Introduction to Unification Theory
Equational Unification

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Overview

Outline

Motivation

- Unifications algorithms are essential components for deduction systems.
- Simple integration of axioms that describe the properties of equality often leads to an unacceptable increase of search space.
- Proposed solution: To build equational axioms into inference, replacing syntactic unification with equational unification.

Motivation

Example

Given: *AI*-theory $\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,x) \approx x\}$. Apply idempotence to the term

$$f(x_0, f(x_1, \ldots, f(x_{n-1}, f(x_n, f(x_0, \ldots, f(x_{n-1}, x_n) \ldots))) \ldots)).$$

- ▶ Exponentially many ways of rearranging the parentheses with the help of associativity: Very time consuming if the prover has to search for the right one.
- ▶ A human mathematician would use words instead of terms, i.e. would work modulo associativity, and apply idempotence $xx = x$ to the word $x_0 \cdots x_n x_0 \cdots x_n$ by unifying x with $x_0 \cdots x_n$.
- ▶ To adopt this way of proceeding for a prover, we must replace the syntactic unification algorithm in the resolution step by associative unification.

Outline

Equational Theory

Equational Theory

- E: a set of equations over $\mathcal{T}(\mathcal{F}, \mathcal{V})$, called identities.
- Equational theory $\doteq_E$ defined by E: The least congruence relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ closed under substitution and containing E
  i.e., $\doteq_E$ is the least binary relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the properties:
  - $E \subseteq \ \doteq_E$.
  - Reflexivity: $s \doteq_E s$ for all s.
  - Symmetry: If $s \doteq_E t$ then $t \doteq_E s$ for all s, t.
  - Transitivity: If $s \doteq_E t$ and $t \doteq_E r$ then $s \doteq_E r$ for all s, t, r.
  - Congruence: If $s_1 \doteq_E t_1, \ldots, s_n \doteq_E t_n$ then $f(s_1, \ldots, s_n) \doteq_E f(t_1, \ldots, t_n)$ for all s, t, n and n-ary f.
  - Closure under substitution: If $s \doteq_E t$ then $s\sigma \doteq_E t\sigma$ for all s, t, σ.

Notation, Terminology

- Identities: $s \approx t$.
- $s \doteq_E t$: The term s is equal modulo E to the term t.
- E will be called an equational theory as well (abuse of the terminology).
- $sig(E)$: The set of function symbols that occur in E.

Example

- $C := \{f(x, y) \approx f(y, x)\}$: f is commutative.
  $sig(C) = f$.
- $f(f(a, b), c) \doteq_C f(c, f(b, a))$.
- $AU := \{f(f(x, y), z) \approx f(x, f(y, z)), f(x, e) \approx x,$
  $\qquad f(e, x) \approx x\}$: f is associative, e is unit.
  $sig(AU) = \{f, e\}$
- $f(a, f(x, f(e, a))) \doteq_{AU} f(f(a, x), a)$.

Notation, Terminology

E-Unification Problem, E-Unifier, E-Unifiability

- E: equational theory.
  $\mathcal{F}$: set of function symbols.
  $\mathcal{V}$: countable set of variables.

- E-Unification problem over $\mathcal{F}$: a finite set of equations

$$\Gamma = \{s_1 \doteq_E^? t_1, \ldots, s_n \doteq_E^? t_n\},$$

  where $s_i, t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

- E-Unifier of Γ: a substitution σ such that

$$s_1\sigma \doteq_E t_1\sigma, \ldots, s_n\sigma \doteq_E t_n\sigma.$$

- $u_E(\Gamma)$: the set of E-unifiers of Γ. Γ is E-unifiable iff $u_E(\Gamma) \neq \emptyset$.

E-Unification vs Syntactic Unification

- ▶ Syntactic unification: a special case of E-unif. with $E = \emptyset$.
- ▶ Any syntactic unifier of an E-unification problem Γ is also an E-unifier of Γ.
- ▶ For $E \neq \emptyset$, $u_E(\Gamma)$ may contain a unifier that is not a syntactic unifier.

