Rational parametrization of algebraic curves
(An appetizer)

Veronika Pillwein

Notation

- K is an algebraically closed field, $\mathbb{Q} \subseteq K$
- $K(t)$ denotes the field of rational functions over K
- $K[x_1, \ldots, x_n]$ denotes the ring of polynomials in the indeterminates $x_1, \ldots, x_n$

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- $\mathbb{A}^n(K) = \{(a_1, \ldots, a_n) \mid a_k \in K\} = \mathbb{A}^n$ the n-dimensional affine space over K
- $\mathcal{C} = \{(a, b) \in \mathbb{A}^2 \mid f(a, b) = 0\}$ is the affine algebraic curve with defining polynomial $f \in K[x, y]$;
  $\mathcal{C}$ is irreducible, iff it has an irreducible defining polynomial

Examples

$$
\begin{aligned}
f_1(x, y) &= x^3 + x^2 - y^2 \\
f_2(x, y) &= y^2 - x^3 + x \\
f_3(x, y) &= (x^2 + 4y + y^2)^2 - 16(x^2 + y^2) \\
f_4(x, y) &= 2x^4 - 3x^2y + y^2 - 2y^3 + y^4
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Question: Is this a good representation for an algebraic curve?

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Answer: Depends on what we want to do!

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Question: Is $P = (-1, 1)$ a point on the curve $\mathcal{C}_i$ defined by f_i?

Examples

$$f_1(x, y) = x^3 + x^2 - y^2 \qquad f_1(-1, 1) = -1$$

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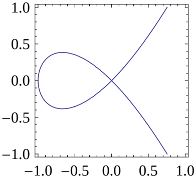
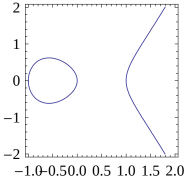
Question: Can we generate arbitrary many real points on $\mathcal{C}_i$?

Examples

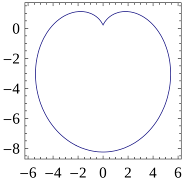
$f_1(x, y)$



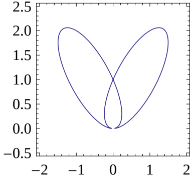
$f_2(x, y)$



$f_3(x, y)$



$f_4(x, y)$



Parametrization

Instead of the implicit representation $f(x, y) = 0$, we seek for a parametrization

$$t \mapsto (r_1(t), r_2(t)) \quad \text{such that} \quad f(r_1(t), r_2(t)) = 0, \quad \forall\, t$$

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Questions:

- When can we rationally parametrize a given curve?
- How do we do it?

Mathematica	Sage	Maple
ContourPlot	implicit_plot	implicitplot

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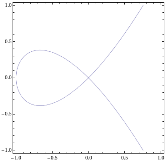

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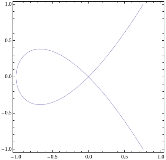

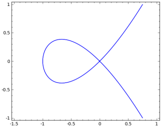

Mathematica

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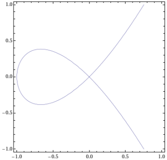

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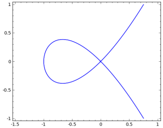

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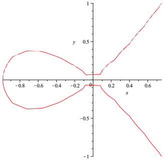

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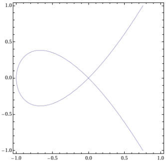

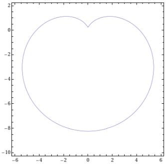

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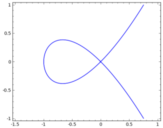

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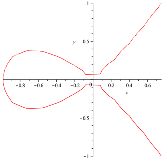

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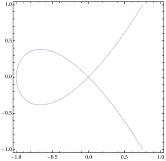

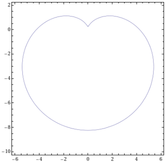

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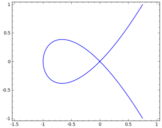

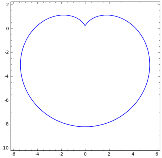

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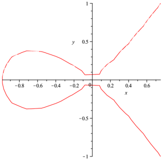

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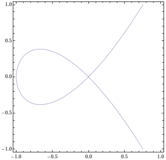

