

Introduction to Unification Theory

Speeding Up

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Overview

Recursive Descent Algorithm is Expensive

Improvement 1: Linear Space, Exponential Time

Improvement 2. Linear Space, Quadratic Time

Improvement 3. Almost Linear Algorithm



Outline

Recursive Descent Algorithm is Expensive

Improvement 1: Linear Space, Exponential Time

Improvement 2. Linear Space, Quadratic Time

Improvement 3. Almost Linear Algorithm



Example from the Previous Lecture

Example

$$s = h(x_1, x_2, \dots, x_n, f(y_0, y_0), f(y_1, y_1), \dots, f(y_{n-1}, y_{n-1}), y_n)$$

$$t = h(f(x_0, x_0), f(x_1, x_1), \dots, f(x_{n-1}, x_{n-1}), y_1, y_2, \dots, y_n, x_n)$$

Unifying s and t will create an mgu where each x_i and each y_i is bound to a term with $2^{i+1} - 1$ symbols:

$$\{x_1 \mapsto f(x_0, x_0), x_2 \mapsto f(f(x_0, x_0), f(x_0, x_0)), \dots, \\ y_0 \mapsto x_0, y_1 \mapsto f(x_0, x_0), y_2 \mapsto f(f(x_0, x_0), f(x_0, x_0)), \dots\}$$

- ▶ Problem: Duplicate occurrences of the same variables cause the explosion in the size of terms.
- ▶ Fix: Represent terms as graphs which share subterms.



Term Dags

Term Dag

A term dag is a directed acyclic graph such that

- ▶ its nodes are labeled with function symbols or variables,
- ▶ its outgoing edges from any node are ordered,
- ▶ outdegree of any node labeled with a symbol f is equal to the arity of f (nodes labeled with variables have outdegree 0).

Term Dags

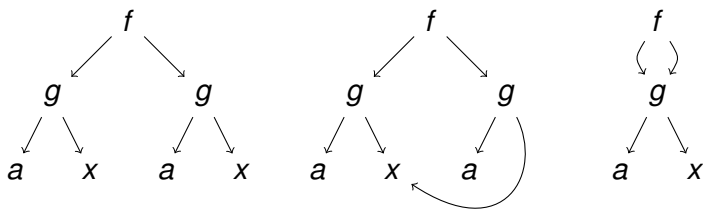
- ▶ Convention: Nodes and terms the term dags represent will not be distinguished.
- ▶ Example: “node” $f(a, x)$ is a node labeled with f and having two arcs to a and to x .

Term Dags

The only difference between various dags representing the same term is the amount of structure sharing between subterms.

Example

Three representations of the term $f(g(a, x), g(a, x))$:



Term Dags

- ▶ It is possible to build a dag for a given term in $O(n)$ where n is the number of symbols in the term.
- ▶ Assumption for the algorithm we plan to consider:
 - ▶ The input is a term dag representing the two terms to be unified, with unique, shared occurrences of all variables.



Term Dags

Representing substitutions involving only subterms of a term dag:

- ▶ Directly by a relation on the nodes of the dag, either
 - ▶ stored explicitly as a list of pairs, or
 - ▶ by storing a link (“substitution arcs”) in the graph itself, and maintaining a list of variables (nodes) bound by the substitution.



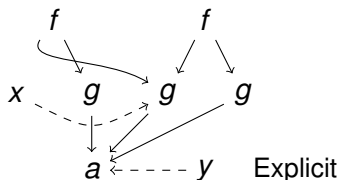
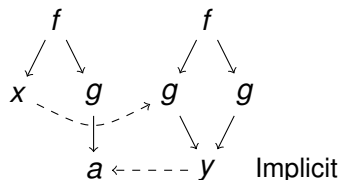
Term Dags

Substitution application. Two alternatives:

1. Implicit: Identifies two nodes connected with a substitution arc, without actually moving any of the subterm links.
2. Explicit: Expresses the substitution by moving any arc (subterm or substitution) pointing to a variable to point to a binding.

Example

A term dag for $f(x, g(a))$ and $f(g(y), g(y))$, and two applications of the substitution $\{x \mapsto g(a), y \mapsto a\}$ on it.



Term Dags

- ▶ With implicit application, the binding for a variable can be determined by traversing the graph depth first, left to right.
- ▶ Explicit application represents a substitution in a direct way.

