II ODEs

Task: Given a differential field K, some elements $a_0, a_1, \dots a_r \in K$, and a differential ring R containing K, find all $y \in R$ st

$$a_0 y + a_1 D(y) + \dots + a_r D^r(y) = 0 \qquad \text{(ODE)}$$

Def: If $a_r \neq 0$, then r is called the order of the equation. If $a_r = 1$, the equation is called monic.

Note: The solutions y form a vector space over $C := \mathrm{Const}\, K$: if $y_1, y_2 \in R$ are solutions and $c_1, c_2 \in C$, then $c_1 y_1 + c_2 y_2$ is a solution as well, because D is a C-linear map. The set of all solutions of (ODE) is therefore called the "solution space".

Thm. Assume that R is (contained in) a field. Let V be the solution space of (ODE) in R. Then $\dim_C V \leq r$.

Proof: Let $y_1, y_2, \dots, y_{r+1} \in V \setminus \{0\}$. To show: They are linearly dependent over C.

First: They are linearly dependent over K, because

$$\underbrace{\begin{pmatrix} y_1 & D(y_1) & \cdots & D^r(y_1) \\ \vdots & \ddots & & \vdots \\ y_{r+1} & D(y_{r+1}) & \cdots & D^r(y_{r+1}) \end{pmatrix}}_{=: A \in R^{(r+1)\times(r+1)}} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_r \end{pmatrix}$$

$\Rightarrow$ rank $A < r+1 \Rightarrow$ rank $A^T < r+1$

$\Rightarrow \exists (b_1 \dots b_{r+1}) \in R^{r+1} \setminus \{0\} : b_1 y_1 + \dots + b_{r+1} y_{r+1} = 0.$

For these b_i, we even have

$$b_1 \begin{pmatrix} y_1 \\ \vdots \\ D^r(y_1) \end{pmatrix} + \dots + b_{r+1} \begin{pmatrix} y_{r+1} \\ \vdots \\ D^r(y_{r+1}) \end{pmatrix} = 0.$$

Next: The vectors $Y_i := \begin{pmatrix} y_i \\ \vdots \\ D^{r-1}(y_i) \end{pmatrix}$ $(i = 1 .. r+1)$

(and hence in particular their first coordinates) are linearly dependent over C.

Let $n \in \{1, \dots r\}$ be such that

$$\left\{ \begin{pmatrix} Y_1 \\ D^r(y_1) \end{pmatrix}, \dots, \begin{pmatrix} Y_{n+1} \\ D^r(y_{n+1}) \end{pmatrix} \right\} \subseteq \mathbb{R}^{r+1}$$

is linearly dependent over K and every proper subset is not. Then there are $u_i \in K$, not all zero, with

$$\begin{pmatrix} Y_{n+1} \\ D^r(y_{n+1}) \end{pmatrix} = \sum_{i=1}^{n} u_i \begin{pmatrix} Y_i \\ D^r(y_i) \end{pmatrix}$$

$$\swarrow \qquad \qquad \searrow$$

$$Y_{n+1} = \sum_{i=1}^{n} u_i Y_i \qquad \qquad D(Y_{n+1}) = \sum_{i=1}^{n} u_i D(Y_i)$$

$$\Downarrow$$

$$D(Y_{n+1}) = \sum_{i=1}^{n} D(u_i) Y_i + \sum_{i=1}^{n} u_i D(Y_i) \quad \Longrightarrow \quad \sum_{i=1}^{n} D(u_i) Y_i = 0$$

$\Rightarrow$ $D(u_i) = 0$ for all i $\Rightarrow$ $u_i \in C$ for all i. $\boxtimes$

<u>Def</u>: Any set of r C-linearly independent solutions of (ODE) is called a fundamental system of the equation.

Depending on the equation and the choice of R, such a system may or may not exist. In fact, $\dim_C V$ can be equal to any number in $\{0, \dots, r\}$.

<u>Ex:</u> Consider $y'' - 2y' + y = 0$.

If $R = C(x, e^x)$, then $V = [e^x, xe^x]$, so $\dim_C V = 2$

If $R = C(e^x)$, then $V = [e^x]$, so $\dim_C V = 1$

If $R = C(x)$, then $V = \{0\}$, so $\dim_C V = 0$.