Example

- ▶ Terms $f(a, x)$ and $f(b, y)$:
  - ▶ Not syntactically unifiable.
  - ▶ Unifiable module commutativity of f.
    C-unifier: $\{x \mapsto b, y \mapsto a\}$
- ▶ Terms $f(a, x)$ and $f(y, b)$:
  - ▶ Have the most general syntactic unifier $\{x \mapsto b, y \mapsto a\}$.
  - ▶ If f is associative, then $u_A(\{f(a, x) \doteq^?_A f(y, b)\})$ contains additional A-unifiers, e.g. $\{x \mapsto f(z, b), y \mapsto f(a, z)\}$.

Notions Adapted

Instantiation Quasi-Ordering (Modified)

- E: equational theory. $\mathcal{X}$: set of variables.
- A substitution σ is *more general modulo E on $\mathcal{X}$* than ϑ, written $\sigma \leq_E^{\mathcal{X}} \vartheta$, if there exists η such that $x\sigma\eta \doteq_E x\vartheta$ for all $x \in \mathcal{X}$.
- ϑ is called an *E-instance* of σ modulo E on $\mathcal{X}$.
- The relation $\leq_E^{\mathcal{X}}$ is quasi-ordering, called *instantiation quasi-ordering*.
- $\simeq_E^{\mathcal{X}}$ is the equivalence relation corresponding to $\leq_E^{\mathcal{X}}$.

No Single MGU

- When comparing unifiers of Γ, the set $\mathcal{X}$ is $vars(\Gamma)$.
- Unifiable E-unification problems might not have an mgu.

Example

- f is commutative.
- $\Gamma = \{f(x, y) \doteq^?_C f(a, b)\}$ has two C-unifiers:

$$\sigma_1 = \{x \mapsto a, y \mapsto b\}$$
$$\sigma_2 = \{x \mapsto b, y \mapsto a\}.$$

- On $vars(\Gamma) = \{x, y\}$, any unifier is equal to either σ_1 or σ_2.
- σ_1 and σ_2 are not comparable wrt $\leq^{\{x,y\}}_C$.
- Hence, no mgu for Γ.

MCSU vs MGU

In E-unification, the role of mgu is taken on by a complete set of E-unifiers.

Complete and Minimal Complete Sets of E-Unifiers

- Γ: E-unification problem over $\mathcal{F}$.
- $\mathcal{X} = vars(\Gamma)$.
- $\mathcal{C}$ is a *complete set of E-unifiers* of Γ iff
    1. $\mathcal{C} \subseteq u_E(\Gamma)$: $\mathcal{C}$'s elements are E-unifiers of Γ, and
    2. For each $\vartheta \in u_E(\Gamma)$ there exists $\sigma \in \mathcal{C}$ such that $\sigma \leq_E^{\mathcal{X}} \vartheta$.
- $\mathcal{C}$ is a *minimal complete set of E-unifiers* ($mcsu_E$) of Γ if it is a complete set of E-unifiers of Γ and
    3. two distinct elements of $\mathcal{C}$ are not comparable wrt $\leq_E^{\mathcal{X}}$.
- σ is an mgu of Γ iff $mcsu_E(\Gamma) = \{\sigma\}$.

MCSU's

- $mcsu_E(\Gamma) = \emptyset$ if Γ is not E-unifiable.
- Minimal complete sets of unifiers do not always exist.
- When they exist, they may be infinite.
- When they exist, they are unique up to $\doteq \frac{\mathcal{X}}{E}$.

Outline

Unification Type

Unification Type of a Problem, Theory.

