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Exercises

Lecture for March 5, 2024

HW 1. Try to apply the Gauß-method to sum

1. $\sum_{k=0}^{n}(2k+1)$
2. $\sum_{k=1}^{n} k^2$
3. $\sum_{k=1}^{n} k^3$

Find and prove a formula for (a), (b) and (c).

HW 2. Prove for all $n \in \mathbb{N}$ that

$$\sum_{k=0}^{n-1} \frac{k}{(k+1)(k+2)} = H_n - \frac{2n}{n+1}.$$

HW 3. Let $f : \mathbb{Z} \to \mathbb{C}$ and $a, b \in \mathbb{Z}$ with $a \leq b$.

1. For
$$S(a,b) := \sum_{k=a}^{b}(f(k+1) - f(k))$$
show that
$$S(a,b) = f(b+1) - f(a).$$

2. Suppose in addition that $f(k) \neq 0$ for all k with $a \leq k \leq b$. For
$$P(a,b) := \prod_{k=a}^{b}(f(k+1) - f(k))$$
show that
$$P(a,b) = \frac{f(b+1)}{f(a)}.$$

HW 4. Use the previous homework to find a closed form for

$$a_n := \prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right).$$

BP 1. Consider the function $\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Prove: there is no rational function $r(x) \in \mathbb{R}(x)$ (i.e., $r(x) = \frac{p(x)}{q(x)}$ for polynomials $p, q \in \mathbb{R}[x]$) such that

$$\exp(x) = r(x) \quad \forall x \in U$$

where $U \subseteq \mathbb{R}$ is some non-empty open interval.

HW 5. Given a tower of n discs, initially stacked in decreasing size on one of three pegs. Transfer the entire tower to one of the other pegs, moving only one disc at each step and never moving a larger one onto a smaller one. Find a_n, the minimal number of moves ($n \geq 0$).

HW 6. How many slices of pizza can a person maximally obtain by making n straight cuts with a pizza knife. Let P_n ($n \geq 0$) be that number.

BP 2. Prove that there is no rational function $r(x) \in \mathbb{C}(x)$ such that

$$H_n = r(n)$$

holds for all $n \in \mathbb{N}$ with $n \geq \lambda$ for some $\lambda \in \mathbb{N}$.

Lecture from March 12, 2024

HW 7. Show that $H_n \sim \log(n)$.

Lecture from March 19, 2024

HW 8. Let $P(n)$ be defined by $P(1) = 1$ and

$$P(n) = \sum_{i=0}^{n-1} \frac{1}{n}\Big(1 + \frac{i}{n}P(i) + \frac{n-i-1}{n}P(n-i-1)\Big).$$

Show that $P(n) = 1 + \frac{2}{n^2}\sum_{i=0}^{n-1} i\,P(i)$.

HW 9. Let $P(n)$ be the sequence from the previous homework. Show for $n \geq 2$ that

$$n^2\,P(n) - (n-1)(n+1)P(n-1) = 2n - 1.$$

HW 10. Find a representation of $P(n)$ in terms of H_n.

HW 11. Show that

1. $P(n) \in O(\log(n))$;
2. $P(n) \sim 2\log(n)$.

Lecture from April 9, 2024

BP 3. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ as defined in the lecture is a vector space over $\mathbb{K}$.

BP 4. Show that $(\mathbb{K}^{\mathbb{N}}, +, \odot)$ with the Hadamard product $\odot$ is a commutative ring with 1, but not an integral domain.

BP 5. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ with the Cauchy product $\cdot$ is a commutative ring with 1.

HW 12. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ with the Cauchy product $\cdot$ has no zero-divisors.

HW 13. Show: For $\lambda \in \mathbb{K}$ and $m \in \mathbb{N}$ we have

$$(\lambda\, x^m) \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \lambda a_n x^{n+m} = \sum_{n=m}^{\infty} \lambda a_{n-m} x^n;$$

here the multiplication on the left hand side is the standard Cauchy product.

HW 14. For $k \in \mathbb{N}$ and $a(x), b(x) \in \mathbb{K}[[x]]$ show

$$[x^k](a(x) + b(x)) = [x^k]a(x) + [x^k]b(x),$$
$$[x^k](\lambda\, a(x)) = \lambda[x^k]a(x).$$

Lecture from April 16, 2024

HW 15. In $(\mathbb{K}[x], +, \cdot)$ prove

1. $\left(\sum_{n=0}^{\infty} c^n x^n\right)(1 - cx) = 1 \qquad (c \in \mathbb{K})$
2. $\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n\right) = 1.$

HW 16. Show for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ that

$$\binom{z+1}{k} = \binom{z}{k} + \binom{z}{k-1}.$$

HW 17. Consider $D_x : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ with

$$D_x\left(\sum_{n=0}^{\infty} a_n x^k\right) = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n.$$

Show that $(\mathbb{K}[[x]], D_x)$ forms a differential ring, i.e., the following two rules hold: for all $a, b \in \mathbb{K}[[x]]$,

1. $D_x(a + b) = D_x(a) + D_x(b)$
2. $D_x(a \cdot b) = D_x(a)b + aD_x(b).$

Lecture from April 23, 2024

HW 18. Let $\exp(c\,x) := \sum_{n=0}^{\infty} \frac{c^n}{n!} x^n$. For $a, b \in \mathbb{K}$ show:

$$\exp(a\,x)\exp(b\,x) = \exp((a+b)x).$$

HW 19. Find a closed form for the coefficients in the multiplicative inverse of $(1-2x)^2 \in \mathbb{K}[[x]]$.

HW 20. Find a closed form for the coefficients in the multiplicative inverse of $(1-x)^3 \in \mathbb{K}[[x]]$.

HW 21. Find a closed form for the coefficients in the multiplicative inverse of $\exp(2x) \in \mathbb{K}[[x]]$.

HW 22. Consider the formal power series $f(x) = \frac{1}{(1-x)^2} \log(1-x) \in \mathbb{Q}[[x]]$. Express the coefficients $f_n \in \mathbb{Q}$ of $f(x) = \sum_{k=0}^{\infty} f_n x^n$ in terms of the harmonic numbers H_n.

HW 23. Let $g(x) \in \mathbb{K}[[x]]$ with $g(0) = 1$. Show that there is an $f(x) \in \mathbb{K}[[x]]$ with $f(x)^2 = g(x)$ and $f(0) = 1$. (Hint: adapt the construction to invert a formal power series.) Further, conclude (during your construction of $f(x)$) that there is exactly one other solution, namely $-f(x)$.