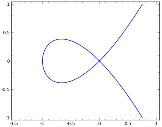

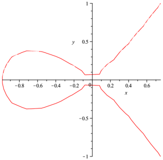

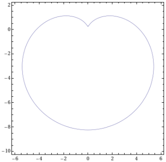

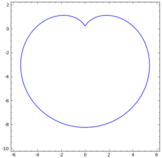

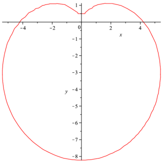

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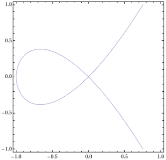

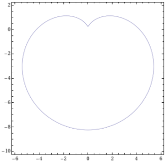

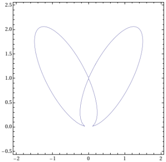

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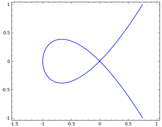

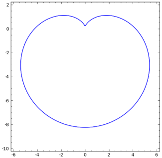

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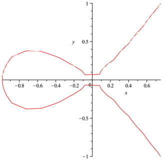

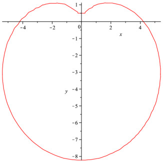

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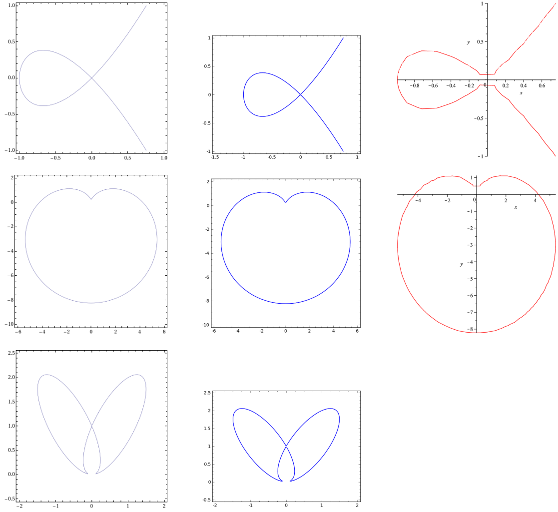

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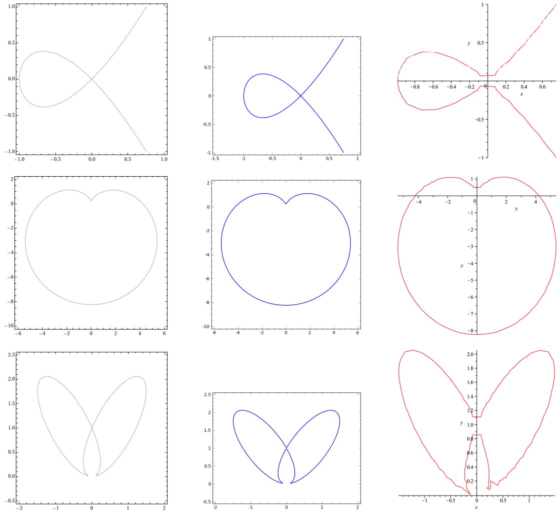

Properties

Definition. Let $\mathcal{C}$ be a curve, $f \in K[x, y]$ its defining polynomial, and $P = (a, b) \in \mathbb{A}^2$. P is a point on $\mathcal{C}$ if $f(a, b) = 0$.

P is a simple point on $\mathcal{C}$, if

$$f(a, b) = 0 \quad \text{and} \quad \left(\frac{\partial f}{\partial x}(a, b) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(a, b) \neq 0 \right).$$

If P is a simple point, then the tangent to $\mathcal{C}$ at P is given by

$$\frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b) = 0.$$

A point on $\mathcal{C}$ that is not simple is called multiple or singular point.

A curve having only simple points is called a non-singular curve.

Properties

Definition. Let $\mathcal{C}$ be a curve, $f \in K[x, y]$ its defining polynomial, and $P = (a, b) \in \mathbb{A}^2$ a point on $\mathcal{C}$. P is a point of multiplicity m, iff

- all partial derivatives $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a, b)$ vanish for $i + j < m$
- at least one of the partial derivatives of order m does not vanish

The multiplicity of P on $\mathcal{C}$ is denoted by $m_P(\mathcal{C})$ or just m_P.