- E: equational theory.
- Γ: E-unification problem over $\mathcal{F}$.
- Γ has *unification type*
  - *unitary,* if $mcsu(\Gamma)$ has cardinality at most one,
  - *finitary,* if $mcsu(\Gamma)$ has finite cardinality,
  - *infinitary,* if $mcsu(\Gamma)$ has infinite cardinality,
  - *zero,* if $mcsu(\Gamma)$ does not exist.
- Abbreviation: type unitary - 1, finitary - ω, infinitary - ∞, zero - 0.
- Ordering: $1 < \omega < \infty < 0$.
- *Unification type* of E wrt $\mathcal{F}$: the maximal type of an E-unification problem over $\mathcal{F}$.

Unification Type

The unification type of an E-equational problem over $\mathcal{F}$ depends both

- on E, and
- on $\mathcal{F}$.

Examples and more details will follow.

Unification Type

Example (Type Unitary)

Syntactic unification.

- The empty equational theory $\emptyset$: Syntactic unification.
- Unitary wrt any $\mathcal{F}$ because any unifiable syntactic unification problem has an mgu.

Unification Type

Example (Type Finitary)

Commutative unification: $\{f(x, y) \approx f(y, x)\}$

- $\{f(x, y) \doteq^?_C f(a, b)\}$ does not have an mgu. C-unification is not unitary.
- Show that it is finitary for any $\mathcal{F}$:
    - Let $\Gamma = \{s_1 \doteq^?_C t_1, \ldots, s_n \doteq^?_C t_n\}$ be a C-unification problem.
    - Consider all possible syntactic unification problems $\Gamma' = \{s'_1 \doteq^? t'_1, \ldots, s'_n \doteq^? t'_n\}$, where $s'_i \doteq_C s_i$ and $t'_i \doteq_C t_i$ for each $1 \leq i \leq n$.
    - There are only finitely many such Γ's, because the C-equivalence class for a given term t is finite.
    - It can be shown that collection of all mgu's of Γ's is a complete set of C-unifiers of Γ. This set if finite.
    - If this set is not minimal (often the case), it can be minimized by removing redundant C-unifiers.

Unification Type

Example (Type Infinary)

Associative unification: $\{f(f(x,y),z) \approx f(x,f(y,z))\}$.

- $\{f(x,a) \doteq_A^? f(a,x)\}$ has an infinite *mcsu*:
  $\{\{x \mapsto a\}, \{x \mapsto f(a,a)\}, \{x \mapsto f(a,f(a,a))\}, \ldots\}$
- Hence, A-unification can not be unitary or finitary.
- It is not of type zero because any A-unification problem has an *mcsu* that can be enumerated by the procedure from

  📄 G. Plotkin.
  Building in equational theories.
  In B. Meltzer and D. Michie, editors, *Machine Intelligence*, volume 7, pages 73–90. Edinburgh University Press, 1972.

- A-unification is infinitary for any $\mathcal{F}$.

Unification Type

Example (Type Zero)

Associative-Idempotent unification:
$\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,x) \approx x\}$.

- $\{f(x,f(y,x)) \doteq^?_{AI} f(x,f(z,x))\}$ does not have a minimal complete set of unifiers, see

  📄 F. Baader.
  Unification in idempotent semigroups is of type zero.
  J. Automated Reasoning, 2(3):283–286, 1986.

- *AI*-unification is of type zero.

Unification Type. Signature Matters

Associative-commutative unification with unit:

$$ACU = \{f(f(x,y),z) \approx f(x,f(y,z)), f(x,y) \approx f(y,x), f(x,e) \approx x\}.$$

- ▶ Any ACU problem built using only f and variables has an mgu (i.e. is unitary).
- ▶ There are ACU problems that contain function symbols other than f and e, which are finitary, not unitary.
  For instance, $mcsu(\{f(x,y) \doteq^?_{ACU} f(a,b)\})$ consists of four unifiers (which ones?).

Kinds of E-unification.

Kinds of E-Unification

One may distinguish three kinds of E-unification problems, depending on the function symbols that are allowed to occur in them.

E-Unification Problems: Elementary, with Constants, General.