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Recursive Descent Algorithm (RDA) on Term Dags

Assumptions:

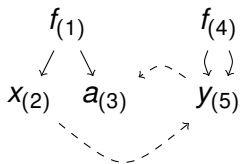
- ▶ Dags consist of nodes.
- ▶ Any node in a given dag defines a unique subdag (consisting of the nodes which can be reached from this node), and thus a unique subterm.
- ▶ Two different types of nodes: variable nodes and function nodes.
- ▶ Information at function nodes:
 - ▶ The name of the function symbol.
 - ▶ The arity n of this symbol.
 - ▶ The list (of length n) of successor nodes (corresponds to the argument list of the function)
- ▶ Both function and variable nodes may be equipped with one additional pointer (displayed as a dashed arrow in diagrams) to another node.



Auxiliary procedures for the RDA on Term Dags

- ▶ `Find`:
Takes a node of a dag as input, and follows the additional pointers until it reaches a node without such a pointer. This node is the output of `Find`.

Example

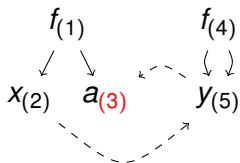


Auxiliary procedures for the RDA on Term Dags

- ▶ `Find`:
Takes a node of a dag as input, and follows the additional pointers until it reaches a node without such a pointer. This node is the output of `Find`.

Example

- ▶ `Find(3)=(3)`

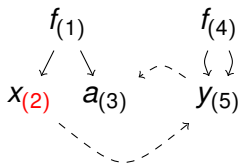


Auxiliary procedures for the RDA on Term Dags

- ▶ `Find`:
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Example

- ▶ `Find(3)=(3)`
- ▶ `Find(2)=`

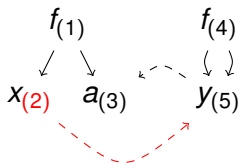


Auxiliary procedures for the RDA on Term Dags

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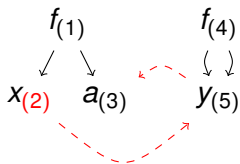


Auxiliary procedures for the RDA on Term Dags

- ▶ `Find`:
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- ▶ `Find(3)=(3)`
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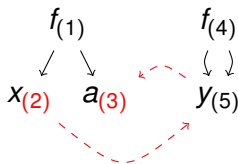


Auxiliary procedures for the RDA on Term Dags

- ▶ `Find`:
Takes a node of a dag as input, and follows the additional pointers until it reaches a node without such a pointer. This node is the output of `Find`.

Example

- ▶ `Find(3)=(3)`
- ▶ `Find(2)= (3)`



Auxiliary procedures for the RDA on Term Dags

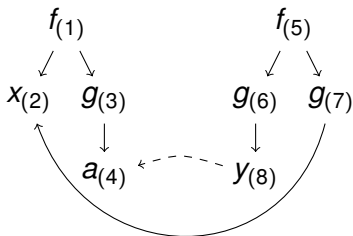
- ▶ `Union`:
Takes as input a pair of nodes u, v that do not have additional pointers and creates such a pointer from u to v .

Auxiliary procedures for the RDA on Term Dags

► `Occur`:

Takes as input a variable node u and another node v (both without additional pointers) and performs the occur check, i.e. it tests whether the variable is contained in the term corresponding to v . The test is performed on the virtual term expressed by the additional pointer structure, i.e. one applies `Find` to all nodes that are reached during the test.

Example



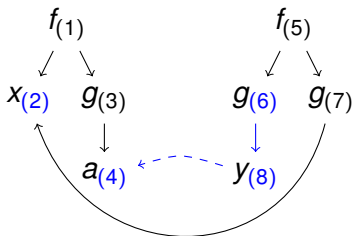
Auxiliary procedures for the RDA on Term Dags

► Occur:

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Example

► $\text{Occur}(2,6)=\text{False}$



Auxiliary procedures for the RDA on Term Dags

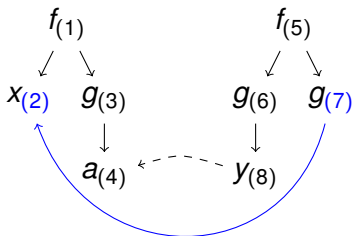
- ▶ `Occur`:

Takes as input a variable node u and another node v (both without additional pointers) and performs the occur check, i.e. it tests whether the variable is contained in the term corresponding to v . The test is performed on the virtual term expressed by the additional pointer structure, i.e. one applies `Find` to all nodes that are reached during the test.

Example

- ▶ `Occur(2,6)=False`

- ▶ `Occur(2,7)=True`



RDA on Term Dags

Input: A pair of nodes k_1 and k_2 in a dag

Output: *True* if the terms corresponding to k_1 and k_2 are unifiable. *False* Otherwise.

Side Effect: A pointer structure which allows to read off an mgu and the unified term.

