

Introduction to Unification Theory

Applications

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Outline

Theorem Proving

Programming

Program Transformation

Computational Linguistics

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Theorem Proving

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Computational Linguistics

Theorem Proving

- ▶ Robinson's unification algorithm was introduced in the context of theorem proving.
- ▶ Unification: Computational mechanism behind the resolution inference rule.

Resolution

- ▶ Resolution is a rule of logical inference that allows one from “ A or B ” and “ $\text{not-}A$ or C ” to conclude that “ B or C ”.
- ▶ Logically

$$\frac{A \vee B \qquad \neg A \vee C}{B \vee C}$$



Resolution

- ▶ Resolution is a rule of logical inference that allows one from “ A or B ” and “ $\text{not-}A$ or C ” to conclude that “ B or C ”.
- ▶ Logically

$$\frac{A \vee B \qquad \neg A \vee C}{B \vee C}$$

- ▶ For instance, from the two sentences
 - ▶ *it rains or it is sunny,*
 - ▶ *it does not rain or trees are wet*
(this is the same as *if it rains then trees are wet*)one concludes that
 - ▶ *it is sunny or trees are wet.*
- ▶ Just take A for *it rains*, B for *it is sunny*, and C for *trees are wet*.



Resolution

- ▶ Resolution for first-order clauses:

$$\frac{A_1 \vee B \quad \neg A_2 \vee C}{B\sigma \vee C\sigma},$$

where $\sigma = mgu(A_1, A_2)$.

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- ▶ For instance, from the two sentences
 - ▶ *Every number is less than its successor.*
 - ▶ *If y is less than x then y is less than the successor of x.*one concludes that
 - ▶ *every number is less than the successor of its successor.*



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where $\sigma = mgu(A_1, A_2)$.

- ▶ For instance, from the two sentences
 - ▶ *Every number is less than its successor.*
 - ▶ *If y is less than x then y is less than the successor of x.*one concludes that
 - ▶ *every number is less than the successor of its successor.*
- ▶ How?



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 $\forall y \forall x \text{ less_than}(y, x) \Rightarrow \text{less_than}(y, s(x))$



Resolution

- ▶ Let's write the sentences as logical formulae.
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 $\forall x \text{ number}(x) \Rightarrow \text{less_than}(x, s(x))$
- ▶ *If y is less than x then y is less than the successor of x:*
 $\forall y \forall x \text{ less_than}(y, x) \Rightarrow \text{less_than}(y, s(x))$
- ▶ Write these formulae in disjunctive form and strip off the quantifiers:
 $\neg \text{number}(x) \vee \text{less_than}(x, s(x))$
 $\neg \text{less_than}(y, x) \vee \text{less_than}(y, s(x))$



Resolution

- ▶ Prepare for the resolution step. Make the clauses variable disjoint:

$$\neg \textit{number}(x) \vee \textit{less_than}(x, s(x))$$

$$\neg \textit{less_than}(y, x') \vee \textit{less_than}(y, s(x'))$$



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- ▶ Unify $\textit{less_than}(x, s(x))$ and $\textit{less_than}(y, x')$. The mgu $\sigma = \{x \mapsto y, x' \mapsto s(y)\}$



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$$\sigma = \{x \mapsto y, x' \mapsto s(y)\}$$

- ▶ Perform the resolution step and obtain the resolvent:

$$\neg \textit{number}(y) \vee \textit{less_than}(y, s(s(y))).$$



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- ▶ Unify $\textit{less_than}(x, s(x))$ and $\textit{less_than}(y, x')$. The mgu

$$\sigma = \{x \mapsto y, x' \mapsto s(y)\}$$

- ▶ Perform the resolution step and obtain the resolvent:

$$\neg \textit{number}(y) \vee \textit{less_than}(y, s(s(y))).$$

- ▶ What would go wrong if we did not make the clauses variable disjoint?



Factoring

- ▶ Another rule in resolution calculus that requires unification.
- ▶ Factoring

$$\frac{A_1 \vee A_2 \vee C}{A_1\sigma \vee C\sigma}$$

where $\sigma = mgu(A_1, A_2)$.