- E: an equational Theory.
  Γ: an E-unification problem over $\mathcal{F}$.
- Γ is an elementary E-unification problem iff $\mathcal{F} = sig(E)$.
- Γ is an E-unification problem with constants iff $\mathcal{F} \setminus sig(E)$ consists of constants.
- Γ is a general E-unification problem iff $\mathcal{F} \setminus sig(E)$ may contain arbitrary function symbols.

Unification Types of Theories wrt Kinds

- Unification type of E wrt elementary unification: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} = sig(E)$.

- Unification type of E wrt unification with constants: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} \setminus sig(E)$ is a set of constants.

- Unification type of E wrt general unification: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} \setminus sig(E)$ is a set of arbitrary function symbols.

Unification Types of Theories wrt Kinds

The same equational theory can have different unification types for different kinds. Examples:

- ACU (Abelian monoids): Unitary wrt elementary unification, finitary wrt unification with constants and general unification.
- AG (Abelian groups): Unitary wrt elementary unification and unification with constants, finitary wrt general unification.

Decision and Unification Procedures

- **Decision procedure** for an equational theory E (wrt $\mathcal{F}$): An algorithm that for each E-unification problem Γ (wrt $\mathcal{F}$) returns *success* if Γ is E-unifiable, and *failure* otherwise.

- E is **decidable** if it admits a decision procedure.

- (Minimal) E**-unification algorithm** (wrt $\mathcal{F}$): An algorithm that computes a (minimal) finite complete set of E-unifiers for all E-unification problems over $\mathcal{F}$.

- E-unification algorithm yields a decision procedure for E.

- (Minimal) E**-unification procedure**: A procedure that enumerates a possible infinite (minimal) complete set of E-unifiers.

- E-unification procedure does not yield a decision procedure for E.

Decidability wrt Kinds

Decidability of an equational theory might depend on the kinds of E-unification.

- ▶ There exists an equational theory for which elementary unification is decidable, but unification with constants is undecidable:

  📄 H.-J. Bürckert.
  Some relationships between unification, restricted unification, and matching.
  In J. Siekmann, editor, *Proc. 8th Int. Conference on Automated Deduction*, volume 230 of *LNCS*. Springer, 1986.

Single Equation vs Systems of Equations

- ▶ In syntactic unification, solving systems of equations can be reduced to solving a single equation.
- ▶ For equational unification, the same holds only for general unification.
- ▶ For elementary unification and for unification with constants it is not the case.

Single Equation vs Systems of Equations

There exists an equational theory E such that

- all elementary E-unification problems of cardinality 1 (single equations) have minimal complete sets of E-unifiers, but

- E is of type zero wrt to elementary unification: There exists an elementary E-unification problem of cardinality that does not have a minimal complete set of unifiers.

📄 H.-J. Bürckert, A. Herold, and M. Schmidt-Schauß.
On equational theories, unification, and decidability.
J. Symbolic Computation **8**(3,4), 3–49. 1989.

Single Equation vs Systems of Equations

There exists an equational theory E such that

- unifiability of elementary E-unification problems of cardinality 1 (single equations) is decidable, but

- for elementary problems of larger cardinality it is undecidable.

📄 P. Narendran and H. Otto.
Some results on equational unification.
In M. E. Stickel, editor, *Proc. 10th Int. Conference on Automated Deduction*, volume 449 of *LNAI*. Springer, 1990.

Three Main Questions in Unification Theory

For a given E, unification theory is mainly concerned with finding answers to the following three questions:

Decidability: Is it decidable whether an E-unification problem is solvable? If yes, what is the complexity of this decision problem?

Unification type: What is the unification type of the theory E?

Unification algorithm: How can we obtain an (efficient) E-unification algorithm, or a (preferably minimal) E-unification procedure?

The answers depend on whether we consider elementary unification, unification with constants, or general unification.

