to be prepared for 11.12.2007

Exercise 40. Consider polyomials $f, g \in k[x]$ of positive degrees m, n respectively. Let I denote the ideal in $k[x]$ generated by f, and let μ denote the multiplication map

$$\mu\colon k[x]/I \longrightarrow k[x]/I, \quad h + I \mapsto gh + I.$$

Demonstrate that $\mathrm{res}_x(f, g) = LC(f)^{\deg(g)} \det(\mu)$.

Exercise 41. Let $f, g \in k[x]$ be polynomials whose roots are $\zeta_1, \ldots, \zeta_m$ and $\eta_1, \ldots, \eta_n$ respectively. Prove that

$$\mathrm{res}_x(f, g) = LC(f)^{\deg(g)} LC(g)^{\deg(f)} \prod_{i=1}^{m} \prod_{j=1}^{n} (\zeta_i - \eta_j)$$

Exercise 42. $f, g \in k[x]$, divide f by g, $f = qg + r$ with $\deg(r) < \deg(g)$. Assuming that $\deg(g) \neq 0$ show that

$$\mathrm{res}_x(f, g) = (-1)^{\deg(f)\deg(g)} LC(g)^{\deg(f)-\deg(r)} \mathrm{res}_x(g, r).$$

Exercise 43. Prove the following lemma [Lemma 4.2.3].[1]
Let $a, b \in \mathbb{Z}[x_1, \ldots, x_{n-2}][y][x]^*$ and $r \in \mathbb{Z}$ such that $y - r$ does not divide both $\mathrm{lc}_x(a)$ and $\mathrm{lc}_x(b)$. Let $c = \gcd(a, b)$.

1. $\deg_x(\gcd(a_{y-r}, b_{y-r})) \geq \deg_x(\gcd(a, b))$.
2. If $y - r \not| \, \mathrm{res}_x(\frac{a}{c}, \frac{b}{c})$ then $\gcd(a_{y-r}, b_{y-r}) = c_{y-r}$.

Exercise 44. Apply the modular algorithm for computing the resultant $\mathrm{res}_y(a, b)$ of the polynomials

$$\begin{aligned} a(x, y) &= xy^2 - x^3y - 2x^2y + xy + 2x^4 - 2x^2 \\ b(x, y) &= 2x^2y^2 - 4x^3y + 4x^4. \end{aligned}$$

[1] F. Winkler, Polynomial Algorithms in Computer Algebra