Name: .... 29 Jan 2008

Studienkennzahl: ....

Matrikelnummer: ....

Final Exam
Computer Algebra (326.017)
(books allowed)

(1) Determine a polynomial $r(x) \in \mathbb{Q}[x]$ which satisfies

$$\begin{aligned} r(x) &\equiv 2x+1 && \bmod x^2+2x+2 \\ r(x) &\equiv -2 && \bmod x-1 \\ r(x) &\equiv -6 && \bmod x+1 \end{aligned}$$

Show also the intermediary steps in the computation.

(2) Factor the polynomial $a(x) = x^4 + 4x^3 + 2x^2 + x + 2$ modulo 5.
Show also the intermediary steps in the computation.

(3) Consider the following polynomials in $\mathbb{Q}[x, y]$:

$$a(x, y) = xy^2 + 2y^2 - xy - 2, \qquad b(x, y) = xy + 2y + x - 4$$

(i) Compute $\mathrm{res}_y(a, b) \in \mathbb{Q}[x]$, the resultant of a and b w.r.t. y;
(ii) Can the solution $x = -2$ of $\mathrm{res}_y(a, b)$ be extended to a solution of the system $a(x, y) = 0 = b(x, y)$? Explain this situation.
(iii) Can the solution $x = 1$ of $\mathrm{res}_y(a, b)$ be extended to a solution of the system $a(x, y) = 0 = b(x, y)$? Explain this situation.

(4) Prove: *If $f, g \in \mathbb{Q}[x, y]$ are relatively prime (i.e. $\gcd(f, g) = 1$), then there are only finitely many pairs $(a, b) \in \mathbb{C}^2$ such that $f(a, b) = 0 = g(a, b)$.*
[Hint: consider the gcd of f and g in $\mathbb{Q}(x)[y]$.]

(5) Consider the polynomials $a(x, y)$ and $b(x, y)$ of Question (3). We want to determine a Gröbner basis G for the ideal generated by $\{a, b\}$ w.r.t. the lexicographic order with $x < y$:
spol(a, b) can be reduced to $c(x, y) = 4y + x - 5$, so we add c to the basis;
spol(a, c) can be reduced to $d(x, y) = x^2 - 7x + 6$, so we add d to the basis;
Complete the algorithm and determine G.

(6) Consider the following polynomials in $\mathbb{Q}[x]$:

$$a(x) = x^3 - x^2 - 3x - 1, \qquad b(x) = x^3 - 3x^2 + x + 1\ .$$

(i) Compute the gcd of a and b;
(ii) compute a Gröbner basis for the ideal generated by a and b;
(iii) how are the results in (i) and (ii) related? Explain.