plane curves without common components and degrees n and m, respectively. Then*

$$n \cdot m = \sum_{P \in \mathcal{C} \cap \mathcal{D}} mult_P(\mathcal{C}, \mathcal{D}).$$

Proof. Let us assume w.l.o.g. that $\mathcal{C}$ and $\mathcal{D}$ are such that $(0 : 0 : 1)$ is not on $\mathcal{C}$ or $\mathcal{D}$ nor on any line connecting two of their intersection points. Let $F(x, y, z)$ and $G(x, y, z)$ be the defining polynomials of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then the resultant $R(x, y)$ of F and G with respect to z is a non-constant homogeneous polynomial of degree $n \cdot m$ (compare the proof of Theorem 7.2.1). Furthermore, if $\{(a_i : b_i : c_i)\}_{i=1,\ldots,r}$ are the intersection points of $\mathcal{C}$ and $\mathcal{D}$ (note that $(0 : 0 : 1)$ is not one of them) then we get

$$R(x, y) = \prod_{i=1}^{r} (b_i x - a_i y)^{n_i},$$

where n_i is, by definition, the multiplicity of intersection of $\mathcal{C}$ and $\mathcal{D}$ at $(a_i : b_i : c_i)$. Therefore, the formula holds. $\square$

Example 7.2.2. We consider the two cubics $\mathcal{C}$ and $\mathcal{D}$ of Figure 7.2 defined by the polynomials

$$F(x,y,z) = \frac{516}{85}z^3 - \frac{352}{85}yz^2 - \frac{7}{17}y^2z + \frac{41}{85}y^3 + \frac{172}{85}xz^2 - \frac{88}{85}xyz + \frac{1}{85}y^2x - 3x^2z + x^2y - x^3,$$

$$G(x,y,z) = -132z^3 + 128yz^2 - 29y^2z - y^3 + 28xz^2 - 76xyz + 31y^2x + 75x^2z - 41x^2y + 17x^3,$$

respectively.

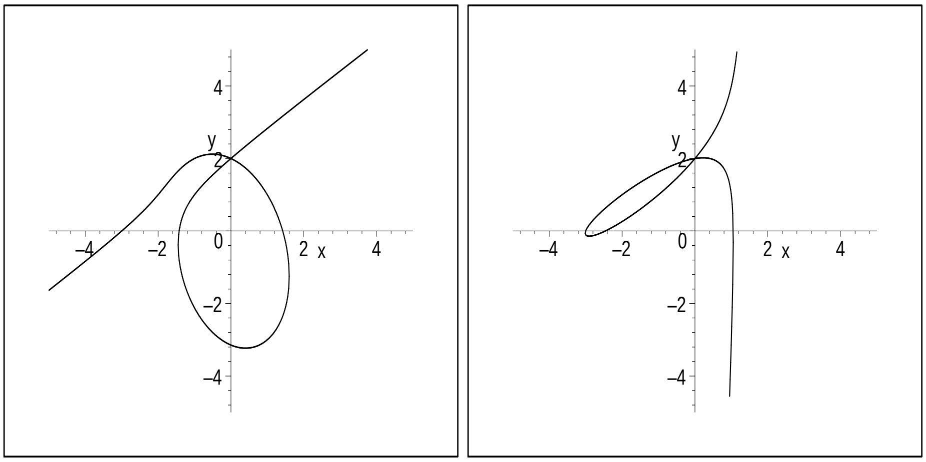

Figure 7.2: Real part of $\mathcal{C}_{\star,z}$ (Left), Real part of $\mathcal{D}_{\star,z}$ (Right)

Let us determine the intersection points of these two cubics and their corresponding multiplicities of intersection. For this purpose, we first compute the resultant

$$R(x,y) = \mathrm{res}_z(F,G) = -\frac{5474304}{25}x^4\, y\,(3x+y)\,(x+2y)\,(x+y)\,(x-y).$$

For each factor $(ax - by)$ of the resultant $R(x,y)$ we obtain the polynomial $D(z) = \gcd(F(a,b,z), G(a,b,z))$ in order to find the intersection points generated by this factor.