Resolution and Factoring in Action

Given:

- ▶ If y is less than x then y is less than the successor of x .
- ▶ If x is not less than a successor of some y , then 0 is less than x .

Prove:

- ▶ 0 is less than its successor.



Resolution and Factoring in Action

Translating into formulae.

Given:

- ▶ $\neg \textit{less_than}(y, x) \vee \textit{less_than}(y, s(x))$.
- ▶ $\textit{less_than}(x, s(y)) \vee \textit{less_than}(0, x)$.

Prove:

- ▶ $\textit{less_than}(0, s(0))$



Resolution and Factoring in Action

Negate the goal and try to derive the contradiction:

1. $\neg \text{less_than}(y, x) \vee \text{less_than}(y, s(x))$.
2. $\text{less_than}(x, s(y)) \vee \text{less_than}(0, x)$.
3. $\neg \text{less_than}(0, s(0))$.



Resolution and Factoring in Action

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2. $\text{less_than}(x, s(y)) \vee \text{less_than}(0, x)$.
3. $\neg \text{less_than}(0, s(0))$.
4. $\text{less_than}(0, s(x)) \vee \text{less_than}(x, s(y))$,
(Resolvent of the renamed copy of 1
 $\neg \text{less_than}(y', x') \vee \text{less_than}(y', s(x'))$) and 2, obtained by
unifying $\text{less_than}(y', x')$ and $\text{less_than}(0, x)$ with
 $\{y' \mapsto 0, x' \mapsto x\}$).



Resolution and Factoring in Action

Negate the goal and try to derive the contradiction:

1. $\neg \text{less_than}(y, x) \vee \text{less_than}(y, s(x))$.
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 $\{y' \mapsto 0, x' \mapsto x\}$).
5. $\text{less_than}(0, s(0))$
(Factor of 4 with $\{x \mapsto 0, y \mapsto 0\}$)



Resolution and Factoring in Action

Negate the goal and try to derive the contradiction:

1. $\neg \text{less_than}(y, x) \vee \text{less_than}(y, s(x))$.
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 $\neg \text{less_than}(y', x') \vee \text{less_than}(y', s(x'))$) and 2, obtained by
unifying $\text{less_than}(y', x')$ and $\text{less_than}(0, x)$ with
 $\{y' \mapsto 0, x' \mapsto x\}$).
5. $\text{less_than}(0, s(0))$
(Factor of 4 with $\{x \mapsto 0, y \mapsto 0\}$)
6. \square
(Contradiction, resolvent of 3 and 5).



Outline

Theorem Proving

Programming

Program Transformation

Computational Linguistics

Logic Programming

- ▶ Unification plays a crucial role in logic programming.
- ▶ Used to perform execution steps.

Logic Programming

- ▶ Logic programs consist of (nonnegative) clauses, written:

$$A \leftarrow B_1, \dots, B_n,$$

where $n \geq 0$ and A, B_i are atoms.

- ▶ Example:
 - ▶ $likes(john, X) \leftarrow likes(X, wine)$.
John likes everybody who likes wine.
 - ▶ $likes(john, wine)$.
John likes wine.
 - ▶ $likes(mary, wine)$.
Marry likes wine.



Logic Programming

- ▶ Goals are negative clauses, written

$$\leftarrow D_1, \dots, D_m$$

where $m \geq 0$.

- ▶ Example:

- ▶ $\leftarrow \text{likes}(\text{john}, X)$.
Who (or what) does John like?
- ▶ $\leftarrow \text{likes}(X, \text{marry}), \text{likes}(X, \text{wine})$.
Who likes both marry and wine?
- ▶ $\leftarrow \text{likes}(\text{john}, X), \text{likes}(Y, X)$.
Find such X and Y that both John and Y like X .



Logic Programming

Inference step:

$$\frac{\leftarrow D_1, \dots, D_m}{\leftarrow D_1\sigma, \dots, D_{i-1}\sigma, B_1\sigma, \dots, B_n\sigma, D_{i+1}\sigma, \dots, D_m\sigma}$$

where $\sigma = mgu(D_i, A)$ for (a renamed copy of) some program clause $A \leftarrow B_1, \dots, B_n$.