`Unify1 (k_1, k_2)`

`if $k_1 = k_2$ then return True;` `/* Trivial */`

`else`

`if function-node(k_2) then`

`$u := k_1; v := k_2$`

`else`

`$u := k_2; v := k_1$;`

`/* Orient */`

`end`

Procedure `Unify1`. Recursive descent algorithm on term dags.
(Continues on the next slide)



Recursive Descent Algorithm on Term Dags

```
if variable-node( $u$ ) then  
  if Occurs ( $u, v$ ) ;                               /* Occur-check */  
  then  
    return False  
  else  
    Union( $u, v$ ) ;                                   /* Variable elimination */  
    return True  
end
```

Procedure *Unify1*. Recursive descent algorithm on term dags.
Continued.

(Continues on the next slide)



Recursive Descent Algorithm on Term Dags

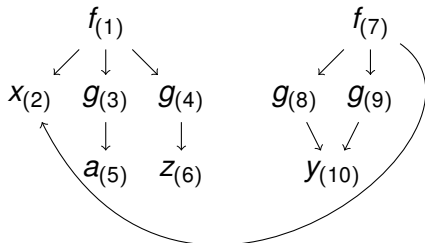
```
else if function-symbol(u)  $\neq$  function-symbol(v)  
then  
    return False;                                /* Symbol clash */  
else  
    n := arity(function-symbol(u));  
    (u1, ..., un) := succ-list(u);  
    (v1, ..., vn) := succ-list(v);  
    i := 0; bool := True;  
  
    while i ≤ n and bool do  
        i := i + 1; bool := Unify1(Find(ui), Find(vi));  
        /* Decomposition */  
    end  
    return bool
```

Procedure *Unify1*. Recursive descent algorithm on term dags.
Finished.



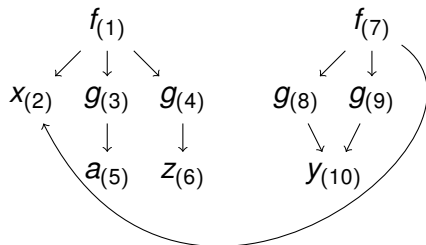
RDA on Term Dags. Example 1

- ▶ Unify $f(x, g(a), g(z))$ and $f(g(y), g(y), x)$.
- ▶ First, create dags.
- ▶ Numbers indicate nodes.



RDA on Term Dags. Example 1

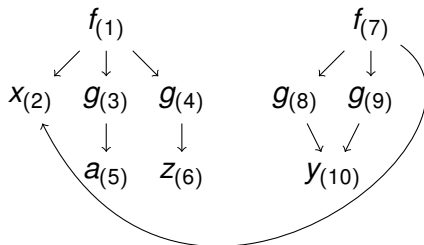
Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

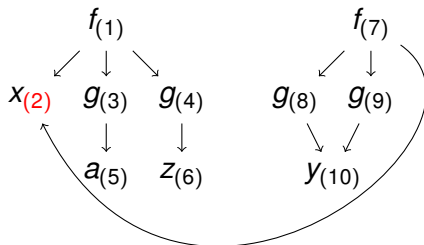


RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$



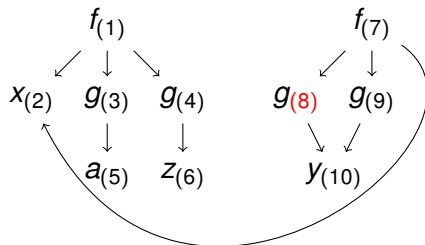
RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$



RDA on Term Dags. Example 1

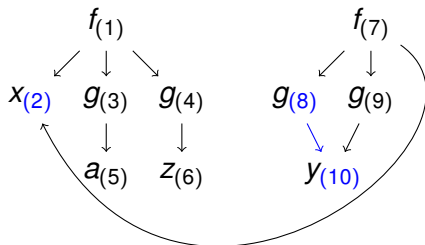
Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

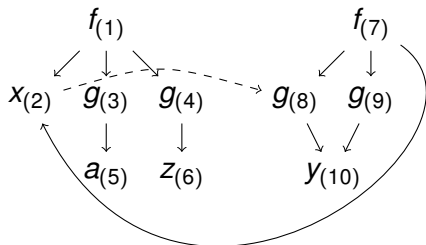
$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

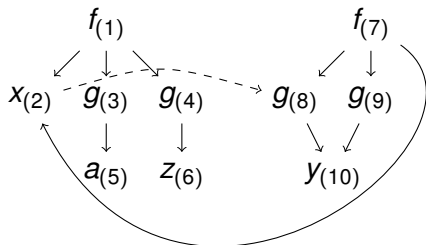
$\text{Union}(2, 8)$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

```
Unify1(Find(2), Find(8))  
  Find(2) = (2)  
  Find(8) = (8)  
  Occur(2, 8) = False  
  Union(2, 8)  
Unify1(Find(3), Find(9))
```



