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Symbolic Integral Operators and Boundary Problems

Lecture Notes (Preliminary Version)

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Contents

Chapter 1
Introduction

Preliminaries on Notation. Rings and algebras are always assumed to have an identity but need not be commutative. A left inverse is called a retraction, a right inverse a section.

(L)ODE = (Linear) Ordinary Differential Equation
(L)PDE = (Linear) Partial Differential Equation

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

In this chapter, we give an overview of the main topics of the lecture.

1.1 The Simplest Boundary Problem

$$\boxed{\begin{aligned} &u'' = f, \\ &u(0) = u(1) = 0. \end{aligned}} \tag{1.1}$$

$$\boxed{\begin{aligned} &Tu = f \\ &\beta_1, \ldots, \beta_n \end{aligned}} \tag{1.2}$$

Proof. asdf □

Chapter 2
Solving Boundary Problems in Analysis

Chapter 3
Integro-Differential Algebras

3.1 Axioms and Basic Properties

For working with boundary problems (in particular for "solving" them) in a symbolic way, the first step is to move to an *algebraic setting*. But which algebraic properties should we distil from the differential / integral / boundary operators occurring in the statements of Chapter 2 such that everything works properly?

Before answering this question, we make a small adaptation in these statements leading to a *smooth formulation* more amenable to algebra. Recall that we viewed the differential operator as a $\mathbb{C}$-linear homomorphism

$$\tfrac{d}{dx}\colon C^1[a,b] \to C[a,b],$$

hence the usual integral operator appears as a $\mathbb{C}$-linear homomorphism

$$\textstyle\int_a^x\colon C[a,b] \to C^1[a,b].$$

Obviously we can restrict both operators to $\mathbb{C}$-linear *endo*morphisms $\mathscr{F} \to \mathscr{F}$, where $\mathscr{F} = C^\infty[a,b]$. This has the advantage that we can focus our attention to the structure $(\mathscr{F}, \frac{d}{dx}, \int_a^x)$, which has only one carrier $\mathscr{F}$ as we are used to from algebra.

Of course this seems to be a pointless restriction from the viewpoint of analysis: If the forcing function of a boundary problem just needs to be continuous, why should we require it to be C^∞? But we should remember that we are mainly interested in the symbolic solution of boundary problems, meaning their Green's operators G. And once we have a *symbolic representation* of $G\colon C^\infty[a,b] \to C^\infty[a,b]$, we can expect that the same representation will also work on larger function spaces. We will see that this is indeed the case.

Now back to the structure $(\mathscr{F}, \frac{d}{dx}, \int_a^x)$. What are its essential algebraic properties? Obviously, it is an *algebra* over the field $\mathbb{C}$. We will generalize this to algebras over coefficient rings. So for the rest of this section, let K be a fixed commutative ring. It remains to impose the right axioms on the "infinitesimal" operations $\partial = \frac{d}{dx}$ and $\int = \int_a^x$, and the following definition tries to accomplish exactly that. (As we

will show a bit later, it also incorporates the boundary operators in an appropriate sense.)

Definition 3.1. We call $(\mathscr{F},\partial,\int)$ an *integro-differential algebra* over K if $\mathscr{F}$ is a commutative K-algebra with K-linear operations ∂ and $\int$ such that the three axioms

$$(\textstyle\int f)' = f, \tag{3.1}$$

$$(fg)' = f'g + fg', \tag{3.2}$$

$$(\textstyle\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g') \tag{3.3}$$

are satisfied, where $\ldots'$ is the usual shorthand notation for ∂.

We refer to ∂ and $\int$ respectively as the *derivation* and *integral* of $\mathscr{F}$. In general, a K-linear operation ∂ is usually called a derivation if it satisfies (3.2). Moreover, we call a section $\int$ of ∂ an integral for ∂ if it satisfies (3.3).

We call Axiom (3.1) the *section axiom* since it says that $\partial\circ\int = 1_{\mathscr{F}}$, so $\int$ is required to be a section of ∂. In differential algebra [27], Axiom (3.2) is commonly called the *Leibniz axiom*, obviously encoding the product rule of differentiation. In contrast, Axiom (3.3) captures integration by parts and is new in this form [23]; we have called it the *differential Baxter axiom* for reasons that will soon be explained.

Note that we have applied operator notation for the latter; otherwise Axiom (3.3) would read $\int(f')\int(g')+\cdots$, which is quite unusual at least for an analyst. We will likewise use operator notation for the derivation, so Axiom (3.2) can also be written as $\partial fg = (\partial f)g + f(\partial g)$. For the future we also introduce the following convention for saving parentheses: Multiplication has precedence over integration, so $\int f\int g$ is to be parsed as $\int(f\int g)$.

Let us also remark that Definition 3.1 can be generalized in two ways [23]: First, no changes are needed for the *noncommutative case* (meaning $\mathscr{F}$ but not K is noncommutative). This would be an appropriate setting for matrices with entries in $\mathscr{F} = C^\infty[a,b]$, providing an algebraic framework for the results on linear systems of ODEs described in (???). Second, one may add a *nonzero weight* in the Leibniz axiom, thus incorporating discrete models where ∂ is the difference operator defined by $(\partial f)_k = f_{k+1} - f_k$. The nice thing is that all other axioms remain unchanged.

As for any retraction/section pair between modules [5, p.209], Axiom (3.1) yields two complementary *associated projectors*.

Definition 3.2. Let $(\mathscr{F},\partial)$ be a differential algebra and $\int$ a section of ∂. Then

$$\mathtt{J} = \textstyle\int\circ\partial \qquad \text{and} \qquad \mathtt{E} = 1 - \mathtt{J}$$

are respectively called the *initialization* and the *evaluation* of $\mathscr{F}$.

Note that they are indeed projectors since $\mathtt{J}\circ\mathtt{J} = \int\circ(\partial\circ\int)\circ\partial = \mathtt{J}$ by (3.1), which implies $\mathtt{E}\circ\mathtt{E} = 1-\mathtt{J}-\mathtt{J}+\mathtt{J}\circ\mathtt{J} = \mathtt{E}$. As is well known, every projector is characterized by its kernel and image—they form a direct decomposition of the module into two submodules, and every such decomposition corresponds to a unique

projector. Since we will need them more often, we will also introduce names for these two *associated modules* associated with an integro-differential algebra.

Definition 3.3. Let $(\mathscr{F}, \partial, \int)$ be an integro-differential algebra. Then the modules

$$\mathscr{C} = \mathrm{Ker}(\partial) = \mathrm{Ker}(\mathtt{J}) = \mathrm{Im}(\mathtt{E}) \quad \text{and} \quad \mathscr{I} = \mathrm{Im}(\textstyle\int) = \mathrm{Im}(\mathtt{J}) = \mathrm{Ker}(\mathtt{E})$$

are respectively called the *constant functions* and the *initialized functions*.

We have therefore a *canonical decomposition*

$$\mathscr{F} = \mathscr{C} \dotplus \mathscr{I},$$

which allows to split off the "constant part" of every function.

Before turning to the other axioms, let us check what all this means in the familiar *standard model* $\mathscr{F} = C^\infty[a,b]$ with $\partial = \frac{d}{dx}$ and $\int = \int_a^x$. Obviously, the elements of $\mathscr{C}$ are then indeed the constant functions $f(x) = c$ with $c \in \mathbb{C}$, while $\mathscr{I}$ consists of those $f \in C^\infty[a,b]$ that satisfy the homogeneous initial condition $f(a) = 0$. This also explains the terminology for the projectors: Here $\mathtt{E} f = f(a)$ evaluates f at the initialization point a, and $\mathtt{J} f = f - f(0)$ enforces the initial condition.

The two other axioms shed some more light on the two submodules $\mathscr{C}$ and $\mathscr{I}$. For understanding this, it is more economic to forget for a moment about integro-differential algebras and turn to the following general observation about *projectors on an algebra*.

Lemma 3.4. *Let E, J be projectors on a K-algebra with $E + J = 1$, set $C = \mathrm{Im}(E) = \mathrm{Ker}(J)$ and $I = \mathrm{Ker}(E) = \mathrm{Im}(J)$. Then the following statements are equivalent:*

1. *The projector E is multiplicative.*
2. *The projector J satisfies the identity $(Jf)(Jg) + J\,fg = f\,Jg + g\,Jf$.*
3. *The module C is a subalgebra and the module I an ideal.*

Proof. 1. $\Leftrightarrow$ 2. Multiplicativity just means $fg - J\,fg = fg - f\,Jg - g\,Jf + (Jf)(Jg)$.
1. & 2. $\Rightarrow$ 3. This follows immediately because C is the image and I the kernel of the algebra endomorphism E.
3. $\Rightarrow$ 1. Let f, g be arbitrary. Since the given K-algebra is a direct sum of C and I, we have $f = f_C + f_I$ and $g = g_C + g_I$ for $f_C = Ef, g_C = Eg \in C$ and $f_I = Jf, g_I = Jg \in I$. Then

$$E\,fg = E(f_C g_C) + E(f_C g_I) + E(f_I g_C) + E(f_I g_I)$$

Since I is an ideal, $f_C g_I, f_I g_C, f_I g_I \in I$ and the last three summands must vanish. Furthermore, C is a subalgebra, so $f_C g_C \in C$. This implies $E(f_C g_C) = f_C g_C$ because E is a projector with image C. □

This lemma is obviously applicable to integro-differential algebras $\mathscr{F}$ with the projectors $E = \mathtt{E}$ and $J = \mathtt{J}$ and with the submodules $C = \mathscr{C}$ and $I = \mathscr{I}$ because the differential Baxter axiom (3.3) is exactly condition 2. From now on, we will therefore refer to $\mathscr{C}$ as the *algebra of constant functions* and to $\mathscr{I}$ as the *ideal of*

initialized functions. Altogether we obtain now the following characterization of integrals (note that the requirement that $\mathscr{C}$ be a subalgebra already follow from the Leibniz axiom).

Corollary 3.5. *Let $(\mathscr{F},\partial)$ be a differential algebra. Then a section $\int$ of ∂ is an integral iff its evaluation* **E** *is multiplicative iff $\mathscr{I} = \mathrm{Im}(\int)$ is an ideal.*

Note that the ideal $\mathscr{I}$ corresponding to an integral is in general *not a differential ideal* of $\mathscr{F}$. We can see this already in the standard example $C^\infty[0,1]$, where $\mathscr{I}$ consists of all $f \in C^\infty[0,1]$ with $f(0) = 0$. Obviously $\mathscr{I}$ is not differentially closed since $x \in \mathscr{I}$ but $x' = 1 \notin \mathscr{I}$.

The above corollary implies immediately that an integro-differential algebra $\mathscr{F}$ can *never be a field* since then the only possibilities for $\mathscr{I}$ would be 0 and $\mathscr{F}$. The former case is excluded since it means that $\mathrm{Ker}(\partial) = \mathscr{F}$, contradicting the surjectivity of ∂. The latter case corresponds to $\mathrm{Ker}(\partial) = 0$, which is not possible because $\partial 1 = 0$.

Corollary 3.6. *An integro-differential algebra is never a field.*

In some sense, this observation ensures that all integro-differential algebras are *fairly complicated*. The result points in the same direction, excluding finite-dimensional algebras.

Proposition 3.7. *The iterated integrals $1, \int 1, \int\int 1, \ldots$ are all linearly independent. Hence every integro-differential algebra is infinite-dimensional.*

Proof. Let (u_n) be the sequence of iterated integrals of 1. We prove by induction on n that $u_0, u_1, \ldots, u_n$ are linearly independent. The base case $n = 0$ is trivial. For the induction step from n to $n+1$, assume $c_0 u_0 + \cdots + c_{n+1} u_{n+1} = 0$. Applying ∂^{n+1} yields $c_{n+1} = 0$. But by the induction hypothesis, we have also $c_0 = \cdots = c_n = 0$. Hence $u_0, \ldots, u_{n+1}$ are linearly independent. □

Let us now return to our discussion of the *differential Baxter axiom* (3.3). We will now offer an equivalent description that is closer to analysis. It is also more compact but less symmetric. (In the noncommutative case one has to add the opposite version—reversing all products—for obtaining an equivalence.)