The next table shows the results of this computation (compare to Figure 7.3.):

factor	$D(z)$	intersection point	multipl. of intersection
x^4	$(2z-1)^2$	$P_1 = (0:2:1)$	4
y	$z+1/3$	$P_2 = (-3:0:1)$	1
$3x+y$	$z-1$	$P_3 = (1:-3:1)$	1
$x+2y$	$z+1$	$P_4 = (-2:1:1)$	1
$x+y$	$z+1$	$P_5 = (-1:1:1)$	1
$x-y$	$z-1$	$P_6 = (1:1:1)$	1

Table 7.2.

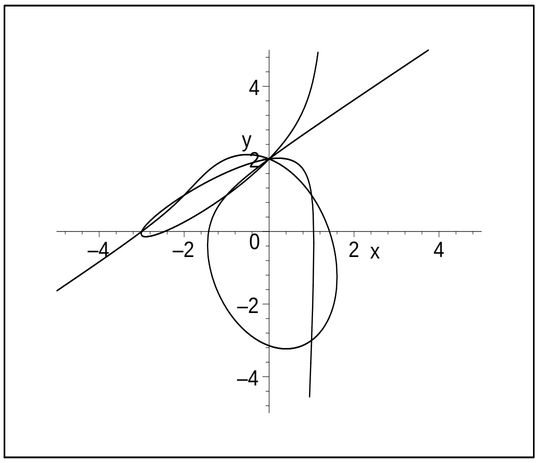

Figure 7.3: Joint picture of the real parts of $\mathcal{C}_{\star,z}$ and $\mathcal{D}_{\star,z}$

Furthermore, since $(0:0:1)$ is not on the cubics nor on any line connecting their intersection points, the multiplicity of intersection is 4 for P_1, and 1 for the other points. It is also interesting to observe that P_1 is a double point on each cubic (compare Theorem 7.2.3. (5)). $\square$

Some authors introduce the notion of multiplicity of intersection axiomatically (see, for instance, [Ful69], Sect. 3.3). The following theorem shows that these axioms are satified for our definition.

Theorem 7.2.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two projective plane curves, without common components, defined by the polynomials F and G, respectively, and let $P \in \mathbb{P}^2(K)$. Then the following holds:*

(1) $\mathrm{mult}_P(\mathcal{C}, \mathcal{D}) \in \mathbb{N}$.

(2) $\mathrm{mult}_P(\mathcal{C}, \mathcal{D}) = 0$ *if and only if* $P \notin \mathcal{C} \cap \mathcal{D}$. *Furthermore,* $\mathrm{mult}_P(\mathcal{C}, \mathcal{D})$ *depends only on those components of $\mathcal{C}$ and $\mathcal{D}$ containing P.*

(3) If T is a linear change of coordinates, and $\mathcal{C}'$, $\mathcal{D}'$, P' are the imagines of $\mathcal{C}$, $\mathcal{D}$, and P under T, respectively, then $\mathrm{mult}_P(\mathcal{C},\mathcal{D}) = \mathrm{mult}_{P'}(\mathcal{C}',\mathcal{D}')$.

(4) $\mathrm{mult}_P(\mathcal{C},\mathcal{D}) = \mathrm{mult}_P(\mathcal{D},\mathcal{C})$.

(5) $\mathrm{mult}_P(\mathcal{C},\mathcal{D}) \geq \mathrm{mult}_P(\mathcal{C}) \cdot \mathrm{mult}_P(\mathcal{D})$. *Furthermore, equality holds if and only $\mathcal{C}$ and $\mathcal{D}$ intersect transversally at P (i.e. if the curves have no common tangents at P).*

(6) Let $\mathcal{C}_1,\dots,\mathcal{C}_r$ and $\mathcal{D}_1,\dots,\mathcal{D}_s$ be the irreducible components of $\mathcal{C}$ and $\mathcal{D}$ respectively. Then

$$\mathrm{mult}_P(\mathcal{C},\mathcal{D}) = \sum_{i=1}^{r}\sum_{j=1}^{s} \mathrm{mult}_P(\mathcal{C}_i,\mathcal{D}_j).$$

(7) $\mathrm{mult}_P(\mathcal{C},\mathcal{D}) = \mathrm{mult}_P(\mathcal{C},\mathcal{D}_H)$, *where $\mathcal{D}_H$ is the curve defined by $G+HF$ for an arbitrary form $H \in K[x,y,z]$ (i.e. the intersection multiplicity does not depend on the particular representative G in the coordinate ring of $\mathcal{C}$).*