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Example. Let $\mathcal{C}$ be defined by the polynomial $f_1(x,y)=x^3+x^2-y^2$. Then $P = (0,0)$ is a double point on $\mathcal{C}$:

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = -2.$$

Tangents at multiple points

Let $1 \leq m = m_P(\mathcal{C})$ be the multiplicity of $P = (a, b)$ on $\mathcal{C}$. The linear factors of

$$\sum_{i=0}^{m} \binom{m}{i} \frac{\partial m f}{\partial x^i \partial y^{m-i}}(a, b)(x - a)^i (y - b)^{m-i}$$

are the tangents to $\mathcal{C}$ at P. An m-fold point is called ordinary, iff all the tangents are different.

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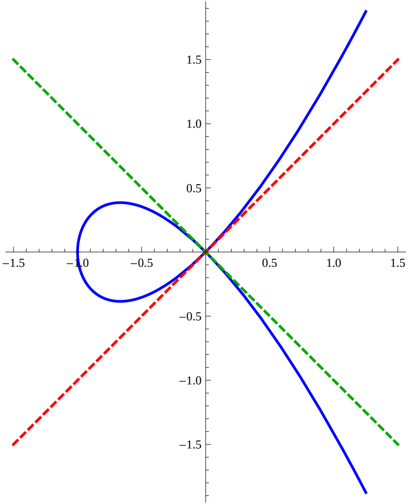
Example. Let $\mathcal{C}$ be defined by the polynomial $f_1(x,y) = x^3 + x^2 - y^2$ and $P = (0, 0)$. Then

$$\sum_{i=0}^{2} \binom{2}{i} \frac{\partial^2 f}{\partial x^i \partial y^{2-i}}(0, 0) x^i y^{2-i} = 2(x-y)(x+y)$$

Loop with tangents



More on multiplicities

Let $\mathcal{C}$ be defined by $f \in K[x, y]$, $\deg f = d$. There is an upper bound for the number of singularities:

$$\frac{1}{2}(d-1)(d-2) \geq \sum_{P \in \mathcal{C}} \frac{1}{2} m_P(m_P - 1).$$

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Projective space

Definition. The n-dimensional projective space over K is defined as

$$\mathbb{P}^n(K) = \{(c_1 : \cdots : c_{n+1}) \mid (c_1 : \cdots : c_{n+1}) \in K^{n+1} \setminus \{(0, \ldots, 0)\}\},$$

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$$\mathcal{C}* = \{(c_1 : c_2 : c_3) \in \mathbb{P}^2 \mid f^*(c_1, c_2, c_3) = 0\}.$$

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Let $f^*, g^* \in K[x, y, z]$ be relatively prime, homogeneous polynomials and let $\mathcal{C}^*, \mathcal{D}^*$ be the corresponding projective curves. Then $\mathcal{C}^*$ and $\mathcal{D}^*$ have exactly $\deg(f^*) \cdot \deg(g^*)$ projective points in common counting multiplicities.

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Let $\mathcal{C}^*$ be an irreducible curve of degree d in $\mathbb{P}^2$ having only ordinary points. Then

$$\text{genus}(\mathcal{C}^*) = \frac{1}{2}\left((d-1)(d-2) - \sum_{P\in\mathcal{C}^*} m_P(m_P-1)\right).$$

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Theorem. An algebraic curve $\mathcal{C}$ (having only ordinary singularities) is rationally parametrizable if and only if $\mathrm{genus}(\mathcal{C}) = 0$.