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

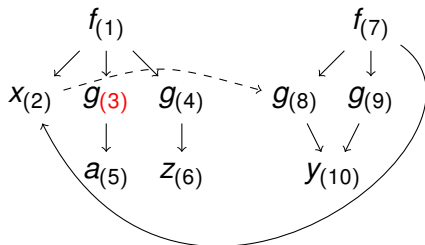
$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

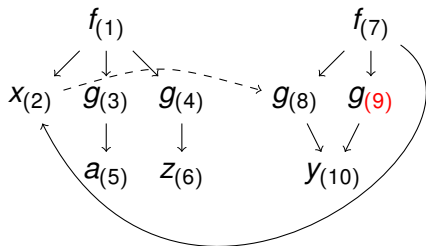
$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

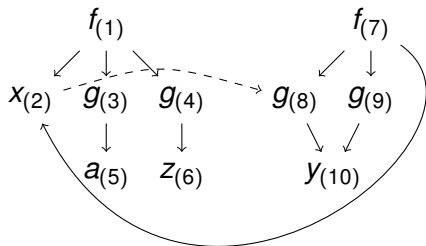
$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$

$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

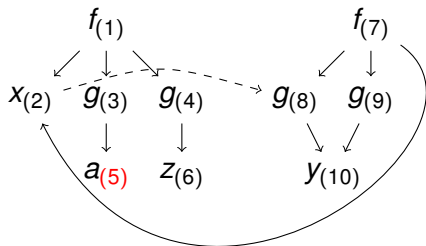
$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$

$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$

$\text{Find}(5) = 5$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

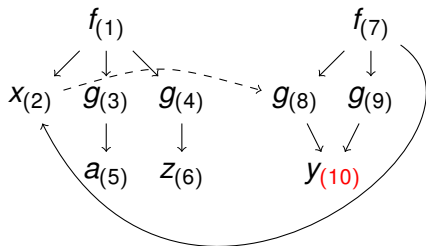
$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$

$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$

$\text{Find}(5) = 5$

$\text{Find}(10) = 10$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

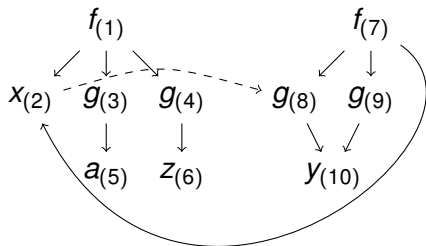
$\text{Find}(9) = (9)$

$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$

$\text{Find}(5) = 5$

$\text{Find}(10) = 10$

$\text{orient}(10, 5)$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \textit{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$

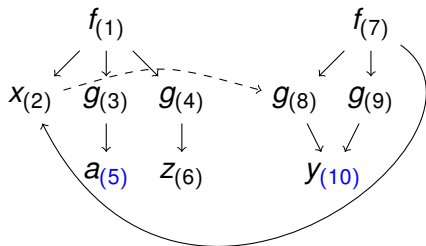
$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$

$\text{Find}(5) = 5$

$\text{Find}(10) = 10$

$\textit{orient}(10, 5)$

$\text{Occur}(10, 5) = \textit{False}$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(2), \text{Find}(8))$

$\text{Find}(2) = (2)$

$\text{Find}(8) = (8)$

$\text{Occur}(2, 8) = \text{False}$

$\text{Union}(2, 8)$

$\text{Unify}_1(\text{Find}(3), \text{Find}(9))$

$\text{Find}(3) = (3)$

$\text{Find}(9) = (9)$

$\text{Unify}_1(\text{Find}(5), \text{Find}(10))$

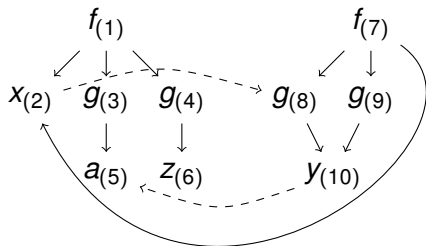
$\text{Find}(5) = 5$

$\text{Find}(10) = 10$

$\text{orient}(10, 5)$

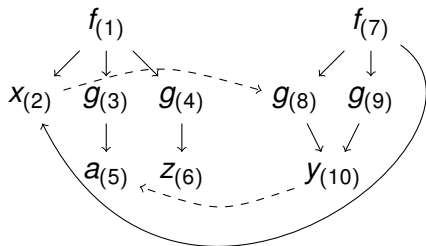
$\text{Occur}(10, 5) = \text{False}$

$\text{Union}(10, 5)$



RDA on Term Dags. Example 1

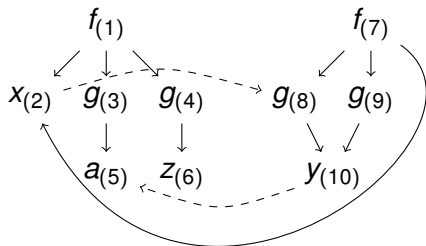
Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

