I Symbolic integration

Fix a differential field (K, D). Then the problem of symbolic integration (aka "integration in finite terms") is as follows:

<u>weak version</u>: given $f \in K$, decide whether there exists $g \in K$ with $D(g) = f$. And if so, find one. ("f is integrable in K")

<u>strong version</u>: given $f \in K$, decide whether there exists an elementary extension E of K and $g \in E$ with $D(g) = f$. And if so, find one. "f is elementary integrable"

<u>Ex</u>:	weak version	strong version
$K = \mathbb{Q}(x)$, $D = \frac{d}{dx}$ $f = \frac{(x+1)(x-2)}{(2x-1)^2}$	YES! $g = \frac{(x+1)^2}{2(2x-1)}$	YES! dto.
same (K, D) $f = \frac{2x}{x^2+1}$	NO! (why not?)	YES! $E = \mathbb{Q}(x)(t)$ with $D(x) = 1$, $D(t) = f$. $g = t$. ("$t \doteq \log(x^2+1)$")
$K = \mathbb{Q}(x)(t)$ $D(x) = 1$, $D(t) = 2xt$ ("$t = e^{x^2}$") $f = t$	NO!	NO! (why not?)

<u>Note:</u>

YES for weak version ⟹ YES for strong version

NO for strong version ⟹ NO for weak version.

<u>want:</u> algorithms that solve these problems (input f, output g or "impossible")

Whether such algorithms exist, depends on K. Main result (Risch / Bronstein):

There is an algorithm which for any given Liouvillean extension K of $\mathbb{Q}$ and any given $f \in K$ decides whether there is an elementary extension E of K and $g \in E$ with $D(g) = f$. And if so, finds one.

The algorithm which solves the problem in full generality is <u>very</u> complicated and has never been implemented completely. However, most CA systems contain an implementation covering a reasonably large subclass: the case when K contains no algebraic extensions. (the "transcendental case")

Goal for the coming weeks: explain the algorithm for this situation.

① Weak version for $(C(x), \frac{d}{dx})$

Here, C is some computable field.

Task: given $f \in C(x)$, find $g \in C(x)$ with $\frac{d}{dx} g = f$, or assert that no such g exist.

In this section, we write also $u' := \frac{d}{dx} u$ for the derivative of some $u \in C(x)$.

[Slide: some facts from computer algebra]

Consider $f = \frac{a}{uv^m}$ for some $a, u, v \in \mathbb{C}[x]$, $m \geq 2$ with $\deg a < \deg u + m \deg v$ and $\gcd(u,v) = \gcd(v,v') = \gcd(a, uv) = 1$.

Idea: reduce the integration problem for f to one of the same type, but with smaller m.

Want: $b, c \in \mathbb{C}[x]$ such that

$$\frac{a}{uv^m} = \underbrace{D\left(\frac{\boxed{b}}{v^{m-1}}\right)} + \frac{\boxed{c}}{uv^{m-1}}$$

$$= \frac{b'v^{m-1} - (m-1)bv^{m-2}v'}{v^{2(m-1)}}$$

$$= \frac{b'v - (m-1)bv'}{v^m}$$

$$\Leftrightarrow \quad a = b'uv - (m-1)buv' + cv \qquad (*)$$

$$\Rightarrow \quad a \equiv -(m-1)bv'u \mod v$$

This equation can be solved for $b \in \mathbb{C}[x]$ with $\deg b < \deg v$ because $\gcd(uv', v) = 1$ and $m \neq 1$.

Once b is known, we can determine $c \in \mathbb{C}[x]$ with $\deg c < \deg u + (m-1)\deg v$ by $(*)$.

This leads to the following algorithm:

<u>Alg 1</u> (Hermite reduction)

Input: $f \in C(x)$

Output: $g, h \in C(x)$ st $f = D(g) + h$ and
h has square free denominator.

(1) Write $f = p + \frac{a}{d}$ for $p, a, d \in C[x]$
with $\deg a < \deg d$.

(2) Let $d = d_1 d_2^2 \cdots d_m^m$ be the square free
decomposition of d

(3) $u = d_1 d_2^2 \cdots d_{m-1}^{m-1}$; $v = d_m$; $g = \int p$

(4) while $m \geq 2$ do

(5) find b, c as described above

(6) $g = g + \frac{b}{v^{m-1}}$; $a = c$

$u = u / d_{m-1}^{m-1}$; $v = d_{m-1}$.

$m = m - 1$

(7) return g and $h := f - D(g)$.

<u>Thm 2</u>: $f \in C(x)$ is integrable in $C(x)$
$\Longleftrightarrow$ Alg 1 applied to f returns (g, h)
with $h = 0$.

<u>Proof:</u> „$\Leftarrow$" clear.

„$\Rightarrow$" Observe that

$$f \text{ is integrable} \iff f - D(g) \text{ is integrable for every } g \in C(x).$$

Therefore it suffices to show that $h \neq 0$ with squarefree denominator is not integrable in $C(x)$. Assume otherwise, say

$$h = D\left(\frac{p}{q}\right)$$

for some $p, q \in C[x]$. Let v be an irred. factor of q ~~of~~ of multiplicity $m \geq 1$, so that $p = uv^m$ for some u with $\gcd(p, v) = \gcd(u, v) = \gcd(v', v) = 1$. Then

$$D\left(\frac{p}{uv^m}\right) = \frac{p'uv^m - pu'v^m - m\,puv^{m-1}v'}{u^2v^{2m}}$$

$$= \frac{p'uv - pu'v - m\,puv'}{u^2v^{m+1}}$$

Because of $\gcd(puv', v) = 1$ and $m \geq 1$, v does not divide the ~~de~~ numerator, and therefore v^{m+1} must divide the denominator of h, in contradiction to its squarefreeness. $\square$