Proposition 3.8. *The differential Baxter axiom* (3.3) *is equivalent to*

$$f \int g = \int f g + \int f' \int g, \tag{3.4}$$

assuming the other two axioms of Definition 3.1.

Proof. For proving (3.4) note that since $\mathscr{I}$ is an ideal, $f\int g$ is invariant under the projector **J** and thus equal to $\int (f\int g)' = \int f' \int g + \int f g$ by the Leibniz axiom (3.2) and the section axiom (3.1). Alternatively, one can also obtain (3.4) from (3.3) if one replaces g by $\int g$ in (3.3). Conversely, assuming (3.4) we see that $\mathscr{I}$ is an ideal of $\mathscr{F}$, so Corollary 3.5 implies that $\int$ satisfies the differential Baxter axiom (3.3). □

For obvious reasons, we refer to (3.4) as *integration by parts*. The usual formulation $\int fG' = fG - \int f'G$ is only satisfied "up to a constant", or if one restricts G to $\mathrm{Im}(\int)$. Substituting $G = \int g$ then lead to (3.4). But note that we have now a more algebraic perspective on this well-known identity of Calculus: It tells us how $\mathscr{I}$ is realized as an ideal of $\mathscr{F}$.

Sometimes a slight variation of (3.4) is useful. Replacing g by g' and using the relation $\mathtt{J} = 1 - \mathtt{E}$, we obtain $f(g - \mathtt{E}g) = \int fg' + \int f'(g - \mathtt{E}g) = \int (fg)' - (\mathtt{E}g)\int f'$. Since also $\int f' = f - \mathtt{E}f$, the term $(\mathtt{E}g)\,f$ cancels, and we get

$$\int fg' = fg - \int f'g - (\mathtt{E}f)(\mathtt{E}g), \tag{3.5}$$

which we call the *evaluation variant* of integration by parts (a form that is also used in Calculus). One can also derive it by applying $\mathtt{J} = 1 - \mathtt{E}$ to the Leibniz axiom 3.2, using the fact that $\mathtt{E}$ is multiplicative. Note that 3.2 is also a strong Baxter axiom just like integration by parts (3.4). In fact, we regain the latter upon replacing g by $\int g$ in 3.5.

If we extract the differential part from an integro-differential algebra $(\mathscr{F}, \partial, \int)$, we obtain a *differential algebra* $(\mathscr{F}, \partial)$, meaning a K-algebra $\mathscr{F}$ with a K-linear operation ∂ that satisfies the Leibniz axiom (3.2). But in general one cannot expand a given differential algebra to an integro-differential algebra: The latter clearly requires the derivation to be surjective. For example, in $(K[x^2], x\partial)$ the image of ∂ does not contain 1. As another example, the algebra of differential polynomials $\mathscr{F} = K\{u\}$ does not admit an integral in the sense of Definition 3.1 since here the image of ∂ does not contain u.

But how can we isolate the *integro part* of an integro-differential algebra? The disadvantage (and also advantage!) of the differential Baxter axiom (3.3) is that it entangles derivation and integral. So how can one express "integration by parts" without referring to the derivation?

Definition 3.9. Let $\mathscr{F}$ be a K-algebra and $\int$ a K-linear operation satisfying

$$(\int f)(\int g) = \int f \int g + \int g \int f. \tag{3.6}$$

Then $(\mathscr{F}, \int)$ is called a *Baxter algebra*.

In the literature, (3.6) is called the *Baxter axiom* named after Glen Baxter [2, 3]; in contrast to the differential Baxter axiom (3.3), we will sometimes call it the *pure Baxter axiom*.

One might now think that an integro-differential algebra $(\mathscr{F}, \partial, \int)$ is a differential algebra $(\mathscr{F}, \partial)$ combined with a Baxter algebra $(\mathscr{F}, \int)$ such that the section axiom (3.1) is satisfied. In fact, such a structure was introduced, independently from us, by Guo and Keigher [22] under the name *differential Rota-Baxter algebras*. But we will see that an integro-differential algebra is a little bit more—this is why we also refer to (3.6) as the "weak Baxter axiom" and to (3.3) and (3.4) as the "strong Baxter axioms".

Proposition 3.10. *Let $(\mathscr{F},\partial)$ be a differential algebra and $\int$ a section for ∂. Then $\int$ satisfies the pure Baxter axiom* (3.6) *iff $\mathscr{I}=\mathrm{Im}(\int)$ is a subalgebra of $\mathscr{F}$. In particular, $(\mathscr{F},\int)$ is a Baxter algebra for an integro-differential algebra $(\mathscr{F},\partial,\int)$.*

Proof. Clearly (3.6) implies that $\mathscr{I}$ is a subalgebra of $\mathscr{F}$. Conversely, if $(\int f)(\int g)$ is contained in $\mathscr{I}$, it is invariant under the projector $\mathtt{J}$ and therefore must be equal to $\int\partial\,(\int f)(\int g)=\int f\int g+\int g\int f$ by the Leibniz axiom (3.2). □

So the strong Baxter axiom (3.3) requires that $\mathscr{I}$ be an ideal, the weak Baxter axiom (3.6) only that it be a subalgebra. We will soon give a counterexample for making sure that (3.3) is indeed asking for more than (3.6). But before this we want to express the difference between the two axioms in terms of a *linearity property*. Recall that both ∂ and $\int$ were introduced as K-linear operations on $\mathscr{F}$. Using the Leibniz axiom (3.2), one sees immediately that ∂ is even $\mathscr{C}$-linear. It is natural to expect the same from $\int$, but this is exactly the difference between (3.3) and (3.6).

Proposition 3.11. *Let $(\mathscr{F},\partial)$ be a differential algebra and $\int$ a section for ∂. Then $\int$ satisfies the differential Baxter axiom* (3.3) *iff it satisfies the pure Baxter axiom* (3.6) *and is $\mathscr{C}$-linear.*

Proof. Assume first that $\int$ satisfies the differential Baxter axiom (3.3). Then the pure Baxter axiom (3.6) holds by Proposition 3.10. For proving $\int cg=c\int g$ for all $c\in\mathscr{C}$ and $g\in\mathscr{F}$, we use again the integration-by-parts formula (3.4) and $c'=0$.

Conversely, assume the pure Baxter axiom (3.6) is satisfied and $\int$ is $\mathscr{C}$-linear. By Lemma 3.8 it suffices to prove the integration-by-parts formula (3.4) for $f,g\in\mathscr{F}$. Since $\mathscr{F}=\mathscr{C}\dotplus\mathscr{I}$, we may first consider the case $f\in\mathscr{C}$ and then the case $f\in\mathscr{I}$. But the first case follows from $\mathscr{C}$-linearity; the second case means $f=\int\tilde{f}$ for $\tilde{f}\in\mathscr{F}$, and (3.4) becomes the pure Baxter axiom (3.6) for $\tilde{f}$ and g. □

Let us now look at some natural *examples of integro-differential algebras*, in addition to our standard example $C^\infty[a,b]$.

Example 3.12. The *analytic functions* on the real interval $[a,b]$ form an integro-differential subalgebra $C^\omega[a,b]$ of $C^\infty[a,b]$ over $K=\mathbb{R}$ or $K=\mathbb{C}$. It contains in turn the integro-differential algebra $K[x,e^{Kx}]$ of *exponential polynomials*, defined as the space of all K-linear combinations of $x^ne^{\lambda x}$, with $n\in\mathbb{N}$ and $\lambda\in K$. Finally, the *ordinary polynomials* $K[x]$ are clearly an integro-differential subalgebra in all cases.

Note also that the exponents of $K[x,e^{Kx}]$ can be restricted in various ways. In general, let S be an additive subset of K, meaning a subsemigroup of $(K,+)$. Then the $x^ne^{\lambda x}$ with $\lambda\in S$ span a subalgebra $K[x,e^{Sx}]$ of $K[x,e^{Kx}]$, which we call *S-restricted exponential polynomials*. Typical choices for S are $\pm\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset K$. □

The three examples above all have *algebraic counterparts*, with integro-differential structures defined in the expected way.

Example 3.13. For a field K of characteristic zero, the *formal power series* $K[[x]]$ are an integro-differential algebra. One sets $\partial x^k=kx^{k-1}$ and $\int x^k=x^{k+1}/(k+1)$; note that the latter needs characteristic zero. The formal power series contain

a highly interesting and important integro-differential subalgebra: the *holonomic power series*, defined as those whose derivatives span a finite-dimensional K-vector space [14, 41].

Of course $K[[x]]$ also contains (an isomorphic copy of) the integro-differential algebra of *exponential polynomials*. In fact, one can define the algebraic version of $K[x, e^{Kx}]$ as a quotient of the free algebra generated by the symbols x^k and $e^{\lambda x}$, with λ ranging over K. Derivation and integration are then defined in the obvious way. The exponential polynomials in turn contain the *polynomial ring* $K[x]$ as an integro-differential subalgebra. Note that we use the notation $K[x]$ and $K[x, e^{Kx}]$ both for the analytic and the algebraic objects. □

In most of these lecture notes, we restrict ourselves to fields of characteristic zero. The following example is a noteworthy exception to this rule—it is a clever way of transferring the previous example to coefficient fields of *positive characteristic*.

Example 3.14. Let K be an arbitrary field (having zero or positive characteristic). Then the algebra $H(K)$ *Hurwitz series* [25] over K is defined as the K-vector space of infinite K-sequences with the multiplication defined as

$$(a_n) \cdot (b_n) = \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right)_n$$

for all $(a_n), (b_n) \in H(K)$. If one introduces derivation and integration by

$$\partial\,(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots),$$
$$\textstyle\int (a_0, a_1, \dots) = (0, a_0, a_1, \dots),$$

the Hurwitz series form an integro-differential algebra $(H(K), \partial, \int)$, as explained by [26] and [21]. Note that as an additive group, $H(K)$ coincides with the formal power series $K[[z]]$, but its multiplicative structure differs: We have an isomorphism

$$\sum_{n=0}^{\infty} a_n z^n \;\mapsto\; (n!\, a_n)$$

from $K[[z]]$ to $H(K)$ if and only if K has characteristic zero. The point is that one can integrate every element of $H(K)$, whereas the formal power series z^{p-1} does not have an antiderivative in $K[[z]]$ if K has characteristic $p > 0$. □

In the analytic integro-differential algebras of Example 3.12, we have allowed complex numbers only in the function values. But every function in $C^\omega[a,b]$ extends to a holomorphic function on some complex domain containing $[a,b]$. This gives us the right hint for considering functions with *complex arguments* as integro-differential algebras.

Example 3.15. Let D be any simply connected domain with a distinguished point $z_0 \in D$, and let $\mathscr{F}$ be the algebra of *holomorphic functions* on D. Set $\partial f = \frac{df}{dz}$ in the sense of the usual derivative and $\int f = \int_{z_0}^{z} f(\zeta)\, d\zeta$ in the sense of a complex

integral along any path within D that connects z_0 and z. Then $(\mathscr{F},\partial,\int)$ is an integro-differential algebra, which contains $C^\omega[a,b]$ if $[a,b]\subseteq D$.

Note also that $\mathscr{F}$ contains the so-called *Hardy spaces* H^p if D is the open unit disk [43]. For any $p>1$, they form a subspace of L^p, for $p=2$ even a reproducing kernel Hilbert space. But of course H^p is not an algebra (let alone an integro-differential one). □

Speaking of *polynomials*: As remarked before, they form an elementary example of an integro-differential algebra. But more is true: They are present within any integro-differential algebra! One may think of this fact as an integro-differential analog of the field situation: Every field of characteristic zero contains $\mathbb{Q}$ as its prime field.