Outline

Results for Specific Theories

General unification:

Theory	Decidability	Type	Algorithm/Procedure
$\emptyset$	Yes	1	Yes
A	Yes	∞	Yes
C	Yes	ω	Yes
I	Yes	ω	Yes
AC	Yes	ω	Yes
AI	Yes	0	?
CI	Yes	ω	Yes
ACI	Yes	ω	Yes
AU	Yes	∞	Yes
AG	Yes	ω	Yes
CRU	No	? (∞ or 0)	?

CRU - Commutative ring with unit

Example. C-Unification

- C-unification algorithm $\mathcal{U}_C$ can be obtained from the inference system $\mathcal{U}$ by adding the C-Decomposition rule:

  C-Decomposition: $\{f(s_1, s_2) \doteq^?_C f(t_1, t_2)\} \cup P'; S \Longrightarrow$
  $$\{s_1 \doteq^?_C t_2, s_2 \doteq^?_C t_1\} \cup P'; S,$$
  if f is commutative.

- **C-Decomposition** and **Decomposition** transform the same system in different ways.
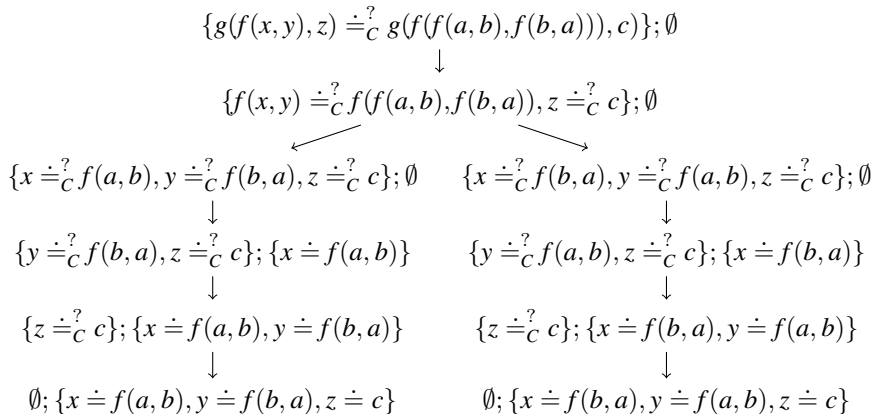
Example. *C*-Unification

In order to C-unify s and t:

1. Create an initial system $\{s \stackrel{?}{=}_C t\}; \emptyset$.
2. Apply successively rules from $\mathcal{U}_C$, building a complete tree of derivations. **C-Decomposition** and **Decomposition** rules have to be applied concurrently and form branching points in the derivation tree.

Example. C-Unification

C-unify $g(f(x,y),z)$ and $g(f(f(a,b),f(b,a)),c)$, commutative f.

$$\{g(f(x,y),z) \doteq_C^? g(f(f(a,b),f(b,a))),c)\}; \emptyset$$
$$\downarrow$$
$$\{f(x,y) \doteq_C^? f(f(a,b),f(b,a)), z \doteq_C^? c\}; \emptyset$$

$$\{x \doteq_C^? f(a,b), y \doteq_C^? f(b,a), z \doteq_C^? c\}; \emptyset \qquad \{x \doteq_C^? f(b,a), y \doteq_C^? f(a,b), z \doteq_C^? c\}; \emptyset$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$\{y \doteq_C^? f(b,a), z \doteq_C^? c\}; \{x \doteq f(a,b)\} \qquad \{y \doteq_C^? f(a,b), z \doteq_C^? c\}; \{x \doteq f(b,a)\}$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$\{z \doteq_C^? c\}; \{x \doteq f(a,b), y \doteq f(b,a)\} \qquad \{z \doteq_C^? c\}; \{x \doteq f(b,a), y \doteq f(a,b)\}$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$\emptyset; \{x \doteq f(a,b), y \doteq f(b,a), z \doteq c\} \qquad \emptyset; \{x \doteq f(b,a), y \doteq f(a,b), z \doteq c\}$$

$\{\{x \mapsto f(b,a), y \mapsto f(a,b), z \mapsto c\}, \{x \mapsto f(a,b), y \mapsto f(b,a), z \mapsto c\}\}$
Not minimal.