Logic Programming

Example

Program:

$likes(john, X) \leftarrow likes(X, wine).$

$likes(john, wine).$

$likes(mary, wine).$

Goal:

$\leftarrow likes(X, marry), likes(X, wine).$

Inference:

- ▶ Unifying $likes(X, marry)$ with $likes(john, X')$ gives $\{X \mapsto john, X' \mapsto marry\}$
- ▶ New goal: $\leftarrow likes(marry, wine), likes(john, marry).$



Prolog

- ▶ Prolog: Most popular logic programming language.
- ▶ Unification in Prolog is nonstandard: Omits occur-check.
- ▶ Result: Prolog unifies terms x and $f(x)$, using the substitution $\{x \mapsto f(f(f(\dots)))\}$.
- ▶ Because of that, sometimes Prolog might draw conclusions the user does not expect:

```
less(X, s(X)).  
foo : -less(s(Y), Y).  
?- foo.  
yes.
```

- ▶ Infinite terms in a theoretical model for real Prolog implementations.



Higher-Order Logic Programming

Example

A λ -Prolog program:

(age bob 24).

(age sue 23).

(age ned 23).

(mapped P nil nil).

(mapped P (X::L) (Y::K)):- (P X Y), (mapped P L K).

mapped maps the predicate P on the lists (X::L) and (Y::K).



Higher-Order Logic Programming

Example

A λ -Prolog program:

```
(age bob 24).
```

```
(age sue 23).
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```
(age ned 23).
```

```
(mapped P nil nil).
```

```
(mapped P (X::L) (Y::K)):- (P X Y), (mapped P L K).
```

mapped maps the predicate P on the lists (X::L) and (Y::K).

The goal `(mapped x\y\ (age x y) L (23::24::nil))` returns two answers:

```
L = (sue::bob::nil)
```

```
L = (ned::bob::nil)
```



Higher-Order Logic Programming

Example (Cont.)

- ▶ On the previous slide, the goal was unified with the head of the (copy of) second mapped clause by the substitution

$$\{P \mapsto x \backslash y \backslash (\text{age } x \ y), L \mapsto (X :: L'), Y \mapsto 23, K \mapsto (24 :: \text{nil})\}$$

$x \backslash y \backslash (\text{age } x \ y)$ is the λ -Prolog notation for $\lambda x. \lambda y. (\text{age } x \ y)$.

- ▶ It made the new goal

$(\text{age } X \ 23), (\text{mapped } x \backslash y \backslash (\text{age } x \ y) \ L' \ (24 :: \text{nil})).$

etc.



Higher-Order Logic Programming

- ▶ The fragment of higher-order unification used in λ -prolog is unification for higher-order patterns.
- ▶ Higher-order pattern is a λ -term where arguments of free variables are distinct bound variables.
- ▶ Higher-order pattern unification is unitary.



Programming in Mathematica

- ▶ Mathematica is a symbolic computation system, a product of Wolfram Research, Inc.
- ▶ It comes with a rule based programming language.
- ▶ An example of Mathematica code to compute factorial:

```
f[0] := 1
f[n_] := n * f[n - 1]; n > 0
```

- ▶ To compute $f[5]$, it first tries to match 0 with 5, which fails.
- ▶ Next, n matches 5 with the substitution $n \mapsto 5$, the condition $5 > 0$ is satisfied and the next goal becomes $5 * f[4]$.
- ▶ n_* indicates that n is a variable that can match an expression.
- ▶ Matching is a special case of unification: $s \doteq^? t$ is a matching problem if t is ground.



Programming in Mathematica

- ▶ Mathematica has a special kind of variable, called sequence variable.
- ▶ Sequence variables can be instantiated by finite sequences.
- ▶ Convenient to write short, elegant programs.
- ▶ Unification with sequence variables is decidable and infinitary, matching is finitary.