Proof. We have already remarked above that the intersection multiplicity is independent of a particular linear change of coordinates. Statements (1),(2),and (4) can be easily deduced from the definition of multiplicity of intersection, and we leave them to the reader.
A proof of (5) can be found for instance in [Wal50], Chap. IV.5, Theorem 5.10.
(6) Let us assume without loss of generality that $\mathcal{C}$ and $\mathcal{D}$ satisfy the requirements of Definition 7.2.1. That is, $(0:0:1)$ is not on the curves nor on any line connecting their intersection points. Let $F_i, i = 1,\dots,r$, and $G_j, j = 1\dots,s$, be the defining polynomials of $\mathcal{C}_i$ and $\mathcal{D}_j$, respectively. Then we use the following fact: if $A,B,C \in D[x]$, where D is an integral domain, then $\mathrm{res}_x(A, B\cdot C) = \mathrm{res}_x(A,B)\cdot\mathrm{res}_x(A,C)$ (see, for instance, [BCL83] Theorem 3 p. 178). Hence, (6) follows immediately from

$$\mathrm{res}_z(\prod_{i=1}^{r} F_i, \prod_{j=1}^{s} G_j) = \prod_{i=1}^{r}\mathrm{res}_z(F_i, \prod_{j=1}^{s} G_j) = \prod_{i=1}^{r}\prod_{j=1}^{s}\mathrm{res}_z(F_i, G_j).$$

(7) Let $H \in K[x,y,z]$ be a form. Obviously $P \in \mathcal{C}\cap\mathcal{D}$ if and only if $P \in \mathcal{C}\cap\mathcal{D}_H$. We assume without loss of generality that $\mathcal{C}$ and $\mathcal{D}$ satisfy the conditions of the Definition 7.2.1, and also $\mathcal{C}$ and $\mathcal{D}_H$ satisfy these conditions. Then, $\mathrm{mult}_P(\mathcal{C},\mathcal{D})$ and $\mathrm{mult}_P(\mathcal{C},\mathcal{D}_H)$ are given by the multiplicities of the corresponding factors in $\mathrm{res}_z(F,G)$ and $\mathrm{res}_z(F, G+H\,F)$, respectively. Now, we use the following property of resultants: if $A,B,C \in D[x]$, where D is an integral domain, and a is the leading coefficient of A, then $\mathrm{res}_x(A,B) = a^{\deg_x(B)-\deg_x(A\,C+B)}\,\mathrm{res}_x(A, A\,C+B)$ (see, for instance, [BCL83] Theorem 4, p. 178). (7) follows directly from this fact, since the leading coefficient of F in z is a non-zero constant (note that $(0:0:1)$ is not on $\mathcal{C}$). □

From Theorem 7.2.3 one can extract an alternative algorithm for computing the intersection multiplicity.

Because of (3) we may assume that $P = (0, 0)$.

Since the intersection multiplicity is a local property, we may work in the affine plane. Consider the curves defined by $f(x, y)$ and $g(x, y)$. We check whether P lies on a common component of f and g, i.e. whether $\gcd(f, g)(P) = 0$. If this is the case we set $\mathrm{mult}_P(f \cap g) = \infty$, because of (1). Otherwise we continue to determine $\mathrm{mult}_P(f \cap g) \in \mathbb{N}_0$.

We proceed inductively. The base case $\mathrm{mult}_P(f \cap g) = 0$ is covered by (2). Our induction hypothesis is

$$\mathrm{mult}_P(a \cap b) \text{ can be determined for } \mathrm{mult}_P(a \cap b) < n.$$

So now let $\mathrm{mult}_P(f \cap g) = n > 0$.

Let $f(x, 0), g(x, 0) \in K[x]$ be of degrees r, s, respectively. Because of (4) we can assume that $r \leq s$.

<u>Case 1, $r = -1$:</u> I.e. $f(x, 0) = 0$. In this case $f = y \cdot h$ for some $h \in K[x, y]$, and because of (6)

$$\mathrm{mult}_P(f \cap g) = \mathrm{mult}_P(y \cap g) + \mathrm{mult}_P(h \cap g).$$

Let m be such that $g(x, 0) = x^m(a_0 + a_1 x + \ldots), a_0 \neq 0$. Note that $g(x, 0) = 0$ is impossible, since otherwise f and g would have the common component y, which we have excluded.

$$\mathrm{mult}_P(y \cap g) =_{(7)} \mathrm{mult}_P(y \cap g(x, 0)) =_{(2)} \mathrm{mult}_P(y \cap x^m) =_{(6)} m \cdot \mathrm{mult}_P(y \cap x) =_{(5)} m.$$

Since $P \in g$, we have $m > 0$, so $\mathrm{mult}_P(h \cap g) < n$. By the induction hypothesis, this implies that $\mathrm{mult}_P(h \cap g)$ can be determined.