Proposition 3.16. *Let $(\mathscr{F},\partial,\int)$ be an integro-differential algebra over a field K of characteristic zero. Then $x\mapsto\int 1$ induces a monomorphism $K[x]\to\mathscr{F}$ in the category of integro-differential algebras.*

Proof. Let $\psi\colon K[x]\to\mathscr{F}$ be the algebra morphism induced by $x\mapsto\int 1$. For proving that ψ is a differential morphism, we compute

$$(\partial_{\mathscr{F}}\psi)x=\partial_{\mathscr{F}}\textstyle\int_{\mathscr{F}}1_{\mathscr{F}}=1_{\mathscr{F}}=\psi 1_{K[x]}=(\psi\partial_{K[x]})x,$$

using the section axiom (3.1) for ∂ in $\mathscr{F}$. Similarly, we verify that ψ is an integro morphism by checking

$$(\textstyle\int_{\mathscr{F}}\psi)x=\int_{\mathscr{F}}\int_{\mathscr{F}}1_{\mathscr{F}}=\frac{1}{2}(\int_{\mathscr{F}}1_{\mathscr{F}})^2=\psi\frac{x^2}{2}=(\psi\textstyle\int_{K[x]})x,$$

this time using $(\int 1)^2=2\int\int 1$ in $\mathscr{F}$, which is an immediate consequence of the Baxter axiom (3.6). Hence ψ is a morphism of integro-differential algebras.

It remains to show that $\mathrm{Ker}(\psi)=0$. Since ψ is a differential morphism, it is clear that $\mathrm{Ker}(\psi)$ is a differential ideal of $K[x]$. But there are only two differential ideals in $K[x]$, either 0 or all of $K[x]$. The latter is excluded because $\psi 1_{K[x]}=1_{\mathscr{F}}$. □

Now for the promised *counterexample* to the claim that the section axiom would suffice for merging a differential algebra $(\mathscr{F},\partial)$ and a Baxter algebra $(\mathscr{F},\int)$ into an integro-differential algebra $(\mathscr{F},\partial,\int)$.

Example 3.17. Set $R=K[y]/y^4$ for K a field of characteristic zero and define ∂ on $\mathscr{F}=R[x]$ as usual. Then $(\mathscr{F},\partial)$ is a differential algebra. Let us define a K-linear map $\int$ on $\mathscr{F}$ by

$$\textstyle\int f=\int^* f+f(0,0)y^2, \tag{3.7}$$

where $\int^*$ is the usual integral on $R[x]$ with $x^k\mapsto x^{k+1}/(k+1)$. Since the second term vanishes under ∂, we see immediately that $\int$ is a section of ∂. For verifying the pure Baxter axiom (3.6), we compute

$$\begin{aligned}&(\textstyle\int f)(\int g)=(\int^* f)(\int^* g)+y^2\int^*\big(g(0,0)\,f+f(0,0)\,g\big)+f(0,0)\,g(0,0)\,y^4,\\&\textstyle\int f\int g=\int f\big(\int^* g+g(0,0)y^2\big)=\int^* f\int^* g+g(0,0)y^2\int^* f.\end{aligned}$$

Since $y^4 \equiv 0$ and the ordinary integral $\int^*$ fulfills the pure Baxter axiom (3.6), this implies immediately that $\int$ does also. However, it does not fulfill the differential Baxter axiom (3.3) because it is not $\mathscr{C}$-linear: Observe that $\mathscr{C}$ is here $\mathrm{Ker}(\partial) = R$, so in particular we should have $\int(y \cdot 1) = y \cdot \int 1$. But one checks immediately that the left-hand side yields xy, while the right-hand side yields $xy + y^3$. □

If one is familiar with the mechanism of *adjoining transcendental functions* in differential algebras, the analogous task in integro-differential algebras will appear very cumbersome. In fact, if we fix the derivative of a new function, the Leibniz axiom settles the derivatives of all sums and products involving the new functions. The Baxter axioms (no matter which) do not provide a similarly convenient device for settling integrals—otherwise the task of integration would be trivial! In fact, closure under integration is often a difficult but interesting question. Consider adjoining $\exp(x^2)$ to $K[x]$. This forces us to adjoin also the error function—but what else?

The canonical process of adjoining elements in an algebraic structure is this: One first constructs the corresponding free objects (some sort of "polynomials" in which the new elements appear as indeterminates) and then takes the quotient modulo a suitable congruence relation (given by an ideal in ring-like structures). For integro-differential algebras, the first step involves the *integro-differential polynomials* [39], which are considerably more complicated than their plain differential analogs so that we will study them only in the second part of these lecture notes. At any rate, one sees that adjunction in integro-differential algebras is rather tedious, also in the canonical process with polynomials and quotients.

We can view this difficulty also from another perspective: Many examples of adjunctions are constructed in the frame of the Risch theory [6] and in differential Galois theory [35]. While the former is concerned with the problem of finding symbolic antiderivatives (solving $u' = f$), the latter may be seen as an extension that studies the structure of solutions for arbitrary linear ordinary differential equations. Restricting our attention to $u' = f$, the adjunction of u seems to give rise to an integral operator $f \mapsto u = \int f$. But the new domain containing u is normally constructed as a *differential field*. Since a field has no proper nontrivial ideals, Lemma 3.4 shows at once that they cannot carry the structure of an integro-differential algebra!

This seems paradoxical. But what happens is that the initial differential field containing f is extended by *some* antiderivative u such that all solutions of $u' = f$ are given by $u + c$, where c ranges over the constants. The situation is somewhat analogous to algebraic field extensions: one adjoins all roots of an irreducible polynomial, but one cannot identify the individual roots. Hence there is *no canonical integral operator* $f \mapsto u$, at any rate none that satisfies the Baxter axiom.

The *trade-off* between differential fields and integro-differential alebras can be seen in the following simple example: Given a field K, the polynomials $K[x]$ have the most integrals ($\int_a^x$ for each evaluation at $a \in K$) and thus are farthest from a field (only the constants are invertible). The other extreme would be the field of formal Laurent series over K, which has no integral at all. But there are many intermediate algebras. Close to $K[x]$ we have the Laurent polynomials, which still has all integrals except one (evaluation at 0 is not defined) and x as the only invertible non-constant. Close to the Laurent series we have the algebra of formal power series, which has only one integral (corresponding to evaluation at 0) and is "almost a field" (a local ring).

It is an interesting research question to develop something like a *extension theory of integro-differential algebras*, along the lines of the Risch theory of differential fields. Perhaps it would make use of a new type of structure that combines the advantages of fields with those of integro-differential algebras?

As a first nontrivial example of such an extension, let us look at the Laurent polynomials mentioned above. They are the localization of $K[x]$ at x, effected by adjoining $1/x$. It is clear that closure under integration then forces also a *logarithm* on us. But is this enough? The answer turns out to be yes. But of course we cannot choose 0 as our initialization point as we did for all our examples up to now. It is convenient to choose 1 instead since that will make the evaluation annihilate the logarithm. So we will sometimes write $\int$ for the integral operator $f(x) \mapsto \int_1^x f(\xi)\, d\xi$.

For creating a suitable algebraic structure, let us first establish the required integral relations for smooth functions on the positive real axis. We write $\lambda_{m,n}$ for the function $x^m(\log x)^n$, where m ranges over the integers and n over the natural numbers. Clearly the $\lambda_{m,n}$ are linearly independent in $C^\infty(0,\infty)$. Let $\mathfrak{L}$ be the subspace generated by the $\lambda_{m,n}$. We will now prove that $\mathfrak{L}$ is an integro-differential subalgebra of $C^\infty(0,\infty)$ by establishing the *integral relations*

$$\int \lambda_{-1,n} = \tfrac{1}{n+1}\,\lambda_{0,n+1} \tag{3.8}$$

$$\int \lambda_{m,n} = \tfrac{1}{m+1}\,\lambda_{m+1,n} - \tfrac{n}{m+1}\int \lambda_{m,n-1} \quad (m \neq -1) \tag{3.9}$$

for the basis elements $\lambda_{m,n}$. For proving (3.8), we just apply the substitution $\eta = \log\xi$ to the left-hand side $\int_1^x (\log\xi)^n\, d\xi/\xi$, obtaining $\int_0^{\log x} \eta^n\, d\eta = (\log x)^{n+1}/(n+1)$ by the substitution rule. Now let us turn to (3.9). Using integration by parts (3.4) on the left-hand side yields

$$\begin{aligned}\int x^m(\log x)^n &= (\log x)^n \int x^m - n\int \tfrac{1}{x}(\log x)^{n-1}\int x^m \\ &= \tfrac{1}{m+1}\,(x^{m+1}-1)(\log x)^n - \tfrac{n}{m+1}\int (x^m - \tfrac{1}{x})(\log x)^{n-1}.\end{aligned}$$

After multiplying out, the second summand $-\frac{1}{m+1}\,\lambda_{-1,n}$ and the last $\frac{n}{m+1}\int\lambda_{-1,n-1}$ cancel due to (3.8), and we what remains is the right-hand side of (3.9).

We can take (3.9) as a recursive definition of $\int$ on $\mathfrak{L}$ for the generic case $m \neq -1$, the recursion basis being the usual integration of powers given by $\int\lambda_{m,0} = \frac{1}{m+1}\,(x^{m+1}-1)$. The special case $m=-1$ is decribed explicitly by (3.8). It is a straight-forward task to *solve the recurrence* (3.9), obtaining the explicit formula

$$\int \lambda_{m,n} = \tfrac{(-1)^{n+1}n!}{(m+1)^{n+1}} + \sum_{k=0}^{n} \tfrac{(-1)^k n^{\underline{k}}}{(m+1)^{k+1}}\,\lambda_{m+1,n-k} \quad (m \neq -1). \tag{3.10}$$

Naturally, we prove this formula by induction on n. Obviously the base case $n=0$ gives the integration formula for powers just mentioned. For the induction step from n to $n+1$, we use (3.9) and the induction hypothesis, which yields with an index transformation

$$\begin{aligned}\int\lambda_{m,n+1} &= \tfrac{1}{m+1}\,\lambda_{m+1,n+1} - \tfrac{n+1}{m+1}\left(\tfrac{(-1)^{n+1}n!}{(m+1)^{n+1}} + \sum_{k=0}^{n}\tfrac{(-1)^k n^{\underline{k}}}{(m+1)^{k+1}}\,\lambda_{m+1,n-k}\right) \\ &= \tfrac{(-1)^n(n+1)!}{(m+1)^{n+2}} + \frac{1}{m+1}\,\lambda_{m+1,n+1} + \tfrac{n+1}{m+1}\sum_{k=1}^{n+1}\tfrac{(-1)^k n^{\underline{k-1}}}{(m+1)^k}\,\lambda_{m+1,n-k+1}.\end{aligned}$$

This is indeed (3.10) for $n+1$ since $(n+1)\,n^{\underline{k-1}} = (n+1)^{\underline{k}}$ and the middle term above contributes the summand for $k=0$.

We can now replace $\mathfrak{L}$ by the *algebraic construction* $K[x,\frac{1}{x},\log x]$. Starting from the differential algebra $K[x,\frac{1}{x}]$ of Laurent polynomials, this is the transcendental extension generated by $\log x$. By setting $\partial\log x = \frac{1}{x}$, we obtain a differential algebra. Finally, we define the integral explicitly by (3.8) and (3.10). Since $K[x,\frac{1}{x},\log x]$ is isomorphic to $\mathfrak{L}$, it is clear that the section axiom (3.1) and the differential Baxter axiom (3.3) are satisfied, but of course on can also verify this directly based on the definitions of ∂ and $\int$. ▲

3.2 Ordinary Integro-Differential Algebras

The fact that every integro-differential algebra contains the "univariate polynomials" does not necessarily mean that it contains "univariate functions". The following

example demonstrates that our current notion of integro-differential algebra contains both *partial and ordinary* ones, in a sense to be defined soon.