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This yields the parametrization

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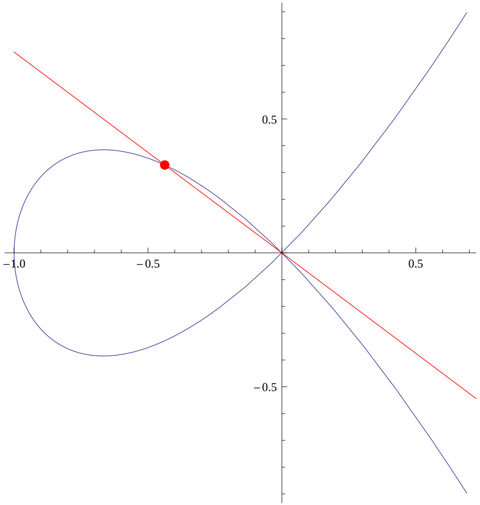

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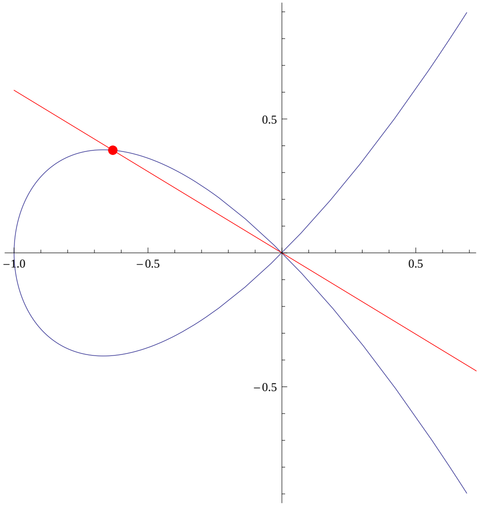

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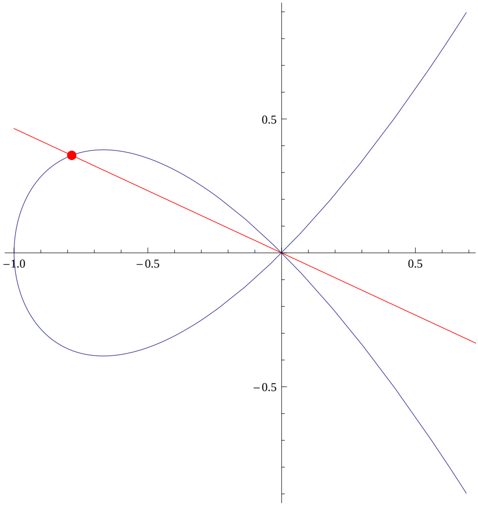

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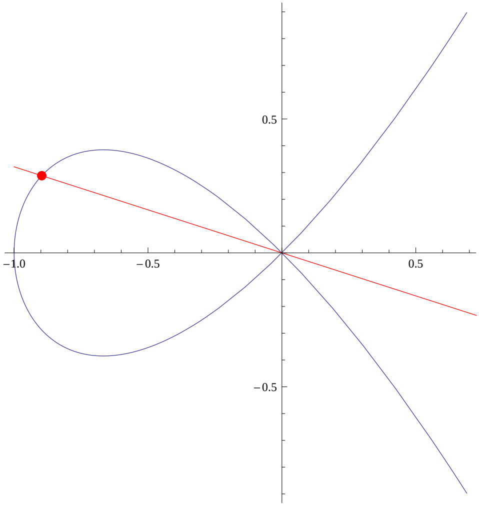

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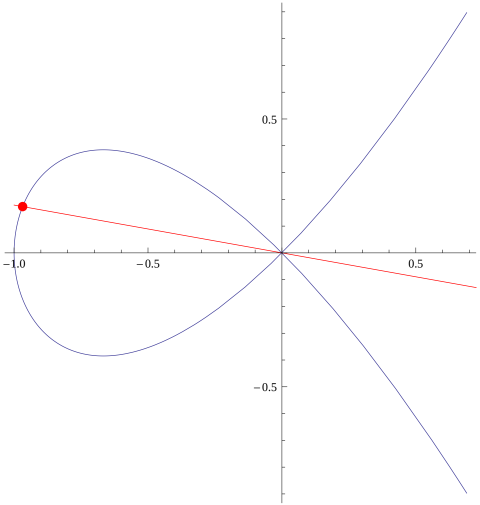