<u>Case 2, $r > -1$:</u> We assume that $f(x, 0), g(x, 0)$ are monic. Otherwise we multiply f and g by suitable constants to make them monic. Let

$$h = g - x^{s-r} \cdot f.$$

Because of (7) we have

$$\mathrm{mult}_P(f \cap g) = \mathrm{mult}_P(f \cap h) \quad \text{and} \quad \deg(h(x, 0)) = t < s.$$

Continuing this process (we might have to interchange f and h, if $t < r$) eventually, after finitely many steps, we reach a pair of curves a, b which can be handled by Case 1.

From these considerations we can immediately extract the following algorithm for computing intersection multiplicities.

Algorithm INT_MULT
in: $f, g \in K[x, y]$, $P = (a, b) \in \mathbb{A}^2(K)$;
out: $M = \text{mult}_P(f \cap g)$;

(1) **if** $P \neq (0, 0)$
 then $\{ P := (0, 0);\ f := f(x + a, y + b);\ g := g(x + a, y + b) \}$;

(2) **if** $\gcd(f, g)(P) = 0$ **then** $\{ M := \infty;$ **return** $\}$;

(3) $f_0 := f(x, 0)$; $r := \deg(f_0)$;
 $g_0 := g(x, 0)$; $s := \deg(g_0)$;
 if $r > s$ **then** interchange (f, f_0, r) and (g, g_0, s);
 while $r > -1$ **do**

  $\{ g := \frac{1}{\text{lc}(g_0)} \cdot g - x^{s-r} \cdot \frac{1}{\text{lc}(f_0)} \cdot f$;

  $g_0 := g(x, 0)$; $s := \deg(g_0)$;

  if $r > s$ **then** interchange (f, f_0, r) and (g, g_0, s) $\}$;

(4) $[r = -1]$
 $h := f/y$; $m :=$ exponent of lowest term in g_0;
 $M := m + \text{INT\_MULT}(h, g, P)$ □

Example 7.2.3. We determine the intersection multiplicity at the origin $O = (0, 0)$ of the affine curves $\mathcal{E}, \mathcal{F}$ defined by

$$\mathcal{E}:\ e(x, y) = (x^2 + y^2)^2 + 3x^2y - y^3, \quad \mathcal{F}:\ f(x, y) = (x^2 + y^2)^3 - 4x^2y^2.$$

For ease of notation we don't distinguish between the curves and their defining polynomials.

We replace f by the following curve g:

$$f(x, y) - (x^2 + y^2)e(x, y) = y \cdot ((x^2 + y^2)(y^2 - 3x^2) - 4x^2y) = y \cdot g(x, y).$$

Now we have

$$\text{mult}_O(e \cap f) = \text{mult}_O(e \cap y) + \text{mult}_O(e \cap g).$$

We replace g by h:

$$g + 3e = y \cdot (5x^2 - 3y^2 + 4y^3 + 4x^2y) = y \cdot h(x, y).$$

So

$$\text{mult}_O(e \cap f) = 2 \cdot \text{mult}_O(e \cap y) + \text{mult}_O(e \cap h).$$

$\text{mult}_O(e \cap y) =_{(4),(7)} \text{mult}_O(x^4 \cap y) =_{(6),(5)} 4$,
$\text{mult}_O(e \cap h) =_{(5)} \text{mult}_O(e) \cdot \text{mult}_O(h) = 6$.
Thus, $\text{mult}_O(\mathcal{E}, \mathcal{F}) = \text{mult}_O(e \cap f) = 14$. $\square$

Theorem 7.2.4. *The line l is tangent to the curve f at the point P if and only if $\text{mult}_P(f \cap l) > \text{mult}_P(f)$.*

Proof: By the property (5), l is tangent to f at P if and only if $\text{mult}_P(f \cap l) > \text{mult}_P(f) \cdot \text{mult}_P(l) = \text{mult}_P(f)$. $\square$