<u>Remark:</u> In general, if $p = \sum_{j=0}^{d} p_j x^j = \prod_{i=1}^{n} (x-u_i)^{e_i} \in C[x]$ for pw dist $u_i \in \overline{C}$, and if R is sufficiently rich, a fundamental system of

$$p_0 y + p_1 y' + \dots + p_r D^r(y) = 0$$

is given by

$$\{x^i e^{u_j x} \mid j = 1..n,\ i = 0..e_j - 1\}.$$

In other words: ODEs with constant coeffs can always be solved completely in closed form. For more general coefficient domains K, this is no longer the case.

① Polynomial Solutions

Here $K = (C(x), \frac{d}{dx})$ and $R = C[x]$.

<u>Task:</u> Given $a_0, \dots, a_r \in C(x)$, find all $y \in C[x]$ st $a_0 y + a_1 y' + \dots + a_r D^r(y) = 0$.

Wlog we may assume that $a_0 \dots a_r \in C[x]$ by clearing denominators.

Solutions $y \in C[x]$ of limited degree (i.e. $\deg y \leq d$ for some fixed $d \in \mathbb{N}$) are easy to find:

(1) make an ansatz $y = \sum_{i=0}^{d} y_i x^i$ with undetermined coeffs $y_0 \dots y_d$.

(2) Plug the ansatz into the equation using that

$$D(y) = \sum_{i=0}^{d} y_i \, i x^{i-1},$$

$$D^2(y) = \sum_{i=0}^{d} y_i \, i(i-1) x^{i-2}, \quad \text{etc}$$

regardless of the values of y_i.

The resulting equation has the form

$$\sum_{j=0}^{n} \left(\sum_{i=0}^{d} c_{ij} y_i \right) x^j = 0$$

for some explicit $c_{ij} \in \mathbb{C}$.

(3) equate the coeffs of x^j to zero, i.e. compute the nullspace of

$$A := \begin{pmatrix} c_{00} & \cdots & c_{d0} \\ \vdots & \ddots & \vdots \\ c_{0n} & \cdots & c_{dn} \end{pmatrix} \in \mathbb{C}^{n \times d}$$

We have

$$(y_0 \dots y_d) \in \ker A$$

$\iff$ $y = y_0 + y_1 x + \dots + y_d x^d$ is a solution

$\square$

Let $V \subseteq \mathbb{C}[x]$ be the set of all solutions and $V_d \subseteq \mathbb{C}[x]$ be the set of solutions of degree at most d. Since $\dim_{\mathbb{C}} V \leq r$, there must be some $d \in \mathbb{N}$ with $V = V_d$. How to find it?

Let $y = y_d x^d + \dots$ be a solution of degree d. Wlog $y_d = 1$ because V is a $\mathbb{C}$-vector space.

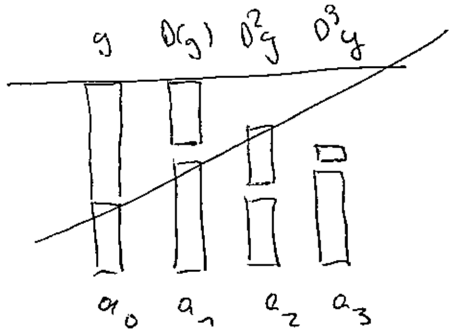
We have
$$y' = d x^{d-1} + \dots$$
$$y'' = d(d-1) x^{d-2} + \dots$$
$$\vdots$$
$$D^r y = d^{\underline{r}} x^{d-r} + \dots$$

Hence $\deg a_i D^i(y) = \deg a_i + (d-i)$.

Let $n = \max_{i=0}^{r} (\deg a_i - i)$.

Then
$$[x^{n+d}]\, a_i D^i y = \begin{cases} lc(a_i)\, d^{\underline{i}} & \text{if } n = \deg a_i - i \\ 0 & \text{else.} \end{cases}$$

By the choice of n, this term will not be zero for all i.



Therefore $$I(d) := \sum_{i=0}^{r} \left([x^{n+i}]a_i\right) \cdot d^{\underline{i}}$$

is a nonzero polynomial in $C[d]$.

It is called the indicial polynomial of the equation.

Feature: If y is a polynomial solution of degree d then d is an integer root of the indicial polynomial.

Hence: $\delta := \max\{d \in \mathbb{N} \mid I(d) = 0\}$ is a valid bound on the degree of the elements in V.