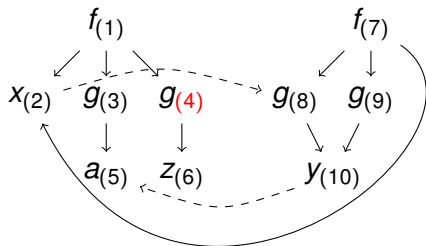


RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

$\text{Find}(4) = 4$



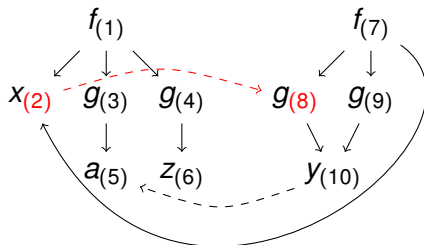
RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

$\text{Find}(4) = 4$

$\text{Find}(2) = 8$



RDA on Term Dags. Example 1

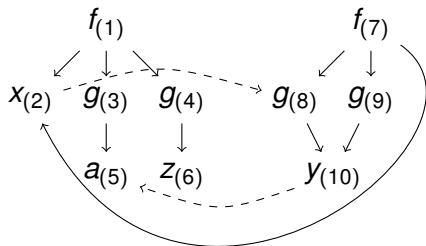
Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

$\text{Find}(4) = 4$

$\text{Find}(2) = 8$

$\text{Unify}_1(4, 8)$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

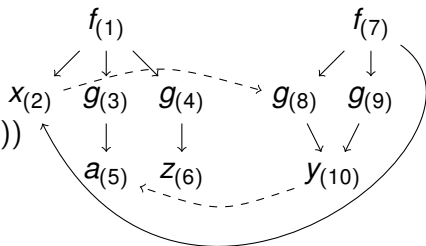
$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

$\text{Find}(4) = 4$

$\text{Find}(2) = 8$

$\text{Unify}_1(4, 8)$

$\text{Unify}_1(\text{Find}(6), \text{Find}(10))$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

$\text{Unify}_1(\text{Find}(4), \text{Find}(2))$

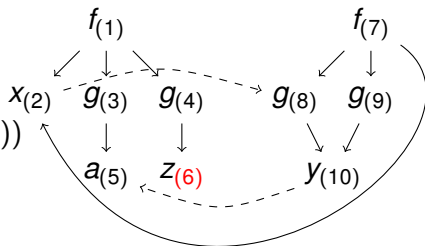
$\text{Find}(4) = 4$

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$\text{Unify}_1(4, 8)$

$\text{Unify}_1(\text{Find}(6), \text{Find}(10))$

$\text{Find}(6) = 6$



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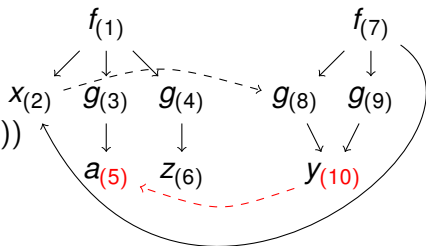
$\text{Find}(2) = 8$

$\text{Unify}_1(4, 8)$

$\text{Unify}_1(\text{Find}(6), \text{Find}(10))$

$\text{Find}(6) = 6$

$\text{Find}(10) = 5$



RDA on Term Dags. Example 1

Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

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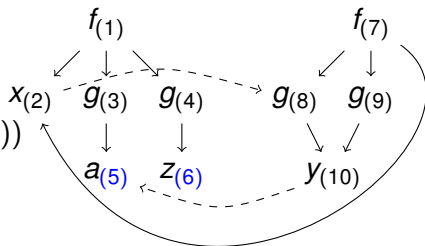
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$\text{Occur}(6, 5) = \text{False}$



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Algorithm run starts with $\text{Unify}_1(1, 7)$ and continues:

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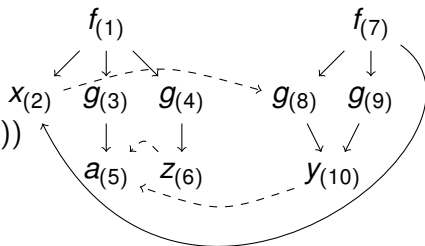
$\text{Unify}_1(\text{Find}(6), \text{Find}(10))$