Theorem 7.2.5. *If the line l is not a component of the curve f, then*

$$\sum_{P \in \mathbb{A}^2} \text{mult}_P(f \cap l) \ \leq \ \deg(f).$$

Proof: Only points in $f \cap l$ can contribute to the sum. Let l be parametrized as $\{x = a + tb,\ y = c + td\}$. Let

$$g(t) = f(a + tb, c + td) = \alpha \cdot \prod_{i=1}^{r} (t - \lambda_i)^{e_i},$$

for $\alpha, e_i \in K$. $g \neq 0$, since l is not a component of f.
$P \in f \cap l$ if and only if $P = P_i = (a + \lambda_i b, c + \lambda_i d)$ for some $1 \leq i \leq r$.
If all the partial derivatives of f of order n vanish at P_i, then by the chain rule also

$$\frac{\partial^n g}{\partial t^n}(\lambda_i) = \sum_{i=0}^{n} \binom{n}{i} \frac{\partial^n f}{\partial x^i \partial y^{n-1}}(a + \lambda_i b, c + \lambda_i d) \cdot b^i \cdot d^{n-i} = 0.$$

So $\text{mult}_{P_i}(f) \leq e_i$.

Summarizing we get

$$\sum_{P \in \mathbb{A}^2} \text{mult}_P(f \cap l) \ = \ \sum_{i=1}^{r} \text{mult}_{P_i}(f \cap l) \ =_{(5)} \ \sum_{i=1}^{r} \text{mult}_{P_i}(f) \cdot \underbrace{\text{mult}_{P_i}(l)}_{=1} \ \leq \ \sum_{i=1}^{r} e_i \ \leq \ \deg(f).$$

$\square$

7.3 Linear Systems of Curves

Throughout this section we take into account the multiplities of components in algebraic curves, i.e. the equations $x^2y = 0$ and $xy^2 = 0$ determine two different curves. As always, we assume that the underlying field K is algebraically closed.

A projective curve $\mathcal{C}$ of degree n is defined by a polynomial equation of the form

$$\sum_{i+j+k=n} a_{ijk}x^iy^jz^k \ = \ 0.$$

The number of coefficients a_{ijk} in this homogeneous equation is $(n+1)(n+2)/2$ [2]. The curve $\mathcal{C}$ is defined uniquely by these coefficients, and conversely the coefficients are determined uniquely by the curve $\mathcal{C}$, up to a constant factor. Therefore, the coefficients can be regarded as projective coordinates of the curve $\mathcal{C}$, and the set $\mathcal{S}_n$ of projective curves of degree n can be regarded as points in a projective space $\mathbb{P}^N$, where

$$N = (\#\text{ of coeff.}) - 1 = n(n+3)/2.$$

Def. 7.3.1. The curves, which constitute an r–dimensional subspace $\mathbb{P}^r$ of $\mathcal{S}_n = \mathbb{P}^N$, form a *linear system of curves* of dimension r. Such a system $\mathcal{S}$ is determined by $r+1$ independent curves $F_0, F_1, \dots, F_r$ in the system. The defining equation of any curve in $\mathcal{S}$ can be written in the form

$$\sum_{i=0}^{r} \lambda_i F_i(x, y, z) \ = \ 0.$$

Since $\mathbb{P}^r$ is also determined by $N - r$ hyperplanes containing $\mathbb{P}^r$, the linear systems $\mathcal{S}$ can also be written as the intersection of these $N - r$ hyperplanes. The equation of such a hyperplane in $\mathbb{P}^N$ is called a *linear condition* on the curves in $\mathcal{S}_n$, and a curve *satisfies* the linear condition iff it is a point in the hyperplane. □

Since N hyperplanes always have at least one point in common, we can always find a curve satisfying N or fewer linear conditions. In general, in an r–dimensional linear system of curves we can always find a curve satisfying r or fewer (additional) linear conditions.