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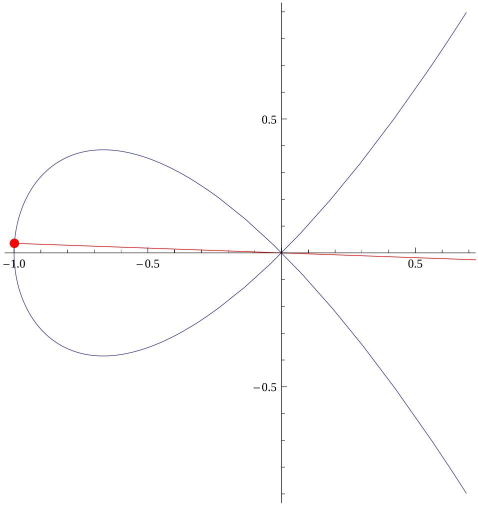

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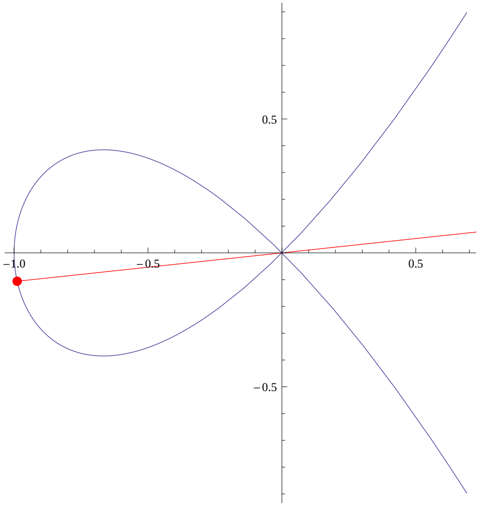

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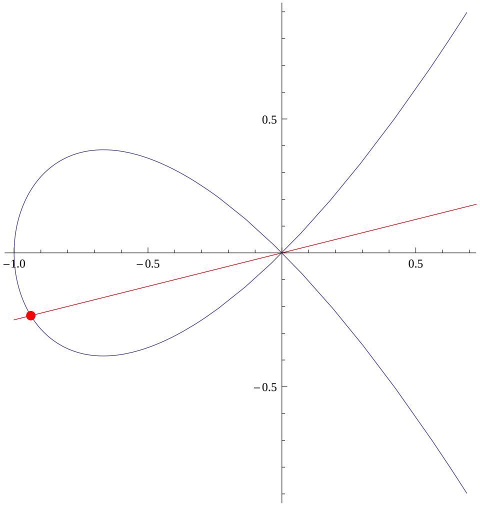

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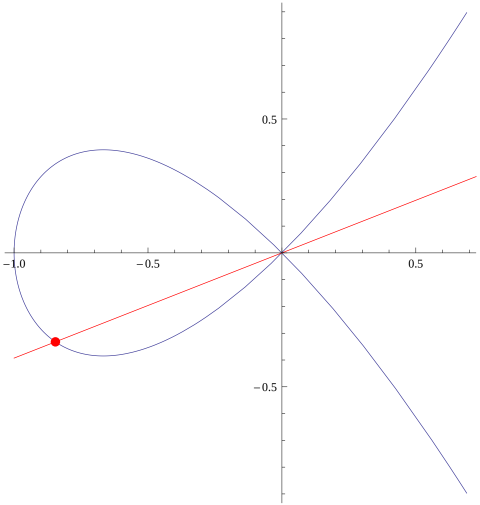

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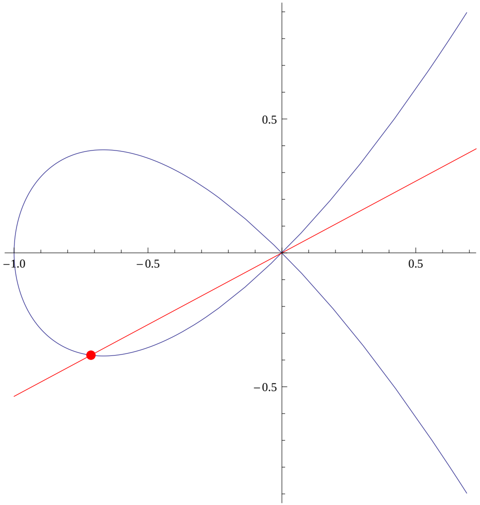

