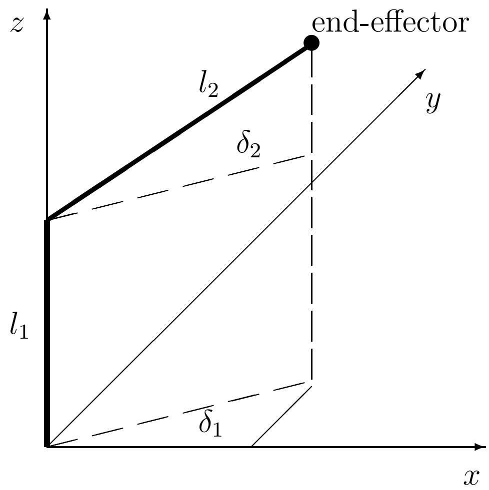
Gröbner bases at work: Inverse Kinematics in Robotics

We consider robots with prismatic and revolute joints. The kinematics of such robots can be described by multivariate polynomial equations, after having represented angles α by their sines and cosines and having added the equation $\sin^2(\alpha) + \cos^2(\alpha) = 1$ to the set of polynomial equations.

- forward kinematics: determines the position of the end-effector for given lengths of prismatic joints and angles of revolute joints
- inverse kinematics: determines possible lengths and angles from a predetermined goal position of the end-effector. Whereas a forward kinematics problem always has exactly one solution, an inverse kinematics problem could have no, exactly one, or several (possibly infinitely many) solutions.

Example:



This is a family of robots (l_1, l_2 are the parameters of the family) with 2 degrees of freedom.

l_1, l_2	......	lengths of the two robot arms
p_x, p_y, p_z	......	x-, y-, and z-coordinates of the position of the end-effector
δ_1, δ_2	......	angles describing the rotations of the revolute joints
s_1, s_2, c_1, c_2	......	sines and cosines of δ_1, δ_2, respectively

Corresponding system of algebraic equations:
given l_1, l_2, p_x, p_z,
solve for s_1, c_1, s_2, c_2, p_y

$$\begin{aligned}
l_2 \cdot c_1 \cdot c_2 - px &= 0, \\
l_2 \cdot s_1 \cdot c_2 - py &= 0, \\
l_2 \cdot s_2 + l_1 - pz &= 0, \\
c_1^2 + s_1^2 - 1 &= 0, \\
c_2^2 + s_2^2 - 1 &= 0.
\end{aligned}$$

Gröbner basis for this system
(in $\mathbb{Q}(l_1, l_2, p_x, p_z)[c_1, c_2, s_1, s_2, p_y]$):

$$
\begin{aligned}
(l_2^2 - l_1^2 + 2l_1p_z - p_z^2 - p_x^2) \cdot c_1 - p_x \cdot s_1 \cdot p_y &= 0, \\
(l_2^3 - l_2l_1^2 + 2l_2l_1p_z - l_2p_z^2 - l_2p_x^2) \cdot c_2 & \\
+(-l_2^2 + l_1^2 - 2l_1p_z + p_z^2) \cdot s_1 \cdot p_y &= 0, \\
(l_2^2 - l_1^2 + 2l_1p_z - p_z^2) \cdot s_1^2 - l_2^2 + l_1^2 - 2l_1p_z & \\
+p_z^2 + p_x^2 &= 0, \\
l_2s_2 + l_1 - p_z &= 0, \\
-l_2^2 + l_1^2 - 2l_1p_z + p_z^2 + p_x^2 + p_y^2 &= 0.
\end{aligned}
$$

In this Gröbner basis the variables “are separated”, i.e. we can solve for 1 variable at a time (starting from the last polynomial up to the first).

So, for instance, for

$l_1 = 30$	......	length of first bar
$l_2 = 45$	......	length of second bar
$p_x = \frac{45 \cdot \sqrt{6}}{4} \simeq 27.5567$	......	x-coordinate of end-eff.
$p_z = \frac{45 \cdot \sqrt{2}}{2} + 30 \simeq 61.8198$	......	z-coordinate of end-eff.

we get (among others) the solution

$$p_y = \frac{45\sqrt{2}}{4},\ s_2 = \frac{\sqrt{2}}{2},\ s_1 = \frac{1}{2},\ c_2 = \frac{\sqrt{2}}{2},\ c_1 = \frac{\sqrt{3}}{2},$$

i.e. the angles have to be set to

$$\delta_1 = 30^\circ, \delta_2 = 45^\circ.$$