[2] $f(x,y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$; so the number of terms is $1 + 2 + \cdots + (n+1) = (n+1)(n+2)/2$

Example 7.3.1. We consider the system $\mathcal{S}_2$ of curves over $\mathbb{C}$. In $\mathcal{S}_2$ we choose the two curves

$$F_0(x,y,z) = x^2 + y^2 - z^2 \qquad\qquad F_1(x,y,z) = yz - x^2$$

F_0 and F_1 determine a 1–dimensional linear subsystem of $\mathcal{S}_2$. The defining equation of a general curve in this subsystem is of the form

$$\lambda_0 F_0 + \lambda_1 F_1 = 0,$$

or

$$(\lambda_0 - \lambda_1)x^2 + \lambda_0 y^2 - \lambda_0 z^2 + \lambda_1 yz = 0.$$

We get a linear condition by requiring that the point $(1:2:1)$ should be a point on any curve. This forces $\lambda_1 = -4\lambda_0$. The following curve ($\lambda_0 = 1, \lambda_1 = -4$) satisfies the condition:

$$5x^2 + y^2 - z^2 - 4yz = 0.$$

□

It is very common for linear conditions to arise from requirements such as "all curves in the subsystem should contain the point P as a point of multiplicity at least r, $r \geq 1$." This means that all the partial derivatives of order $< r$ of the defining equation have to vanish at P. There are exactly $r(r+1)/2$ such derivations, and they are homegeneous linear polynomials in the coefficients a_{ijk}. Thus, such a requirement induces $r(r+1)/2$ linear conditions on the curves of the subsystem.

Def. 7.3.2. A point P of multiplicity $\geq r$ on all the curves in a linear system is called a *base point of multiplicity* r of the system. In particular, the linear subsystem of $\mathcal{S}_n$ which has the different points $P_1, \ldots, P_m$ as base points of multiplicity 1, is denoted by $\mathcal{S}_n(P_1, \ldots, P_m)$. □

Def. 7.3.3. A *divisor* is a formal expression of the type

$$\sum_{i=1}^{m} r_i P_i,$$

where $r_i \in \mathbb{Z}$, and the P_i are different points in $\mathbb{P}^2(K)$. If all integers r_i are non-negative we say that the divisor is *effective* or *positive*.

We define the *linear system* of curves of degree d *generated* by the effective divisor $D = r_1 P_1 + \cdots + r_m P_m$ as the set of all curves $\mathcal{C}$ of degree d such that $\mathrm{mult}_{P_i}(\mathcal{C}) \geq r_i$, for $i = 1, \ldots, m$, and we denote it by $\mathcal{H}(d, D)$. □

Example 7.3.2. We compute the linear system of quintics generated by the effective divisor $D = 3P_1 + 2\,P_2 + P_3$, where $P_1 = (0:0:1)$, $P_2 = (0:1:1)$, and $P_3 = (1:1:1)$. For this purpose, we consider the generic form of degree 5:

$$\begin{aligned} H(x,y,z) \;\; = \;\; & a_0\, z^5 + a_1\, yz^4 + a_2\, y^2z^3 + a_3\, y^3z^2 + a_4\, y^4z + a_5\, y^5 + a_6\, xz^4 + a_7\, xyz^3 \\ & + a_8\, xy^2z^2 + a_9\, xy^3z + a_{10}\, xy^4 + a_{11}\, x^2z^3 + a_{12}\, x^2yz^2 + a_{13}\, x^2y^2z \\ & + a_{14}\, x^2y^3 + a_{15}\, x^3z^2 + a_{16}\, x^3yz + a_{17}\, x^3y^2 + a_{18}\, x^4z + a_{19}\, x^4y + a_{20}\, x^5 \end{aligned}$$

The linear conditions that we have to impose are:

$$\frac{\partial^2 H}{\partial x^{2-i-j}\partial y^i \partial z^j}(P_1) = 0, \quad i+j \leq 2, \quad \frac{\partial H}{\partial x}(P_2) = \frac{\partial H}{\partial y}(P_2) = \frac{\partial H}{\partial z}(P_2) = 0, \quad H(P_3) = 0$$

Solving them one gets that the linear system is defined by:

$$\begin{aligned} H(x,y,z) \;\; = \;\; & a_3\, y^3z^2 - 2\,a_3\, y^4z + a_3\, y^5 + (-a_9 - a_{10})\, xy^2z^2 + a_9\, xy^3z + a_{10}\, xy^4 + \\ & (-a_{13} - a_{14} - a_{15} - a_{16} - a_{17} - a_{18} - a_{19} - a_{20})\, x^2yz^2 + a_{13}\, x^2y^2z + \\ & a_{14}\, x^2y^3 + a_{15}\, x^3z^2 + a_{16}\, x^3yz + a_{17}\, x^3y^2 + a_{18}\, x^4z + a_{19}\, x^4y + a_{20}\, x^5 \end{aligned}$$