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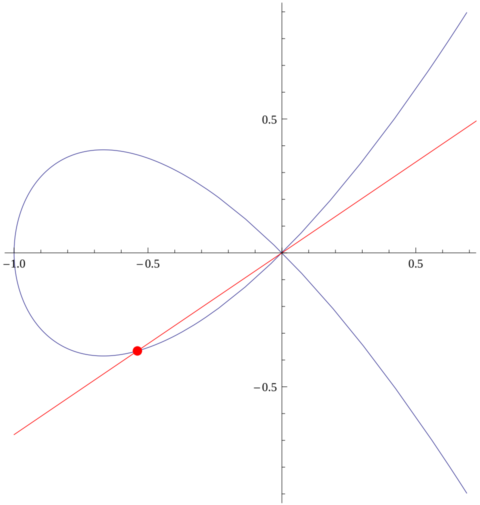

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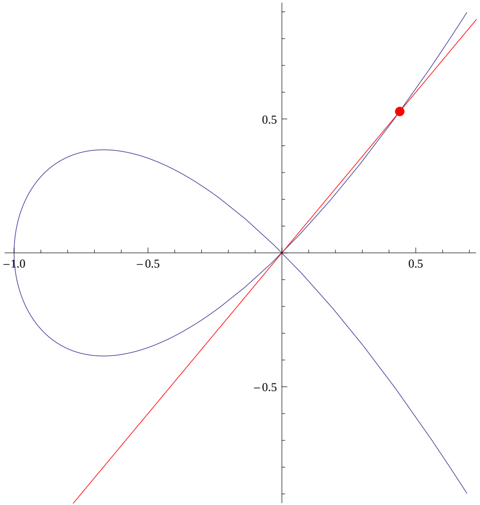

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Example

Let

$$a^*(x,y,z) = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2.$$

with simple points $P_1 = (0:0:1), P_{2,3} = (1:\pm\mathrm{i}:0)$ and additional simple point $Q = (0:-8:1)$. Then

$$\begin{aligned}
a^*(0,0,1) = 0 \quad &\longrightarrow \quad a_5 = 0 \\
a^*(1,\mathrm{i},0) = 0 \quad &\longrightarrow \quad a_0 + \mathrm{i}a_1 - a_3 = 0 \\
a^*(1,-\mathrm{i},0) = 0 \quad &\longrightarrow \quad a_0 - \mathrm{i}a_1 - a_3 = 0 \\
a^*(0,-8,1) = 0 \quad &\longrightarrow \quad 64a_3 - 8a_4 + a_5 = 0
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Example

We have

$$f_3(x,y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2)$$

and the affine adjoint curves

$$a_t(x,y) = \frac{1}{8}tx^2 + \frac{1}{8}ty^2 + ty + x.$$

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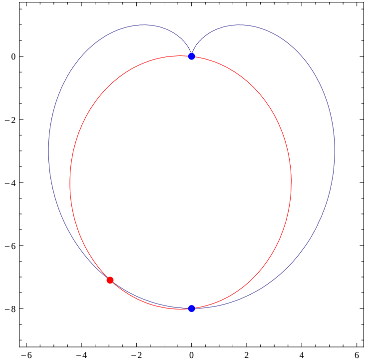

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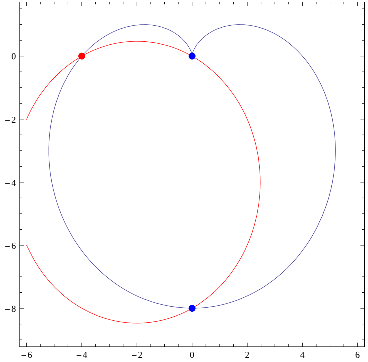

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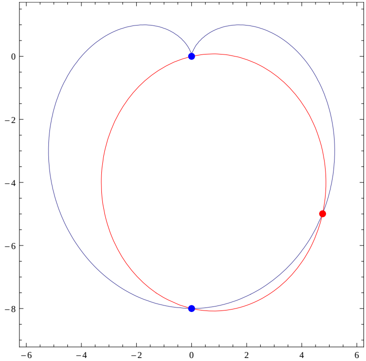

Some open questions

- What is a resultant and how to compute it?
- What is the intersection multiplicity and how to compute it?
- How do I determine simple simple points on the curve?
- How does the choice of simple points affect the parametrization?
- What about real parametrization?
- How do curves enter in cryptography?