Example 3.18. Consider $\mathscr{F} = C^\infty(\mathbb{R}^2)$ with the derivation $\partial u = u_x + u_y$. Finding sections for ∂ means solving the *partial differential equation* $u_x + u_y = f$. Its general solution is given by

$$u(x,y) = \int_a^x f(t, y-x+t)\,dt + g(y-x),$$

where $g \in C^\infty(\mathbb{R})$ and $a \in \mathbb{R}$ are arbitrary. Let us choose $a = 0$ for simplicity. In order to ensure a linear section, one has to choose $g = 0$, arriving at

$$\int f = \int_0^x f(t, y-x+t)\,dt,$$

Using a change of variables, one may verify that $\int$ satisfies the pure Baxter axiom (3.6), so $(\mathscr{F}, \int)$ is a Baxter algebra.

We see that the *constant functions* $\mathscr{C} = \operatorname{Ker}(\partial)$ are given by $(x,y) \mapsto c(x-y)$ with arbitrary $c \in C^\infty(\mathbb{R})$, while the *initialized functions* $\mathscr{I} = \operatorname{Im}(\int)$ are those $F \in \mathscr{F}$ that satisfy $F(0,y) = 0$ for all $y \in \mathbb{R}$. In other words, $\mathscr{C}$ consists of all functions constant on the characteristic lines $x - y = \text{const}$, and $\mathscr{I}$ of those satisfying the homogeneous initial condition on the vertical axis (which plays the role of a "noncharacteristic initial manifold"). This is to be expected since $\int$ integrates along the characteristic lines starting from the initial manifold. The *evaluation* $\mathbf{E}\colon \mathscr{F} \to \mathscr{F}$ maps a function f to the function $(x,y) \mapsto f(0,y-x)$. This means that f is "sampled" only on the initial manifold, effectively becoming a univariate function: the general point (x,y) is projected along the characteristics to the initial point $(0,y-x)$.

Since $\mathbf{E}$ is multiplicative on $\mathscr{F}$, Lemma 3.4 tells us that $(\mathscr{F}, \partial, \int)$ is in fact an *integro-differential algebra*. Alternatively, note that $\mathscr{I}$ is an ideal and that $\int$ is $\mathscr{C}$-linear. Furthermore, we observe that here the polynomials are given by $K[x]$. □

▼ Based on Example 3.18, we can also produce a purely *algebraic construction* with similar features (in particular also giving a partial integro-differential algebra in the sense of Definition 3.19). In fact, one sees immediately that the operations of derivation and integral are closed on the subspace of bivariate polynomial functions in $C^\infty(\mathbb{R}^2)$. Hence we can define these operations directly on $K[x,y]$. For the derivation $\partial = \partial_x + \partial_y$, this is obvious. For the integral $\int$, we sketch three different ways of achieving this.

The straightforward approach is to substitute the basis monomials $f(x,y) = x^n y^m$ into the integral $\int_0^x f(t, y-x+t)\,dt$ and apply the binomial theorem together with the usual integral on univariate polynomials. This yields at first

$$\int x^m y^n = \sum_{k=0}^{n} \tbinom{n}{k} (y-x)^k \int_0^x t^{m+n-k}\,dt = \sum_{k=0}^{n} \sum_{i=0}^{k} \tbinom{n}{k}\tbinom{k}{i} \tfrac{(-1)^i}{m+n-k+1} x^{m+n-k+i-1} y^{k-i},$$

which may be simplified by shifing the inner summation index to $j = n-k+i$ and reflecting the outer summation index by $k \leftrightarrow n-k$. Interchanging the two summations then leads to

$$\int x^m y^n = \sum_{j=0}^{n} \tbinom{n}{j} (-1)^j x^{m+j+1} y^{n-j} \sum_{k=0}^{j} \tbinom{j}{k} \tfrac{(-1)^k}{m+k+1}.$$

One may readily verify that the inner sum gives $m!\,j!/(m+j+1)!$; see e.g. Equation (5.41) of [20]. Thus we end up with the explict integration formula

$$\int \tfrac{x^m}{m!}\tfrac{y^n}{n!} = \sum_{j=0}^{n} (-1)^j \tfrac{x^{m+j+1}}{(m+j+1)!}\tfrac{y^{n-j}}{(n-j)!}, \tag{3.11}$$

where we have divided by $m!n!$ for emphasizing the inherent symmetry.

An alternative method proceeds from the observation that on $K[x]$, the operation $\int$ must agree with the standard integral. Since the rule of integration by parts (3.4) must be satisfied for $\int$, we derive the relation

$$\int x^m y^n = x^m \int y^n - m \int x^{m-1} \int y^n = \tfrac{1}{n+1}\,(x^m y^{n+1} - m \int x^{m-1} y^{n+1}),$$

which can be seen as a recursive defintion, anchored in the obvious recusion basis $\int y^n = \frac{y^{n+1}}{n+1}$. Solving this recurrence leads directly to (3.11).

A third, rather heuristic procedure starts from the generalized Leibniz rule of fractional calculus [34], which states that

$$\partial^s fg = \sum_{j=0}^{\infty} \tbinom{s}{j} (\partial^{s-j} f)(\partial^j g)$$

for all $s \in \mathbb{R}$ and for sufficiently smooth (univariate) functions f and g. Substituting $s = -1$ yields an explicit rule for integrating products, in some sense resolving the recursion gained from integration by parts (3.4), but at the cost of an infinite series. But if the iterated derivatives of g eventually vanish, this series terminates and we can expect to reap an algebraic integration formula. In fact, substituting $f = x^m$ and $g = y^n$ yields (3.11) at one stroke. The idea behind this heuristic is that the bivariate(!) integral operator $\int$ on $K[x,y]$ behaves rather like the usual integral operator since it satisfies the section axiom (3.1) and the differential Baxter axiom (3.3). Hence we employ the Leibniz rule for its fractional generalization (sometimes called "differintegral"), using the fact that $\int$ agrees with the univariate integral operator on $K[x]$ as noted above. ▲

For the first part of these lecture notes, we want to restrict ourselves to boundary problems for *ordinary differential equations*. Hence we want to rule out cases like Example 3.18. The most natural way for distinguishing ordinary from partial differential operators is to look at their kernels: Only the former have finite-dimensional ones. In fact, if we look at the basic differential operator $\frac{d}{dx}$ on $C^\infty(\mathbb{R})$, its kernel has dimension 1 while that of $\frac{\partial}{\partial x}$ on $C^\infty(\mathbb{R}\times\mathbb{R})$ has dimension ∞.

Definition 3.19. A differential algebra $(\mathscr{F},\partial)$ over a field K of characteristic zero is called *ordinary* if $\dim_K \operatorname{Ker}(\partial) = 1$ and *partial* otherwise.

From now on, we restrict the ground ring K to fields of characteristic zero. Note that except for Example 3.18 all our *examples* have been ordinary integro-differential algebras. In fact, we will only turn to partial integro-differential algebras in the second part of these lecture notes.

The requirement of ordinariness has a number of pleasant consequences. First of all, the somewhat tedoius distinction between the weak and strong *Baxter axioms* disappears now since obviously $\mathscr{C} = K$, so now $\mathscr{F}$ is an algebra over its own field of constants. Hence $\int$ is by definition $\mathscr{C}$-linear, and Lemma 3.11 ensures that the pure Baxer axiom (3.6) is now equivalent to the differential Baxter axiom (3.3). Let us summarize this.

Fact 3.20. In an ordinary integro-differential algebra, the constant functions coincide with the ground field, and the strong and weak Baxter axioms are equivalent.

Another nice property of ordinary integro-differential algebras is that the (embedded) *polynomials* behave as expected.

Lemma 3.21. *Let* $(\mathscr{F},\partial,\int)$ *be an ordinary integro-differential algebra. Then we have* $\operatorname{Ker}(\partial^n) = [1,x,\ldots,x^{n-1}]$.

Proof. The inclusion from right to left (which does not need ordinariness) holds because $x^k/k!$ is the k-fold iteration of $\int$ applied to 1; here we use the fact that the polynomials in $x = \int 1$ are isomorphic to $K[x]$ as integro-differential algebras. For the inclusion from left to right, note that $\dim \operatorname{Ker}(\partial^n) = n$; this follows from iterating the identity $\operatorname{Ker}(T^2) = G\operatorname{Ker}(T) \dotplus \operatorname{Ker}(T)$ in [36], generally valid for epimorphisms T and sections G of T. □

Ordinary integro-differential algebras allow us to view their *evaluation as characters* (i.e. multiplicative functionals): One knows from Linear Algebra that a projector P onto a one-dimensional subspace $[w]$ of a K-vector space V can be written as $P(v) = \varphi(v)w$, where $\varphi\colon V \to K$ is the unique functional with $\varphi(w) = 1$. If V is moreover a K-algebra, a projector onto $K = [1]$ is canonically described by the functional φ with normalization $\varphi(1) = 1$. Hence multiplicative projectors like $\mathbf{E}$ can be viewed as characters. In Chapter 6, we will consider other characters on $\mathscr{F}$; for the moment let us just note that $\mathbf{E}$ is as a distinguished character. We write $\mathscr{F}^\bullet$ for the set of all nonzero characters on a K-algebra $\mathscr{F}$, in other words all algebra homomorphisms $\mathscr{F} \to K$.

One calls a K-algebra *augmented* if there exists a character on it. Its kernel $\mathscr{I}$ is then known as an *augmentation ideal* and forms a direct summand of K; see for example [16, p. 132]. Augmentation ideals are always maximal ideals (generalizing the $C^\infty[a,b]$ case) since the direct sum $\mathscr{F} = K \dotplus \mathscr{I}$ induces a ring isomorphism $\mathscr{F}/\mathscr{I} \cong K$. Corollary 3.5 immediately translates to the following characterization of integrals in ordinary differential algebras.

Corollary 3.22. *In an ordinary differential algebra* $(\mathscr{F},\partial)$*, a section* $\int$ *of* ∂ *is an integral iff its evaluation is a character iff* $\mathscr{I} = \operatorname{Im}(\int)$ *is an augmentation ideal.*

▼ One might think that in a "classical" real integro-differential algebra like $\mathscr{F} = C^\infty[0,1]$, the characters are all given by the evaluations $u \mapsto u(\xi)$ with $\xi \in [0,1]$. But this is not so; there are plenty of other evaluations. For seeing this, take the restricted exponential polynomials $\mathbb{R}[x, e^{\mathbb{N}x}]$, which are clearly a subalgebra of $\mathscr{F}$. Then $\varphi(x) = \varphi(e^x) = 1$ defines a character φ, which "mixes" evaluation at 0 and evaluation at 1. Since we have $\varphi(x^n e^{mx}) = 1$, the kernel of this character is given by all exponential polynomials whose coefficients sum to zero. Obviously φ cannot be an evaluation at any ξ since then $\varphi(x) = \xi = 1$ implies $\varphi(e^x) = e^\xi = e$ contradicting $\varphi(e^x) = 1$. ▲

For treating two-point boundary problems, it is convenient to consider *two integral operators* simultaneously—one initialized at the left and the other at the right boundary point. In the standard example $\mathscr{F} = C^\infty[a,b]$, we have $\int^* = \int_a^x$ and $\int_* = \int_x^b$. The following definition captures these ideas in terms of ordinary integro-differential algebras.

Definition 3.23. A *biintegro-differential algebra* is given by $(\mathscr{F},\partial,\int^*,\int_*)$ where both $(\mathscr{F},\partial,\int^*)$ and $(\mathscr{F},\partial,-\int_*)$ are ordinary integro-differential algebras.