Example. *ACU*-Unification

$$ACU = \{f(f(x,y),z) \approx f(x,f(y,z)), f(x,y) \approx f(y,x), f(x,e) \approx x\}$$

Elementary *ACU*-unification problem:

$$\Gamma = \{f(x,f(x,y)) \doteq^?_{ACU} f(z,f(z,z))\}$$

Solving idea:

1. Associate with the equation in Γ a homogeneous linear Diophantine equation. The Diophantine equation states that the number of new variables introduced by a unifier σ in both sides of $\Gamma\sigma$ must be the same:

$$2x + y = 3z.$$

(Continues on the next slide.)

Example. ACU-Unification

Solving (Cont.):

2. Solve $2x + y = 3z$ over nonnegative integers. Three minimal solutions:

$$x = 1, y = 1, z = 1$$
$$x = 0, y = 3, z = 1$$
$$x = 3, y = 0, z = 2$$

Any other solution of the equation can be obtained as a nonnegative linear combination of these three solutions.

(Continues on the next slide.)

Example. ACU-Unification

Solving (Cont.):

3. Introduce new variables v_1, v_2, v_3 for each solution of the Diophantine equation:

	x	y	z
v_1	1	1	1
v_2	0	3	1
v_3	3	0	2

4. Each row corresponds to a unifier of Γ:

$$\sigma_1 = \{x \mapsto v_1, y \mapsto v_1, z \mapsto v_1\}$$
$$\sigma_2 = \{x \mapsto e, y \mapsto f(v_2, f(v_2, v_2)), z \mapsto v_2\}$$
$$\sigma_3 = \{x \mapsto f(v_3, f(v_3, v_3)), y \mapsto e, z \mapsto f(v_3, v_3)\}$$

However, none of them is an mgu.

Example. *ACU*-Unification

Solving (Cont.):

5. To obtain an mgu, we should combine all three solutions:

	x	y	z
v_1	1	1	1
v_2	0	3	1
v_3	3	0	2

Looking at columns: They state that the mgu we are looking for should have

- in the binding for x one v_1, zero v_2, and three v_3's,
- in the binding for y one v_1, three v_2's, and zero v_3,
- in the binding for z one v_1, one v_2, and two v_3's

6. Hence, we can construct the mgu:

$$\sigma = \{x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))),$$
$$z \mapsto f(v_1, f(v_2, f(v_3, v_3)))\}$$

Example. ACU-Unification

Exercise.
Verify that the unifiers σ_1, σ_2 and σ_3 are instances of σ.

Example. E-Unification of Type 0

Example

- Equational theory: $E = \{f(e, x) \approx x, g(f(x, y)) \approx g(y)\}$.
- E-unification problem: $\Gamma = \{g(x) \doteq^?_E g(e)\}$.
- Complete (why?) set of solutions:

$$
\begin{aligned}
\sigma_0 =& \{x \mapsto e\} \\
\sigma_1 =& \{x \mapsto f(x_0, e)\} \\
\sigma_2 =& \{x \mapsto f(x_1, f(x_0, e))\} \\
& \cdots \\
\sigma_n =& \{x \mapsto f(x_{n-1}, x\sigma_{n-1})\} \\
& \cdots
\end{aligned}
$$

- No *mcsu*. $\sigma_i \doteq^{\{x\}}_E \sigma_{i+1}\{x_i \mapsto e\}$. $\sigma_i \not\preceq^{\{x\}}_E \sigma_j$ for $i > j$.
  Infinite descending chain: $\sigma_0 >^{\{x\}}_E \sigma_1 >^{\{x\}}_E \sigma_2 >^{\{x\}}_E \cdots$

Example. E-Unification of Type 0

Example (Cont.)
Why does $\sigma_0 >_E^{\{x\}} \sigma_1 >_E^{\{x\}} \sigma_2 >_E^{\{x\}} \cdots$ imply that there is no *mcsu*?