$\text{Find}(6) = 6$

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$\text{Union}(6, 5)$



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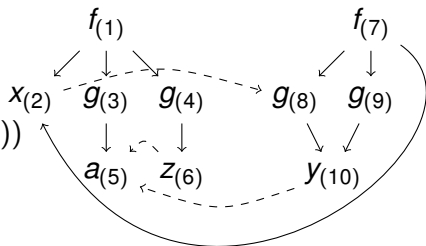
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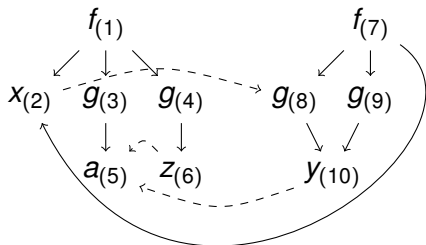
$\text{Occur}(6, 5) = \text{False}$

$\text{Union}(6, 5)$

True



RDA on Term Dags. Example 1 (Cont.)



- ▶ From the final dag one can read off:
 - ▶ The unified term $f(g(a), g(a), g(a))$.
 - ▶ The mgu in triangular form $[x \mapsto g(y); y \mapsto a; z \mapsto a]$.
- ▶ The algorithm does not create new nodes. Only one extra pointer for each variable node.
- ▶ Needs linear space.
- ▶ Time is still exponential. See the next example.



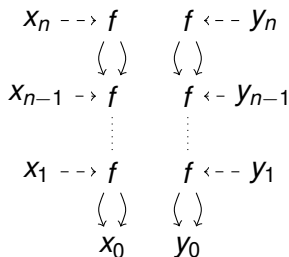
RDA on Term Dags. Example 2

Consider again the problem:

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A dag representation of the term bound to x_n and y_n :



Exponential number of recursive calls.

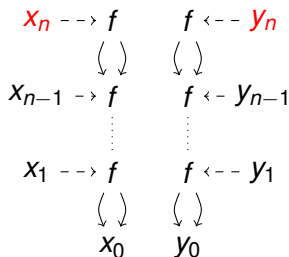
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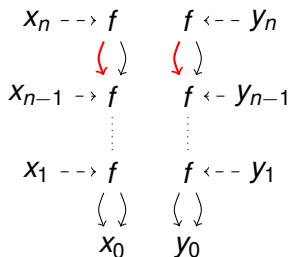
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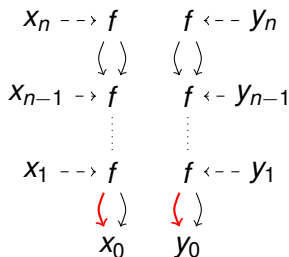
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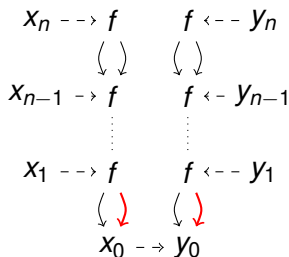
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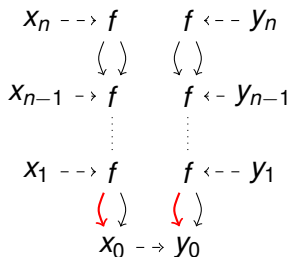
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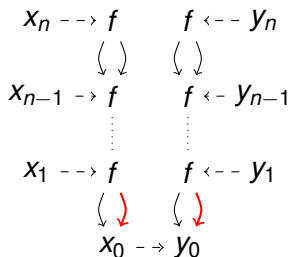
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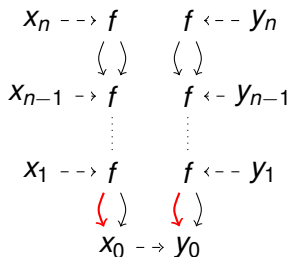
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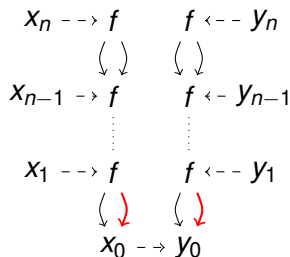
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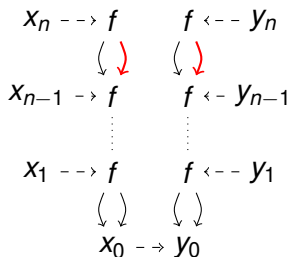
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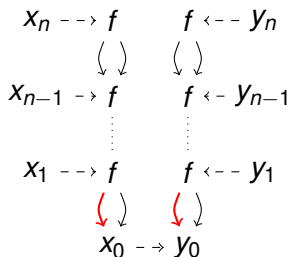
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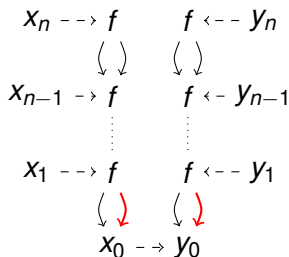
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Correctness of RDA for Term Dags

- ▶ Proof is similar as for the RDA. These two algorithm differ only by the data structure they operate on.