Biintegro-differential algebras were introduced under the name "analytic algebra" in [37] in a different setting. Using the notation from there, we write for their evaluations respectively $\mathbf{E}^*$ and $\mathbf{E}_*$. Obviously we have

$$\mathbf{E}_*\int^* = \int^* + \int_* = \mathbf{E}^*\int_*,$$

where $\int^* + \int_*$ behaves like a definite integral since it evaluates into K. In fact, it is $\int_a^b$ in the standard model $\mathscr{F} = C^\infty[a,b]$, where $\mathbf{E}^* f = f(a)$ and $\mathbf{E}_* f = f(b)$. Taking the analogy further, we introduce now the *inner product* $\langle|\rangle\colon \mathscr{F}\times\mathscr{F}\to K$ on an analytic algebra $(\mathscr{F},\int^*,\int_*)$ by

$$\langle f|g\rangle = (\int^* + \int_*)\, fg.$$

For $\mathscr{F} = C^\infty[a,b]$ this gives the L^2 inner product $\langle f|g\rangle = \int_a^b f(x)g(x)\,dx$.

Proposition 3.24. *In an analytic algebra $(\mathscr{F},\partial,\int^*,\int_*)$, the operator $\int^*$ is the adjoint of $\int_*$ with respect to $\langle|\rangle$.*

Proof. Using the pure Baxter axiom (3.6) for $\int^*$ yields

$$\langle \textstyle\int^* f|g\rangle = \mathbf{E}_*(\int^* g \int^* f) = \mathbf{E}_*(\int^* f)\mathbf{E}_*(\int^* g) - \mathbf{E}_*(\int^* f \int^* g).$$

But $\int^* g = \mathbf{E}_*(\int^* g) - \int_* g$ and $\langle f|\mathbf{E}_*(\int^* g)\rangle = \mathbf{E}_*(\int^* g)\,\langle f|1\rangle = \mathbf{E}_*(\int^* g)\mathbf{E}_*(\int^* f)$, so we can rewrite the last summand as

$$\mathbf{E}_*(\textstyle\int^* f \int^* g) = \langle f|\int^* g\rangle = \mathbf{E}_*(\int^* f)\mathbf{E}_*(\int^* g) - \langle f|\int_* g\rangle,$$

which implies $\langle \int^* f|g\rangle = \langle f|\int_* g\rangle$ as required. □

While the setting of analytic algebras is made for two-point boundary problems, which will treated as a special case in Chapter 6, we will now restrict our attention to the even more specialized setting of *initial value problems*, whose Green's operators will turn out to be the fundamental building blocks for all other Green's operators.

3.3 Initial Value Problems

Looking back to Lemma 3.21, we see that we can at least solve some differential equations. But in general, we cannot assume that the solutions of a differential equation with coefficients in $\mathscr{F}$ are again in $\mathscr{F}$. For example, in $\mathscr{F} = K[x]$, the differential equation $u' - u = 0$ has no solution. In fact, its "actual" solution space is spanned by $u(x) = e^x$ if $K = \mathbb{R}$ or $K = \mathbb{C}$. So in this case we should have taken the exponential polynomials $\mathscr{F} = K[x, e^{Kx}]$ for ensuring that $u \in \mathscr{F}$. But if this is the

case, we can also solve the *inhomogeneous differential equation* $u' - u = f$ whose general solution is $\mathscr{F}e^x + e^x \int e^{-x} f$, with $\int = \int_0^x$ as usual. Of course we can also incorporate the initial condition $u(0) = 0$, which leads to $u = e^x \int e^{-x} f$.

This observation is generally true: Whenever we can solve the homogeneous differential equation within $\mathscr{F}$, we can also solve the initial value problem for the corresponding inhomogeneous problem. The classical tool for achieving this explicitly is the *variation-of-constants formula* [15, p. 74], whose abstract formulation is given in Theorem 3.25 below.

As usual [35], we will write $\mathscr{F}[\partial]$ for the the ring of differential operators with coefficients in $\mathscr{F}$. If $T \in \mathscr{F}[\partial]$ is monic (i.e. having leading coefficient 1) with $\deg T = n$, we call a basis $u_1, \ldots, u_n$ for $\mathrm{Ker}(T)$ a *fundamental system* of solutions for the homogeneous equation $Tu = 0$. The fundamental system will be called *regular* if its associated Wronskian matrix

$$W = \begin{pmatrix} u_1 & \cdots & u_n \\ u_1' & \cdots & u_n' \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{pmatrix}$$

is invertible in $\mathscr{F}^{n\times n}$ or equivalently [29, p. 518] if $\det W$ is invertible in $\mathscr{F}$. Of course this alone implies already that $u_1, \ldots, u_n$ are linearly independent.

Theorem 3.25. *Let $(\mathscr{F}, \partial, \int)$ be an ordinary integro-differential algebra. Given a monic differential operator $T \in \mathscr{F}[\partial]$ with $\deg T = n$ and a regular fundamental system of solutions $u_1, \ldots, u_n \in \mathscr{F}$ for $Tu = 0$, the initial value problem*

$$\boxed{\begin{aligned} &Tu = f \\ &\mathbf{E}u = \mathbf{E}u' = \cdots = \mathbf{E}u^{(n-1)} = 0 \end{aligned}} \tag{3.12}$$

has the unique solution

$$u = \sum_{i=1}^{n} u_i \int d^{-1} d_i f \tag{3.13}$$

for every $f \in \mathscr{F}$, where d is the determinant of the Wronskian matrix W associated with $u_1, \ldots, u_n$, and d_i the determinant of the maxtrix W_i obtained from W by replacing the i-th column by the n-th unit vector.

Proof. We can use the usual technique of reformulating $Tu = f$ as a system of linear first-order differential equations with companion matrix $A \in \mathscr{F}^{n\times n}$. We extend the action of the operators $\int, \partial, \mathbf{E}$ componentwise to $\mathscr{F}^n$. Setting now

$$\hat{u} = W \int W^{-1} \hat{f}$$

with $\hat{f} = (0, \ldots, 0, f)^\top \in \mathscr{F}^n$, we check that $\hat{u} \in \mathscr{F}^n$ is a solution of the first-order system $\hat{u}' = A\hat{u} + \hat{f}$ with initial condition $\mathbf{E}\hat{u} = 0$. Indeed we have $\hat{u}' = W' \int W^{-1} \hat{f} + WW^{-1} \hat{f}$ by the Leibniz rule and $AW = W'$ since $u_1, \ldots, u_n$ are solutions of $Tu = 0$;

so the differential system is verified. For checking the initial condition, note that $\mathbf{E}\int W^{-1}\hat{f}$ is already the zero vector, so we have also $\mathbf{E}\hat{u}=0$ since $\mathbf{E}$ is multiplicative.

Writing u for the first component of $\hat{u}$, we obtain a solution of the initial value problem (3.12), due to the construction of the companion matrix. Let us now compute $\hat{g}=W^{-1}\hat{f}$. Obviously $\hat{g}$ is the solution of the linear equation system $W\hat{g}=\hat{f}$. Hence Cramer's rule, which is also applicable for matrices over rings [29, p. 513], yields $\hat{g}_i$ as $d^{-1}d_i f$ and hence

$$u=(W\textstyle\int\hat{g})_1=\sum_{i=1}^{n}u_i\int d^{-1}\,d_i f$$

since the first row of W is $(u_1,\dots,u_n)$.

For proving uniqueness, it suffices to show that the homogeneous initial value problem only has the trivial solution. So assume u solves (3.12) with $f=0$ and choose coefficients $c_1,\dots,c_n\in K$ such that

$$u=c_1u_1+\cdots+c_nu_n.$$

Then the initial conditions yield $\mathbf{E}(Wc)=0$ with $c=(c_1,\dots,c_n)^\top\in K^n$. But we have also $\mathbf{E}(Wc)=(\mathbf{E}W)c$ because $\mathbf{E}$ is linear, and $\det\mathbf{E}W=\mathbf{E}(\det W)$ because it is moreover multiplicative. Since $\det W\in\mathscr{F}$ is invertible, $\mathbf{E}W\in K^{n\times n}$ is regular, so $c=(\mathbf{E}W)^{-1}0=0$ and $u=0$. □

▼ A few words about the requirements in Theorem 3.25. If a differential equation lacks solutions (i.e. when there are fewer linearly independent solutions than prescribed by the order of the differential operator), they can be adjoined; see the remarks in Section 3.1. For ensuring regularity, we need an *invertible Wronskian* d. This could also be enforced by a suitable localization of $\mathscr{F}$, as for Picard-Vessiot rings [35, p. 12]. But in many applications, this condition will come out naturally: The Wronskian d is always an exponential over $\mathscr{F}$ since it satisfies the differential equation $d'=ad$, where a is the trace of the system matrix A. In our case A is the companion matrix for $T=\partial^n+a_{n-1}\partial^{n-1}+\cdots+a_1\partial+a_0$, so the trace is given by $a=-a_{n-1}$.

Sometimes it is practical to work over integro-differential algebras that are large enough for ensuring these requirements for all differential operators. In order to have some finer control on which differential equations we want to have solutions, we will allow to specify the *coefficients* of the pertinent linear differential operators [40].

Definition 3.26. A differential subalgebra $\mathscr{F}_0\le\mathscr{F}$ is called *saturated* for a differential algebra $\mathscr{F}$ if $\dim\operatorname{Ker}(T)=n$ for every monic $T\in\mathscr{F}_0[\partial]$ with $\deg T=n$ and if all nonzero solutions u of $u'=au$, with $a\in\mathscr{F}_0$ have $\mathbf{E}u\neq 0$. In this context, we call $\mathscr{F}$ the *ground algebra* and $\mathscr{F}_0$ the *coefficient algebra*. If $\mathscr{F}_0$ coincides with $\mathscr{F}$, we simply speak of a saturated integro-differential algebra.

In our original definition [40] we have required that all nonzero solutions u of $u'=au$, with $a\in\mathscr{F}_0$, are *invertible* in $\mathscr{F}$. But this condition follows from the requirement on the evaluation since u has the inverse v/c, where v is a solution of $v'=-av$ and $c=(\mathbf{E}u)(\mathbf{E}v)$.

Some further remarks on the definition. First of all, we point out that we need $\mathscr{F}_0$ to be differentially closed such that we can multiply within $\mathscr{F}_0[\partial]$, which will be needed later for multiplying boundary problems (Section 6.2). The first condition on solvability ensures that *homogeneous equations* $Tu=0$ have a fundamental system with the appropriate number of solutions, while the second condition means that *exponentials* behave as usual. Note also that $\mathscr{F}$ is an ordinary differential algebra as soon as it possesses a saturated coefficient algebra.

Not every integro-differential algebra has a saturated coefficient algebra, e.g. the polynomial algebra $(K[x], \partial, \int)$ does not. We do not know any useful criteria for settling this question. However, there are several important *typical examples* of integro-differential algebras with saturated coefficient algebras. The most obvious example is furnished by $C^\infty[a,b]$ or $C^\omega[a,b]$, with coefficient algebra either itself or any differential subalgebra like $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{C}[x]$.

A less demanding but practically important example of a saturated integro-differential algebra is given by the *exponential polynomials* $K[x, e^{Kx}]$, with $\mathbb{C}$ as a coefficient algebra. This reflects the fact that every solution of a LODE with constant coefficients can be expressed in terms of exponentials.