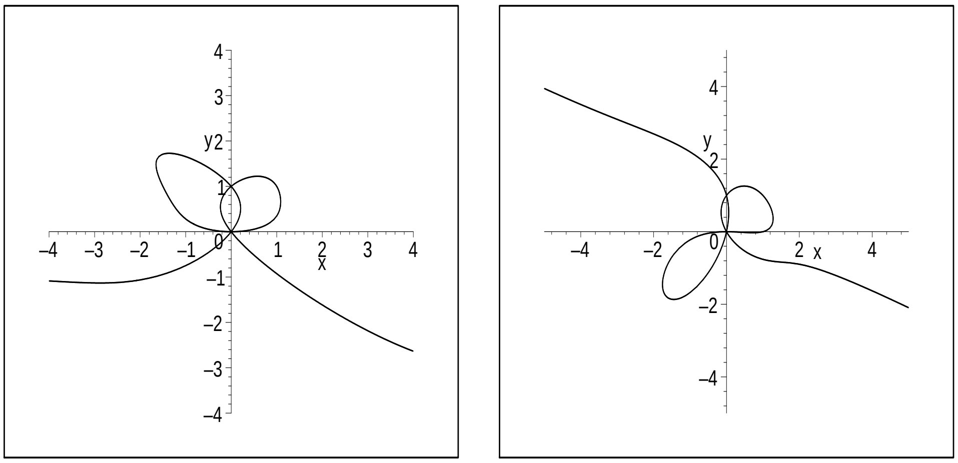

Figure 7.4: real part of $\mathcal{C}_{1_{\star,z}}$ (left), real part of $\mathcal{C}_{2_{\star,z}}$ (right)

Hence, the dimension of the system is 10. Finally, we take two particular curves in the system, $\mathcal{C}_1$ and $\mathcal{C}_2$, defined by the polynomials:

$$\begin{aligned} H_1(x,y,z) \;\; = \;\; & 3\,y^3\,z^2 - 6\,y^4\,z + 3\,y^5 - x\,y^3\,z + x\,y^4 - 5\,x^2\,y\,z^2 + 2\,x^2\,y^2\,z + x^3\,y^2 \\ & + x^4\,z + y\,x^4, \\ H_2(x,y,z) \;\; = \;\; & y^3\,z^2 - 2\,y^4\,z + y^5 - \tfrac{8}{3}\,z^2\,x\,y^2 + 3\,x\,y^3\,z - \tfrac{1}{3}\,x\,y^4 - 8\,x^2\,y\,z^2 + 2\,x^2\,y^2\,z \\ & + y^3\,x^2 + 2\,y\,x^3\,z + x^3\,y^2 - x^4\,z + 2\,y\,x^4 + x^5, \end{aligned}$$

respectively. In the figure the real part of the affine curves $\mathcal{C}_{1_{\star,z}}$ and $\mathcal{C}_{2_{\star,z}}$ are plotted, respectively. □

Theorem 7.3.1. *If two curves $F_1, F_2 \in \mathcal{S}_n$ have n^2 points in common, and exactly mn of these lie on an irreducible curve of degree m, then the other $n(n-m)$ common points lie on a curve of degree $n-m$.*

Proof: Let G be an irreducible curve of degree m containing exactly mn of the n^2 common points of F_1 and F_2. We choose an additional point P on G and determine a curve F in the system

$$\lambda_1 F_1 + \lambda_2 F_2 = 0$$

containing P. So $F = \overline{\lambda}_1 F_1 + \overline{\lambda}_2 F_2$, for some $\overline{\lambda}_1, \overline{\lambda}_2 \in K$. G and F have at least $mn+1$ points in common, so by Theorem 7.2.2 (Bézout's Theorem) they must have a common component, which must be G, since G is irreducible. $F = G \cdot H$ contains the n^2 points, so the curve $H \in \mathcal{S}_{n-m}$ contains the $n(n-m)$ points not on G. □

As a special case we get Pascal's Theorem.