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- ▶ Idea: Keep from revisiting already-solved problems in the graph.
- ▶ The algorithm of Corbin and Bidoit:



J. Corbin and M. Bidoit.

A rehabilitation of Robinson's unification algorithm.

In R. Mason, editor, *Information Processing 83*, pages 909–914. Elsevier Science, 1983.



Outline

Recursive Descent Algorithm is Expensive

Improvement 1: Linear Space, Exponential Time

Improvement 2. Linear Space, Quadratic Time

Improvement 3. Almost Linear Algorithm



Quadratic Algorithm on Term Dags

Input: A pair of nodes k_1 and k_2 in a dag

Output: *True* if the terms corresponding to k_1 and k_2 are unifiable. *False* Otherwise.

Side Effect: A pointer structure which allows to read off an mgu and the unified term.

`Unify2(k_1, k_2)`

if $k_1 = k_2$ **then return** *True*; /* Trivial */

else

if *function-node*(k_2) **then**

$u := k_1; v := k_2$

else

$u := k_2; v := k_1$; /* Orient */

end

Procedure `Unify2`. Quadratic Algorithm.

(No difference from `Unify1` so far. Continues on the next slide)



Quadratic Algorithm

```
if variable-node(u) then  
  if Occurs (u, v) ;                               /* Occur-check */  
  then  
    return False  
  else  
    Union(u, v) ;                                   /* Variable elimination */  
    return True  
end
```

Procedure *Unify2*. Quadratic Algorithm. Continued.
(No difference from *Unify1* so far. Continues on the next slide)



Quadratic Algorithm

```
else if function-symbol(u)  $\neq$  function-symbol(v)  
then  
    return False;                               /* Symbol clash */  
else  
    n := arity(function-symbol(u));  
    (u1, ..., un) := succ-list(u);  
    (v1, ..., vn) := succ-list(v);  
    i := 0; bool := True;  
  
    Union(u,v);  
  
    while i ≤ n and bool do  
        i := i + 1; bool := Unify2(Find(ui), Find(vi));  
        /* Decomposition */  
  
    end  
  
    return bool
```

Procedure *Unify2*. Quadratic Algorithm. Finished.

(The only difference from *Unify1* is *Union*(*u*,*v*.)



Quadratic Algorithm. Example

The same example that revealed exponential behavior of RDA:

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Properties of the Quadratic Algorithm

- ▶ Correctness can be shown in the similar way as for the RDA.
- ▶ The algorithm is quadratic in the number of symbols in original terms:
 - ▶ Each call of `Unify2` either returns immediately, or makes one more node unreachable for the `Find` operation.
 - ▶ Therefore, there can be only linearly many calls of `Unify2`.
 - ▶ Quadratic complexity comes from the fact that `Occur` and `Find` operations are linear.



Outline

Recursive Descent Algorithm is Expensive

Improvement 1: Linear Space, Exponential Time

Improvement 2. Linear Space, Quadratic Time

Improvement 3. Almost Linear Algorithm

Almost Linear Algorithm

How to eliminate two sources of nonlinearity of `Unify2`?

- ▶ `Occur`: Just omit the occur check during the execution of the algorithm.
 - ▶ Consequence: The data structure may contain cycles.
 - ▶ Since the occur-check failures are not detected immediately, at the end an extra check has to be performed to find out whether the generated structure is cyclic or not.
 - ▶ Detecting cycles in a directed graph can be done by linear search.
- ▶ `Find`: Use more efficient union-find algorithm from



[R. Tarjan.](#)

Efficiency of a good but not linear set union algorithm.

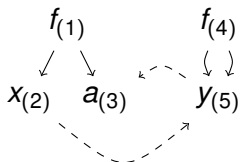
J. ACM, 22(2):215–225, 1975.



Auxiliary Procedures for the Almost Linear Algorithm

- ▶ Collapsing-find:
 - ▶ Like `Find` it takes a node k of a dag as input, and follows the additional pointers until the node `Find(k)` is reached.
 - ▶ In addition, `Collapsing-find` relocates the pointer of all the nodes reached during this process to `Find(k)`.