Programming in Mathematica

- ▶ An example of Mathematica code for bubble sort:

```
sort[{x___, u_, y___, v_, z___}] := sort[{x, v, y, u, z}]/; u > v  
sort[{x___}] := {x}
```

- ▶ `x___` indicates that `x` is a sequence variable.
- ▶ `sort[{x___, u_, y___, v_, z___}]` matches `sort[{1, 2, 3, 4, 1}]` in various ways.
- ▶ The one that satisfies the condition `u > v` is

$$\{x \mapsto 1, u \mapsto 2, y \mapsto (3, 4), v \mapsto 1, z \mapsto ()\}$$

- ▶ The next goal becomes `sort[{1, 1, 3, 4, 2}]`, and so on.



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Program Transformation

- ▶ Program transformation is the act of changing one program into another.
- ▶ Some techniques describe transformation as rewriting systems for program schemas, together with constraints on the instances of the schemas that must be met in order for the transformation to be valid.
- ▶ When a rewriting rule is applied to a particular program, the schema in the left hand side of the rule should match the program.
- ▶ Usually schemas are expressed in a higher-order language.
- ▶ Leads to higher-order matching.



Program Transformation

Example (Schema Matching)

- ▶ Schema:

$$F(x) \Leftarrow \text{if } A(x) \text{ then } B(x) \text{ else } H(D(x), F(E(x))).$$

- ▶ Instance program:

$$\text{fact}(x) \Leftarrow \text{if } x = 0 \text{ then } 1 \text{ else } x * \text{fact}(x - 1)$$

- ▶ The schema matches the instance with the substitution:

$$\{F \mapsto \lambda x. \text{fact}(x), A \mapsto \lambda x. x = 0, B \mapsto \lambda x. 1, \\ H \mapsto \lambda x. \lambda y. x * y, D \mapsto \lambda x. x, E \mapsto \lambda x. x - 1\}$$



Program Transformation

Example (Schema Matching)

The same schema, different instance.

- ▶ Schema:

$$F(x) \Leftarrow \text{if } A(x) \text{ then } B(x) \text{ else } H(D(x), F(E(x)))$$

- ▶ Instance:

$$\text{rev}(x) \Leftarrow \text{if Null}(x) \text{ then } x \text{ else app}(\text{rev}(\text{Cdr}(x)), \text{Cons}(\text{Car}(x), \text{nil}))$$

- ▶ Matching substitution:

$$\{F \mapsto \lambda x.\text{rev}(x), A \mapsto \lambda x.\text{Null}(x), B \mapsto \lambda x.x, \\ H \mapsto \lambda x.\lambda y.\text{app}(y, x), D \mapsto \lambda x.\text{Cons}(\text{Car}(x), \text{nil}), \\ E \mapsto \lambda x.\text{Cdr}(x)\}$$



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Ellipsis Resolution

- ▶ An elliptical construction involves two phrases (clauses) that are parallel in structure in some sense.
- ▶ The source clause is complete.
- ▶ The target clause is missing material found in the source.
- ▶ Goal: To recover the property of the parallel element in the target the missing material stands for.



Ellipsis Resolution

Example

- ▶ Dan likes golf, and George does too.

Ellipsis Resolution

Example

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- ▶ “Dan” and “George” are parallel elements.



Ellipsis Resolution

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- ▶ Semantic interpretation of “Dan likes golf”: *like(dan, golf)*.



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- ▶ dan is called a primary occurrence.



Ellipsis Resolution

Example

- ▶ Dan likes golf, and George does too.
- ▶ “Dan” and “George” are parallel elements.
- ▶ Semantic interpretation of “Dan likes golf”: $like(\underline{dan}, golf)$.
- ▶ dan is called a primary occurrence.
- ▶ To interpret “George does too”, we require the property P such that, when applied to the interpretation of the subject of “Dan likes golf”, i.e. dan , gives the interpretation of “Dan likes golf”.