- Let $S = \{\sigma_0, \sigma_1, \ldots\}$.
- Let S' be an arbitrary complete set of unifiers of Γ.
- Since S is complete, for any $\vartheta \in S'$ there exists $\sigma_i \in S$ such that $\sigma_i \leq_E^{\{x\}} \vartheta$.
- Since $\sigma_{i+1} <_E^{\{x\}} \sigma_i$, we get $\sigma_{i+1} <_E^{\{x\}} \vartheta$.
- On the other hand, since S' is complete, there exists $\eta \in S'$ such that $\eta \leq_E^{\{x\}} \sigma_{i+1}$.
- Hence, $\eta <_E^{\{x\}} \vartheta$ which implies that S' is not minimal.

Specific vs General Results

For each specific equational theory separately studying

- decidability,
- unification type,
- unification algorithm/procedure.

Can one study these problems for bigger classes of equational theories?

Outline

General Results

In general, unification modulo equational theories

- is undecidable,
- unification type of a given theory is undecidable,
- admits a complete unification procedure (Gallier & Snyder, called an universal E-unification procedure).

General Results

Universal E-unification procedure $\mathcal{U}_E$.

Rules:

- **Trivial**, **Orient**, **Decomposition**, **Variable Elimination**
  from $\mathcal{U}$, plus

- **Lazy Paramodulation:**

$$\{e[u]\} \cup P'; S \Longrightarrow \{l \stackrel{?}{=} u, e[r]\} \cup P'; S,$$

for a fresh variant of the identity $l \approx r$ from $E \cup E^{-1}$, where

  - $e[u]$ is an equation where the term u occurs,
  - u is not a variable,
  - if l is not a variable, then the top symbol of l and u are the same.

General Results

Universal E-unification procedure. Control.

In order to solve a unification problem Γ modulo a given E:

- ▶ Create an initial system $\Gamma; \emptyset$.
- ▶ Apply successively rules from $\mathcal{U}_E$, building a complete tree of derivations.
- ▶ No other inference rule may be applied to the equation $l \doteq^? u$ that is generated by the Lazy Paramodulation rule before it is subjected to a Decomposition step.

General Results

Universal E-unification procedure.

Example

$E = \{f(a, b) \approx a, a \approx b\}$.

Unification problem: $\{f(x, x) \doteq^?_E x\}$.

Computing a unifier $\{x \mapsto a\}$ by the universal procedure:

$$
\begin{aligned}
\{f(x, x) \doteq^?_E x\}; \emptyset &\Longrightarrow_{LP} \{f(a, b) \doteq^?_E f(x, x), a \doteq^?_E x\}; \emptyset \\
&\Longrightarrow_D \{a \doteq^?_E x, b \doteq^?_E x, a \doteq^?_E x\}; \emptyset \\
&\Longrightarrow_O \{x \doteq^?_E a, b \doteq^?_E x, a \doteq^?_E x\}; \emptyset \\
&\Longrightarrow_S \{b \doteq^?_E a, a \doteq^?_E a\}; \{x \doteq a\} \\
&\Longrightarrow_{LP} \{a \doteq^?_E a, b \doteq^?_E b, a \doteq^?_E a\}; \{x \doteq a\} \\
&\Longrightarrow^+_T \emptyset; \{x \doteq a\}
\end{aligned}
$$

Pros and cons of the universal procedure:

- ▶ Pros: Is sound and complete. Can be used for any E.
- ▶ Cons: Very inefficient. Usually does not yield a decision procedure or a (minimal) E-unification algorithm even for unitary or finitary theories with decidable unification.

General Results

More useful results can be obtained by imposing additional restrictions on equational theories:

- ▶ Syntactic approaches: Restricting syntactic form of the identities defining equational theories.
- ▶ Semantic approaches: Depend on properties of the free algebras defined by the equational theory.