Similar to its analysis counterpart, also the formal power series $K[[z]]$ are a saturated integro-differential algebra. More generally, the *Hurwitz series* of Example 3.14 are saturated. Defining the exponential function $\exp = (1, 1, 1, \dots)$, we obtain immediately $\partial \exp = \exp$. One can introduce a composition $f \circ g$ for $f, g \in H(K)$ whenever g has vanishing constant term, and the usual chain rule is satisfied for this composition [26]. Then the first-order homogeneous equation $u' = au$ with $a \in H(K)$ is solved by

$$u = c \exp \circ (\textstyle\int a),$$

which is easily seen to be invertible in $H(K)$. By Corollary 4.3 in [26], we know also that all monic homogeneous differential equations of order n have an n-dimensional kernel. Hence $H(K)$ is a saturated integro-differential algebra. ▲

Chapter 4
Solving Boundary Problems in Linear Algebra

4.1 Sections between Modules

Let M and N be modules over a ring R. Let $T\colon M \to N$ and $G\colon N \to M$ be linear maps such that $TG = 1$ meaning that G is a *section* of T. Then T is surjective and G injective, respectively, and GT is a projector since $(GT)^2 = G(TG)T = GT$. Hence

$$\mathrm{Ker}(GT) = \mathrm{Ker}(T) \quad \text{and} \quad \mathrm{Im}(GT) = \mathrm{Im}(G), \tag{4.1}$$

and we have

$$M = \mathrm{Ker}(T) \dotplus \mathrm{Im}(G) \tag{4.2}$$

as a direct sum.

Conversely, we can begin with a given surjective linear map $T\colon M \to N$ and a complement of $\mathrm{Ker}(T)$, and ask if there exists a corresponding right inverse. This is a special case of algebraic generalized inverses as for example in Nashed and Votruba [33].

Proposition 4.1. *Let $T\colon M \to N$ be a surjective linear map and $\mathscr{I}$ a complement of $\mathrm{Ker}(T)$ in M, so that $M = \mathrm{Ker}(T) \dotplus \mathscr{I}$. Then there exists a unique section G of T with $\mathrm{Im}(G) = \mathscr{I}$. Moreover, G is the unique solution of the equation*

$$GT = 1 - P,$$

where P is the projector with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(P) = \mathscr{I}$.

Proof. By (4.1), any section with $\mathrm{Im}(G) = \mathscr{I}$ satisfies $GT = 1 - P$. Hence G is unique since T is surjective. For proving existence, let $\iota\colon \mathscr{I} \to M$ be the canonical injection. Then $T \circ \iota$ is an isomorphism since $\mathrm{Ker}(T) \cap \mathscr{I} = 0$. Its inverse G, considered as a map from N to M, is a section of T with $\mathrm{Im}(G) = \mathscr{I}$.

Now assume that $GT = 1 - P$. Let $w \in N$. Since T is surjective, $w = Tv$ with $v \in M$. Then

$$TGw = TGTv = T(v - Pv) = Tv = w$$

since $\mathrm{Im}(P) = \mathrm{Ker}(T)$. So G is the unique section of T with $\mathrm{Im}(G) = \mathscr{I}$. □

So we have a bijection between the set of complements of $\mathrm{Ker}(T)$ in M and the set of sections of T. The next proposition allows us to describe all sections in terms of a fixed one.

Corollary 4.2. *Given any section $\tilde{G}$ of T, the section corresponding to a complement $\mathscr{I}$ of* $\mathrm{Ker}(T)$ *is given by*

$$G = (1-P)\tilde{G},$$

where P is the projector with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ *and* $\mathrm{Ker}(P) = \mathscr{I}$.

Proof. Let $\tilde{P}$ be the projector with $\mathrm{Im}(\tilde{P}) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(\tilde{P}) = \mathrm{Im}(\tilde{G})$. The claim follows by Proposition 4.1, since

$$GT = (1-P)\tilde{G}T = (1-P)(1-\tilde{P}) = 1-\tilde{P}-P+P\tilde{P} = 1-P,$$

where we use that $P\tilde{P} = \tilde{P}$ because $\mathrm{Im}(P) = \mathrm{Im}(\tilde{P}) = \mathrm{Ker}(T)$.

For integro-differential algebras, we can now describe all integrals in terms of a fixed one using also Corollary 3.5 and the characterization of multiplicative projectors in Lemma 3.4.

Corollary 4.3. *Let $(\mathscr{F}, \partial, \int)$ be an integro-differential algebra. Let $\mathscr{I}$ be a complement of $\mathscr{C} = \mathrm{Ker}(\partial)$ that is also an ideal in $\mathscr{F}$, and let P be the projector with* $\mathrm{Im}(P) = \mathscr{C}$ *and* $\mathrm{Ker}(P) = \mathscr{I}$. *Then P is a multiplicative projector and*

$$\textstyle\int_P = (1-P)\int$$

is an integral for ∂ with evaluation P.

This establishes a bijection between the set of complements of $\mathscr{C}$ that are also ideals in $\mathscr{F}$ and the set of integrals for ∂; each such complement corresponds to a multiplicative projector onto $\mathscr{C}$. Specializing to ordinary integro-differential algebras, we can reformulate this result using Corollary 3.22, describing a bijection between the set of characters and the set of integrals for ∂.

Corollary 4.4. *Let $(\mathscr{F}, \partial, \int)$ be an ordinary integro-differential algebra over a field K. Let φ be a character on $\mathscr{F}$. Then*

$$\textstyle\int_\varphi = (1-\varphi)\int$$

is an integral for ∂ with evaluation φ.

The last proposition in this subsection describes the inverse image of a composition of an arbitrary and a surjective linear map in terms of one of its sections.

Proposition 4.5. *Let L, M and N be modules over a ring R. Let $A\colon M \to N$ and $T\colon L \to M$ be respectively an arbitrary and a surjective linear map. Let G be a section of T and $N_1 \leq N$ a submodule. Then we have*

$$(AT)^{-1}(N_1) = GA^{-1}(N_1) \dotplus \operatorname{Ker}(T)$$

for the inverse image of the composite. In particular, we have

$$\operatorname{Ker}(AT) = G\operatorname{Ker}(A) \dotplus \operatorname{Ker}(T)$$

for the kernel of the composite and

$$T^{-1}(N_1) = G(N_1) \dotplus \operatorname{Ker}(T)$$

for the inverse image.

Proof. For the inclusion $\supseteq$ just observe that

$$AT(GA^{-1}(N_1) + \operatorname{Ker}(T)) = AA^{-1}(N_1) + 0 \subseteq N_1.$$

Conversely, let $u \in (AT)^{-1}(N_1)$. Then $Tu = v$ with $v \in A^{-1}(N_1)$. Hence

$$T(u - Gv) = Tu - v = 0$$

and therefore $u \in GA^{-1}(N_1) + \operatorname{Ker}(T)$. The sum is direct since we have even $G(M) \cap \operatorname{Ker}(T)) = 0$ by (4.2). □

4.2 Abstract Boundary Conditions

First we recall the notion of orthogonality for a bilinear map of modules. Let M and N be modules over a ring R and $\langle|\rangle\colon M \times N \to R$ be a bilinear map. Two vectors $x \in M$ and $y \in N$ are called *orthogonal* with respect to $\langle|\rangle$ if $\langle x|y\rangle = 0$. Let $X^\perp$ denote the set of all $y \in N$ that are orthogonal to X for a fixed bilinear map $\langle|\rangle$. This is obviously a submodule of N, which we call the *orthogonal* of X. We define the orthogonal on the other side in the same way.

It follows directly from the definition that for any subsets $X_1, X_2 \subseteq M$ we have the two characteristic properties

$$X_1 \subseteq X_2 \Rightarrow X_1^\perp \supseteq X_2^\perp \quad \text{and} \quad X_1 \subseteq X_1^{\perp\perp}. \tag{4.3}$$

These statements hold also for subsets of N. The two properties (4.3) for orthogonality are those of an (order-reversing) Galois connection:

Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets (posets). An (order-reversing) *Galois connection* of these posets consists of two order-reversing maps $F\colon A \to B$ and $G\colon B \to A$ such that

$$a \leq GF(a) \quad \text{and} \quad b \leq FG(b) \tag{4.4}$$

for $a \in A$ and $b \in B$.

For a Galois connection, we see that

$$F = FGF \quad \text{and} \quad G = GFG. \tag{4.5}$$

From $a \leq GF(a)$ it follows that $F(a) \geq FGF(a)$ since F reverses the order. From $b \leq FG(b)$ we also obtain $F(a) \leq FGF(a)$, so that $F = FGF$. Similarly, one sees that $G = GFG$. So if $a = G(b) \in \mathrm{Im}(G)$, then $GF(a) = GFG(b) = G(b) = a$, so that $GF = 1$ on $\mathrm{Im}(G)$. This holds analogously for FG, and hence F and G induce mutually inverse *order-reversing bijections*

$$\mathrm{Im}(G) = \{a \in A \mid a = GF(a)\} \rightleftarrows \mathrm{Im}(F) = \{b \in B \mid b = FG(b)\}. \tag{4.6}$$

In concrete cases, the difficult part is usually to determine the sets $\mathrm{Im}(G)$ and $\mathrm{Im}(F)$.

Note also that for any Galois connection, the maps FG and GF are *closure operators*, meaning extensive, order-preserving and idempotent self-maps. This follows immediately from Equations (4.4), (4.5).

The concept of Galois connection generalizes the correspondence between subfields and subgroups in *Galois theory*. Another well-known example of a Galois connection is the correspondence between affine varieties and ideals in algebraic geometry. For further details and references on Galois connections we refer to Erné et al. [19].

Returning to the Galois connection related to orthogonality, let $\mathbb{P}(M)$ denote the *projective geometry* of a module M defined as the poset of all submodules ordered by inclusion. Then orthogonality gives a Galois connection between the projective geometries $\mathbb{P}(M) \rightleftarrows \mathbb{P}(N)$ defined by

$$M_1 \mapsto M_1^{\perp} \quad \text{and} \quad N_1 \mapsto N_1^{\perp}. \tag{4.7}$$

So by Equation (4.5) we know that $S^{\perp} = S^{\perp\perp\perp}$ for any submodule S of M or N.

We call a submodule S *orthogonally closed* if $S = S^{\perp\perp}$. As noted in (4.6), the Galois connection restricted to orthogonally closed submodules is an order-reversing bijection.

Let now V be a vector space over a field K, and let V^* denote its dual space. In the following, we study orthogonality and the corresponding Galois connection induced by the *canonical bilinear form*

$$V \times V^* \to K$$

defined by $\langle v|\beta\rangle = \beta(v)$. Then the orthogonals of subspaces $\mathscr{B}_1 \leq V^*$ and $V_1 \leq V$ are respectively the subspaces

$$\mathscr{B}_1^{\perp} = \{v \in V \mid \beta(v) = 0 \text{ for all } \beta \in \mathscr{B}_1\}$$

and

$$V_1^\perp = \{\beta \in V^* \mid \beta(v) = 0 \text{ for all } v \in V_1\}.$$

We consider first subspaces of the vector space V. The next proposition tells us that in this case the situation is simple.

Proposition 4.6. *Every subspace of V is orthogonally closed.*

Proof. Let $V_1 \leq V$ be a subspace and $v \in V \backslash V_1$. Since $V_1 \subseteq V_1^{\perp\perp}$ it is enough to show that v is not in $V_1^{\perp\perp}$. Using the fact that any basis of a subspace can be extended to a basis for V, we can construct a linear form $\beta \in V^*$ with $\beta(v_1) = 0$ for all $v_1 \in V_1$, that is, $\beta \in V_1^\perp$, and $\beta(v) = 1$. Hence v is not in $V_1^{\perp\perp}$. □

For finite codimensional subspaces, we can also compute the dimension of the orthogonal. If $V_1 \leq V$, we have a natural isomorphism

$$V_1^\perp \cong (V/V_1)^*.$$

Indeed, each $\beta \in V_1^\perp$ induces a well-defined linear form on V/V_1 since it vanishes on V_1, and it is easy to see that this gives an isomorphism between $V_1^\perp$ and $(V/V_1)^*$. This implies in particular that

$$\operatorname{codim} V_1 = \dim V_1^\perp \tag{4.8}$$

if $\operatorname{codim} V_1 < \infty$.