Theorem 7.3.2. (Pascal's Theorem) *Let C be an irreducible conic (curve of degree 2). The opposite sides of a hexagon inscribed in C meet in 3 collinear points.*

Proof: Let $P_1, \ldots, P_6$ be different points on C. Let L_i be the line through P_i and P_{i+1} for $1 \leq i \leq 6$ (for this we set $P_7 = P_1$). The 2 cubic curves $L_1L_3L_5$ and $L_2L_4L_6$ meet in the 6 vertices of the hexagon and in the 3 intersection points of opposite sides. The 6 vertices lie on an irreducible conic, so by Theorem 7.3.1 the remaining 3 points must lie on a line. □

Example 7.3.3. Consider the following particular example of Pascal's Theorem:

Now we want to apply these results for deriving bounds for the number of singularities on plane algebraic curves.

Theorem 7.3.3. *Let F, G be curves of degree m, n, respectively, having no common components and having the multiplicities $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ in their common points $P_1, \ldots, P_k$, respectively. Then*

$$\sum_{i=1}^{k} r_i s_i \ \leq \ mn.$$

Proof: The statement follows from Theorem 7.2.2 (Bézout's Theorem) and proposition (5) in Theorem 7.2.3. □

Theorem 7.3.4. *Let the curve F of degree n have no multiple components and let $P_1, \ldots, P_m$ be the singularities of F with multiplicities $r_1, \ldots, r_m$, respectively. Then*

$$n(n-1) \ \geq \ \sum_{i=1}^{m} r_i(r_i - 1).$$

Proof: Choose the coordinate system such that the point $(0:0:1)$ is not contained in F (i.e. the term z^n occurs with non-zero coefficient in F). Then

$$F_z(x, y, z) := \frac{\partial F}{\partial z} \neq 0.$$

Moreover, F does not have a factor independent of z. Since F is squarefree, F can have no factor in common with F_z.

The curve F_z has every point P_i as a point of multiplicity at least $r_i - 1$, since every j–th derivative of F_z is a $(j+1)$-st derivative of F. Thus, by application of Theorem 7.3.3 we get the result. □

This bound can be sharpened even more for irreducible curves.

Theorem 7.3.5. *Let the irreducible curve F of degree n have the singularities $P_1, \ldots, P_m$ of multiplicities $r_1, \ldots, r_m$, respectively. Then*

$$(n-1)(n-2) \ \geq \ \sum_{i=1}^{m} r_i(r_i - 1).$$

Proof: From Theorem 7.3.4 we get

$$\frac{\sum r_i(r_i-1)}{2} \ \leq \ \frac{n(n-1)}{2} \ \leq \ \frac{(n-1)(n+2)}{2}.$$

(Note: one can always determine a curve of degree $n-1$ satisfying $(n-1)(n+2)/2$ linear conditions. By $r(r-1)/2$ linear conditions we can force a point P to be an $(r-1)$–fold point on a curve.)

So, there is a curve G of degree $n-1$ having P_i as $(r_i - 1)$–fold point, $1 \leq i \leq m$, and additionally passing through

$$\frac{(n-1)(n+2)}{2} - \frac{\sum r_i(r_i-1)}{2}$$

simple points of F. Since F is irreducible and $\deg(G) < \deg(F)$, the curves F and G can have no common component. Therefore, by Theorem 7.3.3,

$$n(n-1) \; \geq \; \sum r_i(r_i-1) \; + \; \frac{(n-1)(n+2)}{2} \; - \; \frac{\sum r_i(r_i-1)}{2},$$

and consequently

$$(n-1)(n-2) \; \geq \; \sum r_i(r_i-1). \qquad \square$$

Indeed, the bound in Theorem 7.3.5 is sharp, i.e. it is actually achieved by a certain class of irreducible curves. These are the curves of genus 0.

Def. 7.3.4. Let F be an irreducible curve of degree n, having only ordinary singularities of multiplicities $r_1, \ldots, r_m$. The *genus* of F, genus(F), is defined as

$$\text{genus}(F) = \frac{1}{2}[(n-1)(n-2) \; - \; \sum_{i=1}^{m} r_i(r_i-1)]. \qquad \square$$

Example 7.3.4. Every irreducible conic has genus 0. An irreducible cubic has genus 0 if and only if it has a double point. $\square$

For curves with non-ordinary singularities the formula

$$\frac{1}{2}[(n-1)(n-2) \; - \; \sum_{i=1}^{m} r_i(r_i-1)]$$

is just an upper bound for the genus. Compare the tacnode curve of Example 1.3, which is a curve of genus 0. We will see this in the chapter on parametrization.