Ellipsis Resolution

Example

- ▶ Dan likes golf, and George does too.
- ▶ “Dan” and “George” are parallel elements.
- ▶ Semantic interpretation of “Dan likes golf”: $like(\underline{dan}, golf)$.
- ▶ \underline{dan} is called a primary occurrence.
- ▶ To interpret “George does too”, we require the property P such that, when applied to the interpretation of the subject of “Dan likes golf”, i.e. \underline{dan} , gives the interpretation of “Dan likes golf”.
- ▶ Find P such that $P(\underline{dan}) \doteq? like(\underline{dan}, golf)$.



Ellipsis Resolution

Example

- ▶ Dan likes golf, and George does too.
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- ▶ Find P such that $P(\underline{dan}) \doteq? like(\underline{dan}, golf)$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, golf)\}$, $\sigma_2 = \{P \mapsto \lambda x. like(x, golf)\}$.



Ellipsis Resolution

Example

- ▶ Dan likes golf, and George does too.
- ▶ “Dan” and “George” are parallel elements.
- ▶ Semantic interpretation of “Dan likes golf”: $like(\underline{dan}, golf)$.
- ▶ \underline{dan} is called a primary occurrence.
- ▶ To interpret “George does too”, we require the property P such that, when applied to the interpretation of the subject of “Dan likes golf”, i.e. dan , gives the interpretation of “Dan likes golf”.
- ▶ Find P such that $P(dan) \doteq? like(\underline{dan}, golf)$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, golf)\}$, $\sigma_2 = \{P \mapsto \lambda x. like(x, golf)\}$.
- ▶ Constraint: Solution should not contain the primary occurrence. Hence, σ_2 is the only solution.



Ellipsis Resolution

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- ▶ Dan likes golf, and George does too.
- ▶ “Dan” and “George” are parallel elements.
- ▶ Semantic interpretation of “Dan likes golf”: $like(\underline{dan}, golf)$.
- ▶ \underline{dan} is called a primary occurrence.
- ▶ To interpret “George does too”, we require the property P such that, when applied to the interpretation of the subject of “Dan likes golf”, i.e. \underline{dan} , gives the interpretation of “Dan likes golf”.
- ▶ Find P such that $P(\underline{dan}) \doteq? like(\underline{dan}, golf)$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, golf)\}$, $\sigma_2 = \{P \mapsto \lambda x. like(x, golf)\}$.
- ▶ Constraint: Solution should not contain the primary occurrence. Hence, σ_2 is the only solution.
- ▶ Interpretation: $like(\underline{dan}, golf) \wedge P(\underline{dan})\sigma_2$ that gives $like(\underline{dan}, golf) \wedge like(\underline{dan}, golf)$.



Ellipsis Resolution

- ▶ Higher-order unification generates multiple solutions.
- ▶ Leads to multiple readings.
- ▶ Constraints help to filter out some.
- ▶ Still, several may remain.
- ▶ Strict and sloppy reading.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
*like(dan, wife-of(*dan*)).*



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
*like(dan, wife-of(*dan*)).*
- ▶ dan is a primary occurrence, *dan* is secondary, because it came from the pronoun which is not a parallel element.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
*like(dan, wife-of(*dan*)).*
- ▶ *dan* is a primary occurrence, *dan* is secondary, because it came from the pronoun which is not a parallel element.
- ▶ Find P such that $P(dan) \doteq? \textit{like}(\underline{dan}, \textit{wife-of}(dan))$.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
 $like(\underline{dan}, wife-of(dan))$.
- ▶ \underline{dan} is a primary occurrence, dan is secondary, because it came from the pronoun which is not a parallel element.
- ▶ Find P such that $P(dan) \doteq? like(\underline{dan}, wife-of(dan))$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(dan))\}$,
 $\sigma_2 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(x))\}$,
 $\sigma_3 = \{P \mapsto \lambda x. like(x, wife-of(dan))\}$,
 $\sigma_4 = \{P \mapsto \lambda x. like(x, wife-of(x))\}$