In the following, we discuss orthogonality for subspaces of the dual vector space V^*. We first recall some results for biorthogonal systems. Two families $(v_i)_{i\in I}$ of vectors in V and linear forms $(\beta_i)_{i\in I}$ in V^* are called *biorthogonal* or said to form a *biorthogonal system* if

$$\langle v_i | \beta_j \rangle = \delta_{ij}.$$

For a biorthogonal system $(v_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ we can easily compute the coefficients of a linear combination $v = \sum a_i v_i$. Applying β_j, we obtain

$$\langle v | \beta_j \rangle = \sum a_i \langle v_i | \beta_j \rangle = a_j. \tag{4.9}$$

Evaluating a linear combination $\beta = \sum a_j \beta_j$ at v_i gives analogously

$$\langle v_i | \beta \rangle = \sum a_j \langle v_i | \beta_j \rangle = a_i. \tag{4.10}$$

This implies in particular that the v_i and β_i are linearly independent. Moreover, we can easily compute projectors onto finite dimensional vector spaces from a finite biorthogonal system.

Proposition 4.7. *Let $v_1, \ldots, v_n \in V$ and $\beta_1, \ldots, \beta_n \in V^*$ be biorthogonal. Let $V_1 = [v_1, \ldots, v_n]$ and $\mathscr{B}_1 = [\beta_1, \ldots, \beta_n]$. Then $P\colon V \to V$ defined by*

$$v \mapsto \sum_{i=1}^{n} \langle v | \beta_i \rangle v_i$$

is a projector with $\operatorname{Im}(P) = V_1$ and $\operatorname{Ker}(P) = \mathscr{B}_1^\perp$ so that

$$V = \mathscr{B}_1^\perp \dot{+} V_1 \quad \textit{and} \quad \operatorname{codim} \mathscr{B}_1^\perp = n.$$

Moreover, for any $\beta \in \mathscr{B}_1^{\perp\perp}$ we have

$$\beta = \sum_{i=1}^{n} \langle v_i | \beta \rangle \beta_i,$$

so that $\mathscr{B}_1$ is orthogonally closed.

Proof. With Equation (4.9) we see that $\langle \sum_{i=1}^n \langle v|\beta_i\rangle v_i | \beta_j \rangle = \langle v|\beta_j\rangle$, hence $P^2 = P$. Obviously, $\operatorname{Im}(P) = V_1$ and $\operatorname{Ker}(P) = \mathscr{B}_1^\perp$ by the definition of the orthogonal. Hence we have

$$V = \operatorname{Ker}(P) \dot{+} \operatorname{Im}(P) = \mathscr{B}_1^\perp \dot{+} V_1$$

and therefore $\operatorname{codim} \mathscr{B}_1^\perp = n$ since the v_i are linearly independent. Let now $\beta \in \mathscr{B}_1^{\perp\perp}$ and define

$$\tilde{\beta} = \sum_{i=1}^{n} \langle v_i | \beta \rangle \beta_i.$$

Then, owing to (4.10), we have

$$\tilde{\beta}(v_j) = \langle v_j | \tilde{\beta} \rangle = \langle v_j | \beta \rangle = \beta(v_j).$$

So the linear forms β and $\tilde{\beta}$ coincide on V_1 and since both vanish on $\mathscr{B}_1^\perp$, they are equal. Hence $\mathscr{B}_1^{\perp\perp} \subseteq \mathscr{B}_1$. The sets are equal since we always have $\mathscr{B}_1 \subseteq \mathscr{B}_1^{\perp\perp}$, so that $\mathscr{B}_1$ is indeed orthogonally closed. □

We have already seen that linear functionals in a biorthogonal system are linearly independent. In fact, linear independence can be characterized in this way.

Proposition 4.8. *Let $\beta_1, \ldots, \beta_n \in V^*$. Then the β_i are linearly independent iff there exist $v_1, \ldots, v_n \in V$ such that (v_i) and (β_i) are biorthogonal.*

Proof. It suffices to prove the converse implication; we use induction on n. For $n = 1$ we choose $u \in V$ with $\beta_1(u) \neq 0$ and set $v_1 = \beta_1(u)^{-1} u$. For the induction step, let $\tilde{v}_1, \ldots, \tilde{v}_{n-1} \in V$ be such that $\langle \tilde{v}_i | \beta_j \rangle = \delta_{ij}$. There exists

$$u \in [\beta_1, \ldots, \beta_{n-1}]^\perp, \quad \text{with } \beta_n(u) \neq 0.$$

Otherwise we would have $\beta_n \in [\beta_1, \ldots, \beta_{n-1}]^{\perp\perp}$, and the β_i would be linearly dependent by the previous proposition. We set

$$v_n = \beta_n(u)^{-1} u \quad \text{and} \quad v_i = \tilde{v}_i - \beta_n(\tilde{v}_i) v_n, \text{ for } i = 1, \ldots, n-1.$$

Then $\langle v_i | \beta_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$. □

Combining the two previous propositions, we obtain the analog of (4.8): for a finite dimensional subspace $\mathscr{B}_1 \leq V^*$, we have

$$\dim \mathscr{B}_1 = \operatorname{codim} \mathscr{B}_1^\perp. \tag{4.11}$$

For the dual vector space V^*, characterizing the orthogonally closed subspace is considerably more involved. For our present purposes it suffices to look at the finite dimensional subspaces.

Corollary 4.9. *Every finite dimensional subspace of V^* is orthogonally closed.*

But if V is an infinite dimensional vector space, there are always linear subspaces, and indeed hyperplanes in V^*, that are not orthogonally closed; see e.g. [28, p. 71]. Nevertheless, since all subspaces of V are orthogonally closed, we have via the Galois connection (4.7) an order-reversing bijection between $\mathbb{P}(V)$ and the poset of all orthogonally closed subspaces of V^*. So we can describe any subspace $V_1 \leq V$ implicitly by the corresponding orthogonally closed subspace $V_1^\perp$. (This Galois connection will be investigated in more detail in the second part of the lecture. ???)

We have already seen (4.8), (4.11) that if $\operatorname{codim} V_1 < \infty$ and $\dim \mathscr{B}_1 < \infty$, then

$$\operatorname{codim} V_1 = \dim V_1^\perp \quad \text{and} \quad \dim \mathscr{B}_1 = \operatorname{codim} \mathscr{B}_1^\perp. \tag{4.12}$$

So we can consider the restriction of the Galois connection induced by the canonical bilinear form to *finite codimensional* subspaces of V and *finite dimensional* subspaces of V^*, which we denote respectively by $\mathbb{P}_{\text{cof}}(V)$ and $\mathbb{P}_{\text{fin}}(V^*)$. Since all subspaces of V and finite dimensional subspaces of V^* are orthogonally closed, we have an order-reversing bijection

$$\mathbb{P}_{\text{cof}}(V) \rightleftarrows \mathbb{P}_{\text{fin}}(V^*).$$

induced by orthogonality.

Note also that $\mathbb{P}_{\text{cof}}(V)$ and $\mathbb{P}_{\text{fin}}(V^*)$ are closed under finite intersection and sum of subspaces, so they are *lattices*, meaning posets in which any two elements have a unique supremum (called join) and a unique infimum (called meet). See for example [17] for more on lattices. Since an order-preserving bijection between lattices preserves join and meet, we obtain the following proposition, which can also be verified directly using the properties of a Galois connection and the definitions.

Proposition 4.10. *We have*

$$(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp, \quad (\mathscr{B}_1 \cap \mathscr{B}_2)^\perp = \mathscr{B}_1^\perp + \mathscr{B}_2^\perp$$

and

$$(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp, \quad (\mathscr{B}_1 + \mathscr{B}_2)^\perp = \mathscr{B}_1^\perp \cap \mathscr{B}_2^\perp$$

for subspaces $V_1, V_2 \in \mathbb{P}_{\text{cof}}(V)$ and $\mathscr{B}_1, \mathscr{B}_2 \in \mathbb{P}_{\text{fin}}(V^*)$.

We conclude this section with some general remarks on the dimension and codimension of the intersection and sum of two subspaces. We use these observations for characterizing regular abstract boundary problems in the next section.

Recall that for subspaces V_1 and V_2 of a vector space V we have

$$\dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim V_1 + \dim V_2$$

and analogously for the codimension

$$\operatorname{codim}(V_1+V_2)+\operatorname{codim}(V_1\cap V_2)=\operatorname{codim}V_1+\operatorname{codim}V_2.$$

Note that if V is finite dimensional, the second equation is a consequence of the first and the equation $\dim V_1+\operatorname{codim}V_1=\dim V$. For V finite dimensional, we obtain similarly the equation

$$\operatorname{codim}(V_1+V_2)+\dim V_1=\dim(V_1\cap V_2)+\operatorname{codim}V_2$$

relating the codimension of the sum to the dimension of the intersection of two subspaces. We show that this equation holds for arbitrary vector spaces.

Proposition 4.11. *We have*

$$\operatorname{codim}(V_1+V_2)+\dim V_1=\dim(V_1\cap V_2)+\operatorname{codim}V_2$$

for subspaces V_1 and V_2 of a vector space V.

Proof. Let $\tilde{V}_1$ and $\tilde{V}_2$ be complements of $V_1\cap V_2$ in V_1 and V_2, respectively, so that $V_1=\tilde{V}_1\dotplus(V_1\cap V_2)$ and $V_2=\tilde{V}_2\dotplus(V_1\cap V_2)$. Then one sees that we have a direct sum

$$V_1+V_2=\tilde{V}_1\dotplus\tilde{V}_2\dotplus(V_1\cap V_2).$$

Let $\tilde{W}$ be a complement of V_1+V_2 in V so that

$$V=(V_1+V_2)\dotplus\tilde{W}=\tilde{V}_1\dotplus\tilde{V}_2\dotplus(V_1\cap V_2)\dotplus\tilde{W}.$$

Hence $\operatorname{codim}(V_1+V_2)=\dim\tilde{W}$ and $\operatorname{codim}V_2=\dim(\tilde{W}+\tilde{V}_1)$. Computing the dimension of the subspace $\tilde{W}+\tilde{V}_1+(V_1\cap V_2)$ in two different ways, we obtain

$$\begin{aligned}\operatorname{codim}(V_1+V_2)+\dim V_1&=\dim\tilde{W}+\dim(\tilde{V}_1+(V_1\cap V_2))\\&=\dim(V_1\cap V_2)+\dim(\tilde{W}+\tilde{V}_1)=\dim(V_1\cap V_2)+\operatorname{codim}V_2,\end{aligned}$$

and the proposition is proved. □

If V_1 is finite dimensional and V_2 finite codimensional, all dimensions and codimensions in the above proposition are finite, and we obtain the following corollaries.

Corollary 4.12. *Let V_1,V_2 be subspaces of a vector space V with $\dim V_1<\infty$ and $\operatorname{codim}V_2<\infty$. Then*

$$\operatorname{codim}(V_1+V_2)-\dim(V_1\cap V_2)=\operatorname{codim}V_2-\dim V_1.$$

In particular, we have $\dim(V_1\cap V_2)=\operatorname{codim}(V_1+V_2)$ iff $\dim V_1=\operatorname{codim}V_2$.

Corollary 4.13. *Let V_1 and V_2 be subspaces of a vector space V with $\dim V_1<\infty$ and $\operatorname{codim}V_2<\infty$. Then $V_1\dotplus V_2=V$ iff ($V_1\cap V_2=0$ and $\dim V_1=\operatorname{codim}V_2$) iff ($V_1+V_2=V$ and $\dim V_1=\operatorname{codim}V_2$).*

So for testing whether two subspaces V_1 and V_2 with $\dim V_1 = \operatorname{codim} V_2 < \infty$ establish a direct decomposition $V = V_1 \dotplus V_2$, we have to check only one of the two defining conditions $V_1 \cap V_2 = 0$ and $V_1 + V_2 = V$.

