to be prepared for Januar 16

Exercise 47. Consider polyomials $f, g \in k[x]$ of positive degrees m, n respectively. Let I denote the ideal in $k[x]$ generated by f, and let μ denote the multiplication map

$$\mu\colon k[x]/I \longrightarrow k[x]/I, \quad h + I \mapsto gh + I.$$

Demonstrate that $\mathrm{res}_x(f, g) = LC(f)^{\deg(g)} \det(\mu)$.

Exercise 48. Let $f, g \in k[x]$ be polynomials whose roots are $\zeta_1, \ldots, \zeta_m$ and $\eta_1, \ldots, \eta_n$ respectively. Prove that

$$\mathrm{res}_x(f, g) = LC(f)^{\deg(g)} LC(g)^{\deg(f)} \prod_{i=1}^{m} \prod_{j=1}^{n} (\zeta_i - \eta_j)$$

Exercise 49. Use Gröbner bases for solving over $\mathbb{C}$:

$$\begin{aligned} f_1(x, y, z) &= xz - xy^2 - 4x^2 - \frac{1}{4} = 0, \\ f_2(x, y, z) &= y^2 z + 2x + \frac{1}{2} = 0, \\ f_3(x, y, z) &= x^2 z + y^2 + \frac{1}{2}x = 0. \end{aligned}$$

Exercise 50. Consider the polynomials

$$\begin{aligned} f_1(x, y) &= x^2 y + xy + 1, \\ f_2(x, y) &= y^2 + x + y \end{aligned}$$

in $\mathbb{Z}_3[x, y]$. Compute a Gröbner basis for the ideal $\langle f_1, f_2 \rangle$ w.r.t. the graduated lexicographical ordering with $x < y$. Show intermediate results.

Exercise 51. Use Gröbner bases to find the implicit representation of the parametrized surface

$$\begin{aligned} x &= 1 + s + t + st \\ y &= 2 + s + st + t^2 \\ z &= s + t + s^2 \end{aligned}$$