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
 $like(\underline{dan}, wife-of(dan))$.
- ▶ \underline{dan} is a primary occurrence, dan is secondary, because it came from the pronoun which is not a parallel element.
- ▶ Find P such that $P(dan) \doteq? like(\underline{dan}, wife-of(dan))$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(dan))\}$,
 $\sigma_2 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(x))\}$,
 $\sigma_3 = \{P \mapsto \lambda x. like(x, wife-of(dan))\}$,
 $\sigma_4 = \{P \mapsto \lambda x. like(x, wife-of(x))\}$
- ▶ Constraint: Solution should not contain the primary occurrence. Hence, σ_1 and σ_2 are discarded.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
 $like(\underline{dan}, wife-of(dan))$.
- ▶ \underline{dan} is a primary occurrence, dan is secondary, because it came from the pronoun which is not a parallel element.
- ▶ Find P such that $P(dan) \doteq? like(\underline{dan}, wife-of(dan))$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(dan))\}$,
 $\sigma_2 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(x))\}$,
 $\sigma_3 = \{P \mapsto \lambda x. like(x, wife-of(dan))\}$,
 $\sigma_4 = \{P \mapsto \lambda x. like(x, wife-of(x))\}$
- ▶ Constraint: Solution should not contain the primary occurrence. Hence, σ_1 and σ_2 are discarded.
- ▶ Strict reading: $P(george)\sigma_3 = like(george, wife-of(dan))$.



Ellipsis Resolution

Example

- ▶ Dan likes his wife, and George does too.
- ▶ Semantic interpretation of “Dan likes his wife”:
 $like(\underline{dan}, wife-of(dan))$.
- ▶ \underline{dan} is a primary occurrence, dan is secondary, because it came from the pronoun which is not a parallel element.
- ▶ Find P such that $P(dan) \doteq? like(\underline{dan}, wife-of(dan))$.
- ▶ $\sigma_1 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(dan))\}$,
 $\sigma_2 = \{P \mapsto \lambda x. like(\underline{dan}, wife-of(x))\}$,
 $\sigma_3 = \{P \mapsto \lambda x. like(x, wife-of(dan))\}$,
 $\sigma_4 = \{P \mapsto \lambda x. like(x, wife-of(x))\}$
- ▶ Constraint: Solution should not contain the primary occurrence. Hence, σ_1 and σ_2 are discarded.
- ▶ Strict reading: $P(george)\sigma_3 = like(george, wife-of(dan))$.
- ▶ Sloppy reading: $P(george)\sigma_4 = like(george, wife-of(george))$.



Brief Summary of the Course

- ▶ First-order syntactic unification
- ▶ First-order equational unification
- ▶ Higher-order unification
- ▶ Applications related to logic, language, and information



Brief Summary of the Course

- ▶ First-order syntactic unification
 - ▶ Most general unifier.
 - ▶ Unification algorithm.
 - ▶ Improvements of the algorithm.
- ▶ First-order equational unification

- ▶ Higher-order unification

- ▶ Applications related to logic, language, and information



Brief Summary of the Course

- ▶ First-order syntactic unification
 - ▶ Most general unifier.
 - ▶ Unification algorithm.
 - ▶ Improvements of the algorithm.
 - ▶ First-order equational unification
 - ▶ Minimal complete set of unifiers.
 - ▶ Decidability/Undecidability, Unification type.
 - ▶ Results for particular theories.
 - ▶ Universal E-unification procedure.
 - ▶ Narrowing.
 - ▶ Higher-order unification
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- ▶ Applications related to logic, language, and information



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 - ▶ Theorem proving.
 - ▶ Programming, program transformation.
 - ▶ Ellipsis resolution.



Open Problems

Some open problems in unification theory:

- ▶ Is there an equational theory for which unification with constants is decidable, but general unification is undecidable? (Baader and Schulz, 1992).
- ▶ Is unification of patterns decidable for equational theories whose axioms have the same set of variables on left and right hand side? (Jouannaud, 1994).
- ▶ Are context unification and linear second order unification decidable? (Comon, 1991, Schmidt-Schauß, 1994, Levy, 1996).
- ▶ What is the exact complexity of word unification? (Schulz, 1998).

The RTA list of open problems:

<http://rtaloop.pps.jussieu.fr/>

