Algebraic and Discrete Methods in Biology

First-Order Predicate Logic

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Outline

Motivation

Example

Everything greater than 0 has a square root.

This statement can be interpreted, for example, in arithmetic on $\mathbb{R}$, the set of real numbers, where it is true.

It can be also interpreted, for example, in arithmetic on $\mathbb{Q}$, the set of rational numbers, where it is false.

Formalization

For all x ($0 < x \implies$ there is some y such that $x = y \cdot y$).

Better Formalization

$\forall x\ (0 < x \implies \exists y\ (x = y \cdot y))$

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Terms and Formulae

Terms

- Constants: 0, 1, 230, *John*, $\mathbb{N}$, … ;
- Variables: $x, y, z, \ldots$;
- Function symbols: $+$, $-$, *Son_of*, … .

Formulae

- Predicates: $=$, $>$, $\not\leq$, *Is_Son_of*, … ;
- Connectives: $\wedge$, $\vee$, $\neg$, $\Longrightarrow$, $\Longleftrightarrow$;
- Quantifiers: $\forall$, $\exists$.

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- Constants are terms.
- Variables are terms.
- If f is a function symbol of arity n and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.
- All terms are generated by applying the above rules.

Examples

- x, $z + 32$, $f(7, a, x(y + 3))$ are terms.
- $x + 5 \not< 7$, $\forall x \exists y\,(x = y)$ are not terms.

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In the formula:

$$\forall x\, (0 < x \implies \exists y\, (x = y \cdot y))$$

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$\forall x \, (0 < x \implies \exists y \, (x = y \cdot y))$

What is the semantical meaning of the formula? Does it express what we want?

When interpreted in arithmetic on $\mathbb{R}$, is it really true?

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Intuitively: it is the place where everything happens.
All constant symbols from the language will be *interpreted* as some elements from D.
All variables from the language will range over D, etc.

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Let D be non-empty set.
Interpretation I over D is a function defined in the following way:

- for any constant symbol c, $\; c^I \in D$;
- for any function symbol f of arity n, $\; f^I : D^n \to D$;
- for any predicate symbol p of arity n, $\; p^I : D^n \to \{\mathbb{T}, \mathbb{F}\}$.

Truth Evaluation Under Interpretation

Definition

Let D be a domain and I be an interpretation over D. Then:

- $\langle \mathbb{T} \rangle^I = \mathbb{T}$ and $\langle \mathbb{F} \rangle^I = \mathbb{F}$.
- If p is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms, then $\langle p(t_1, \ldots, t_n) \rangle^I = p^I(\langle t_1 \rangle^I, \ldots, \langle t_n \rangle^I)$.
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  $\langle \varphi \wedge \psi \rangle^I = \langle \varphi \rangle^I \wedge \langle \psi \rangle^I$,
  $\langle \varphi \vee \psi \rangle^I = \langle \varphi \rangle^I \vee \langle \psi \rangle^I$, etc.
- If φ is a formula and x is a variable, then $\langle \forall x \varphi \rangle^I = \mathbb{T}$ iff for all $d \in D$, $\langle \varphi \rangle^I_{\{x \leftarrow d\}} = \mathbb{T}$.
- If φ is a formula and x is a variable, then $\langle \exists x \varphi \rangle^I = \mathbb{T}$ iff for some $d \in D$, $\langle \varphi \rangle^I_{\{x \leftarrow d\}} = \mathbb{T}$.

Truth Evaluation Under Interpretation

Example

Consider the formula $\forall x\ \exists y(x \leqq y)$.

Let $D = \{0, 1\}$ and I be an interpretation over D, such that $\langle \leqq \rangle^I = \leq$.

$\langle \forall x\ \exists y(x \leqq y) \rangle^I = \mathbb{T}$ iff for all $d \in D$, $\langle \exists y(d \leqq y) \rangle^I = \mathbb{T}$

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  $\langle \exists y(0 \leqq y) \rangle^I = \mathbb{T}$ iff for some $e \in D$, $\langle (0 \leqq e) \rangle^I = \mathbb{T}$.
  Let $e = 0$.
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Example

Consider again the formula $\forall x\ (0 < x \implies \exists y\ (x = y \cdot y))$ on two different domains $\mathbb{Q}$ and $\mathbb{R}$ under *standard* interpretation.

An interpretation I is normally called *standard* for a domain when it interprets the constants, the function symbols and predicate symbols with their standard meaning, e.g.,
$\langle 0 \rangle^I = 0$,
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$\langle \forall x\, (0 < x \implies \exists y\, (x = y \cdot y)) \rangle^I = \mathbb{T}$ iff
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Let r_1 be arbitrary but fixed real, such that $r_1 > 0$.
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Let $r_2 = \sqrt{r_1}$.
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Does the formula hold? May we use the same proof as in the other domain to show it holds?

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