Example

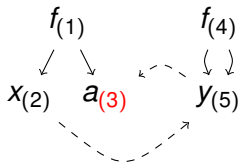


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- ▶ $CF(3)=(3)$

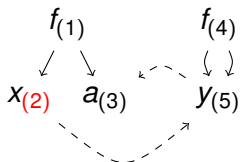


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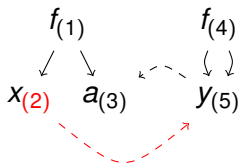


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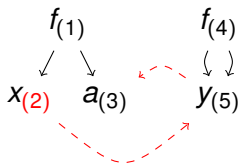


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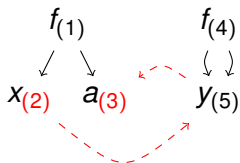


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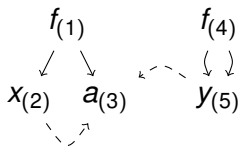


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Auxiliary Procedures for the Almost Linear Algorithm

- ▶ Union-with-weight:
 - ▶ Takes as input a pair of nodes u, v that do not have additional pointers.
 - ▶ If the set $\{k \mid \text{Find}(k) = u\}$ larger than the set $\{k \mid \text{Find}(k) = v\}$ then it creates an additional pointer from v to u .
 - ▶ Otherwise, it creates an additional pointer from u to v .

Weighted union does not apply when we have a variable node and a function node.



Almost Linear Algorithm

One more auxiliary procedure:

- ▶ `Not-cyclic`:
 - ▶ Takes a node k as input, and tests the graph which can be reached from k for cycles.
 - ▶ The test is performed on the virtual graph expressed by the additional pointer structure, i.e. one first applies `Collapsing-find` to all nodes that are reached during the test.



Almost Linear Algorithm

Input: A pair of nodes k_1 and k_2 in a directed graph.

Output: *True* if k_1 and k_2 correspond unifiable terms. *False* Otherwise.

Side Effect: A pointer structure which allows to read off an mgu and the unified term.

`Unify3(k_1, k_2)`

if *Cyclic-unify*(k_1, k_2) and *Not-cyclic*(k_1) **then**
 return *True*

else
 return *False*

end

Procedure `Unify3`. Almost Linear Algorithm.
(Continues on the next slide)



Almost Linear Algorithm

```
Cyclic-unify ( $k_1, k_2$ )  
if  $k_1 = k_2$  then return True;           /* Trivial */  
else  
  if function-node( $k_2$ ) then  
     $u := k_1; v := k_2$   
  else  
     $u := k_2; v := k_1$ ;                 /* Orient */  
end
```

Procedure *Cyclic-unify*.
(Continues on the next slide)



Almost Linear Algorithm

```
if variable-node( $u$ ) then  
  if variable-node( $v$ ) then  
    Union-with-weight( $u, v$ )  
  else  
    Union( $u, v$ );    /* No occur-check. Variable elimination */  
  return True  
end
```

Procedure *Cyclic-unify*.
(Continues on the next slide)



Almost Linear Algorithm

```
else if function-symbol(u)  $\neq$  function-symbol(v)  
then  
    return False;                               /* Symbol clash */  
else  
    n := arity(function-symbol(u));  
    (u1, ..., un) := succ-list(u);  
    (v1, ..., vn) := succ-list(v);  
    i := 0; bool := True;  
  
    Union-with-weight (u,v);  
    while i ≤ n and bool do  
        i := i + 1;  
        bool := Cyclic-unify(Collapsing-find(ui)  
                             Collapsing-find(vi)); /* Decomposition */  
    end  
return bool
```

Procedure Cyclic-unify. Finished.



Almost Linear Algorithm

The algorithm is very similar to the one described in Gerard Huet's thesis:



G. Huet.

Résolution d'Équations dans des Langages d'ordre
 $1, 2, \dots, \omega$.

Thèse d'État, Université de Paris VII, 1976.



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 - ▶ Therefore, complexity of `Cyclic-unify` is $O(n * \alpha(n))$.
 - ▶ Complexity of `Not-cyclic` is linear.
 - ▶ Hence, complexity of `Unify3` is $O(n * \alpha(n))$.



Summary

- ▶ Recursive Descent Algorithm for unification is exponential in time and space.
- ▶ Using term dags reduces space complexity to linear.
- ▶ Making the union pointer between function nodes before unifying their arguments reduces time complexity to quadratic.
- ▶ Using collapsing-find and union-with-weight further reduces time complexity to almost linear.