The hypothesis that the dimensions are finite is necessary. Let K be a field, $V = K^{\mathbb{N}}$, and consider for example the two subspaces

$$V_1 = \{(0, x_1, 0, x_2, 0, x_3, \ldots) \mid (x_n) \in V\}$$
$$V_2 = \{(0, 0, x_1, 0, x_2, 0, x_3, \ldots) \mid (x_n) \in V\}.$$

Then $\dim V_1 = \operatorname{codim} V_2 = \dim V = \infty$, $V_1 \cap V_2 = 0$ but $\operatorname{codim}(V_1 + V_2) = 1$.

We use the following corollary in Section 4.3 as a regularity test for abstract boundary problems.

Corollary 4.14. *Let $V_1 = [v_1, \ldots, v_m]$ be a subspace of a vector space V and $\mathscr{B}_1 = [\beta_1, \ldots, \beta_n]$ a subspace of V^* with β_i and v_j linearly independent. Then*

$$V = V_1 \dotplus \mathscr{B}_1^{\perp}$$

iff $m = n$ and the so-called evaluation matrix

$$\beta(u) = \begin{pmatrix} \beta_1(v_1) & \cdots & \beta_1(v_n) \\ \vdots & \ddots & \vdots \\ \beta_n(v_1) & \cdots & \beta_n(v_n) \end{pmatrix} \in K^{n \times n}$$

is regular.

Proof. By Equation (4.12), $\operatorname{codim} \mathscr{B}_1^{\perp} = \dim \mathscr{B}_1$, so we know from the previous corollary that $V = V_1 \dotplus \mathscr{B}_1^{\perp}$ iff $V_1 \cap \mathscr{B}_1^{\perp} = 0$ and $m = n$. Let $B = (\beta_i(v_j))$ with columns b_j. Now note that B is singular iff there exists a linear combination $\sum \lambda_j b_j = 0$ with at least one $\lambda_j \neq 0$ iff there exists a nonzero $u = \sum \lambda_j v_j$ in $V_1 \cap \mathscr{B}_1^{\perp}$. □

4.3 Abstract Boundary Problems

In this section, we first define *abstract boundary problems* and their Green's operators in a linear algebra setting. We use the notion of orthogonally closed subspaces and the results on sections from the two previous sections. This setting includes also boundary problems for LPDEs as exemplified for the wave equation below. Then we discuss algorithmic aspects for abstract boundary problems, where the corresponding linear maps have finite dimensional kernels and the spaces of boundary conditions are finite dimensional. Note that this includes boundary problems for (systems of) ordinary differential equations and systems of partial differential equations with finite dimensional solution space.

An *abstract boundary problem* is given by a pair $(T,\mathscr{B})$, where $T\colon V\to W$ is a surjective linear map between vector spaces V, W and $\mathscr{B}\leq V^*$ an orthogonally closed subspace of *boundary conditions*. We say that $u\in V$ is a solution of $(T,\mathscr{B})$ for a given $f\in W$ if

$$Tu=f \quad \text{and} \quad u\in\mathscr{B}^\perp.$$

A boundary problem $(T,\mathscr{B})$ is called *regular* if $\mathscr{B}^\perp$ is a complement of $\operatorname{Ker}(T)$ so that

$$V=\operatorname{Ker}(T)\dotplus\mathscr{B}^\perp.$$

Then by Proposition 4.1 there exists a unique section $G\colon W\to V$ of T with $\operatorname{Im}(G)=\mathscr{B}^\perp$. We call G the *Green's operator* for the boundary problem $(T,\mathscr{B})$. Since $TGf=f$ and $Gf\in\mathscr{B}^\perp$, we see that the Green's operator maps every forcing function $f\in W$ to its unique solution $u=Gf\in V$. Hence we say that G solves the boundary problem $(T,\mathscr{B})$, and we use the notation

$$G=(T,\mathscr{B})^{-1}.$$

Conversely, if there exists any section G of T for a boundary problem $(T,\mathscr{B})$ such that $\operatorname{Im}(G)=\mathscr{B}^\perp$, it is regular by (4.2). Given any section $\tilde{G}$ of T, we know with Corollary 4.2 that the Green's operator for a regular boundary problem $(T,\mathscr{B})$ is given by

$$G=(1-P)\tilde{G}, \tag{4.13}$$

where P is the projector with $\operatorname{Im}(P)=\operatorname{Ker}(T)$ and $\operatorname{Ker}(P)=\mathscr{B}^\perp$.

If T is invertible, then $(T,0)$ is the only regular boundary problem for T, and its Green's operator is $(T,0)^{-1}=T^{-1}$. In particular, we have

$$(1,0)^{-1}=1 \tag{4.14}$$

for the identity operator.

Example 4.15. As an example of how boundary problems for LPDEs can be seen as abstract boundary problems, we consider the following problem for the *wave equation* on the domain $\Omega=\mathbb{R}\times\mathbb{R}_{\geq 0}$, writing V for $C^\infty(\Omega)$: Given $f\in V$, find $u\in V$ such that

$$\boxed{\begin{aligned} &u_{tt}-u_{xx}=f,\\ &u(x,0)=u_t(x,0)=0. \end{aligned}} \tag{4.15}$$

The boundary conditions in (4.15) can be expressed by the infinite family of linear functionals $\beta_x\colon u\mapsto u(x,0)$, $\gamma_x\colon u\mapsto u_t(x,0)$ with x ranging over $\mathbb{R}$. So we can represent the boundary problem by the pair $(T,\mathscr{B})$ consisting of the differential operator $T=D_t^2-D_x^2$ and the (here infinite dimensional) boundary space $\mathscr{B}=[\beta_x,\gamma_x]_{x\in\mathbb{R}}\leq V^*$. Here $[\ldots]$ denotes the orthogonal closure of the subspace spanned by $\ldots$; for example, the functionals $u\mapsto\int_0^x u(\xi,0)\,d\xi$ and $u\mapsto u_x(x,0)$, for arbitrary $x\in\mathbb{R}$, are in the orthogonal closure but not in the span. □

For the rest of this section we consider boundary problems $(T,\mathscr{B})$ where T has a finite dimensional kernel and the space of boundary conditions $\mathscr{B}=[\beta_1,\ldots,\beta_n]$ is

also finite dimensional. We can rewrite the condition that $u \in V$ is a solution of the boundary problem $(T, \mathscr{B})$ for a given $f \in W$ in the following traditional form

$$\boxed{\begin{array}{l} Tu = f, \\ \beta_1(u) = \ldots = \beta_n(u) = 0. \end{array}}$$

By Corollary 4.14, a necessary condition for the regularity of $(T, \mathscr{B})$ is

$$\dim \mathrm{Ker}(T) = \dim \mathscr{B},$$

meaning that we have the "correct" number of boundary conditions. Moreover, we get the following algorithmic regularity test for boundary problems (to be found in Kamke [24, p. 184] for the special case of two-point boundary conditions). It hinges on the *evaluation matrix* $\beta(u) \in K^{n \times n}$ of Corollary 4.14.

Proposition 4.16. *A boundary problem $(T, \mathscr{B})$ with $\dim \mathrm{Ker}(T) = \dim \mathscr{B} < \infty$ is regular iff the evaluation matrix $\beta(u)$ is regular, where the (β_i) and (u_j) are any basis of respectively $\mathscr{B}$ and $\mathrm{Ker}(T)$.*

We can now compute the Green's operator for a boundary problem, given any section of the defining operator and a basis for its kernel.

Theorem 4.17. *Let $(T, \mathscr{B})$ be a regular boundary problem with $\dim \mathrm{Ker}(T) = \dim \mathscr{B} < \infty$. Let $u_1, \ldots, u_n$ and $\beta_1, \ldots, \beta_n$ be respectively a basis for $\mathrm{Ker}(T)$ and $\mathscr{B}$, and let $\tilde{G}$ be any section of T. Then*

$$(T, \mathscr{B})^{-1} = (1 - P)\tilde{G}$$

is the corresponding Green's operator, where $P \colon V \to V$ is the projector defined by

$$Pv = \sum_{i=1}^{n} \langle v | \tilde{\beta}_i \rangle u_i$$

with $(\tilde{\beta}_1, \ldots, \tilde{\beta}_n)^t = \beta(u)^{-1}(\beta_1, \ldots, \beta_n)^t$.

Proof. By Equation (4.13), the Green's operator for a regular boundary problem $(T, \mathscr{B})$ is given by $G = (1 - P)\tilde{G}$, where P is the projector with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(P) = \mathscr{B}^{\perp}$. We can invert B since it is regular by the previous proposition. Then the $(\tilde{\beta}_i)$ and (u_j) are obviously biorthogonal. Hence P is the projector with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(P) = \mathscr{B}^{\perp}$ by Proposition 4.7.

Chapter 5
Integro-Differential Operators

We have now built up an algebraic frame for treating boundary problems (currently only for LODEs). This is clearly an indispensible ingredient for a Symbolic Computation treatment, but it is not sufficient. Our next goal is to build up an algorithmic structure for encoding the three objects involved in boundary problems: For stating them, we have to specify differential equation and the boundary conditions; for solving them we must use integral operators. The *integro-differential operators* provide a unified language for expressing these three objects in a single operator algebra.

5.1 The Algebra of Integro-Differential Operators

Let us recall the familiar algebra of *differential operators*. Given a differential algebra $(\mathscr{F},\partial)$ over a ground field K, they are usually introduced as the "operators" that we shall now write as

$$\sum_{i=0}^{n} c_i D^i$$

for arbitrary degree $n \in \mathbb{N}$ and coefficients $c_0,\ldots,c_n \in \mathscr{F}$. The collection of these "operators" becomes a K-algebra by defining addition and scalar multiplication in the obvious way, and by introducing a multiplication through the commutator relation $Dc = cD + \partial(c)$. Due to its obvious connection to the Leibniz axiom (3.2), this relation is known as Leibniz rule.

One can now introduce an *action* of $\mathscr{F}[D]$ on $\mathscr{F}$ by declaring $f \cdot g = fg$ and $D \cdot g = \partial g$ for all $f,g \in \mathscr{F}$. Since D acts on $\mathscr{F}$ just as the original derivation ∂ on $\mathscr{F}$, one usually writes ∂ instead of D and consequently $\mathscr{F}[\partial]$ instead of $\mathscr{F}[D]$. This "abus de langage" means that the commutator relation now reads $\partial c = c\partial + \partial \cdot c$. This notation is very intuitive, but we emphasize that one should be clear about the different roles of the symbol ∂; we will commit the same abuse in what follows.

In a certain sense (that could be made precise), the algebra $\mathscr{F}[\partial]$ contains all arithmetic terms in ∂ like $\partial^2 c_0(\partial + 2\partial c_1(\partial^2 c_2 + (\partial c_3 - c_4\partial^3 c_5)) - \partial c_4) + c_5\partial$, and

it provides *canonical forms* for them. Furthermore, the Leibniz rule extracts all the essential algebraic properties that we know from analysis. (Exercise: Consider the free $\mathbb{R}$-algebra generated by $C^\infty(\mathbb{R})$ and ∂ modulo the relations induced by the multiplication in $C^\infty(\mathbb{R})$. Then the substitution $\partial \mapsto \frac{d}{dx}$ induces a homomorphism whose kernel encodes an ideal of relations. Is this ideal generated by the instances of the Leibniz rule?)

We want to do the same for integro-differential algebras $(\mathscr{F}, \partial, \int)$, so our notion of *integro-differential operator* should capture arithmetic terms in ∂ and $\int$, possibly supplemented by various evaluations at potential boundary points (such that we can formulate boundary problems). One canonical choice for the latter is to take all characters on $\mathscr{F}$ since characters are the obvious algebraic counterparts of point evaluation. In a concrete model like $\mathscr{F} = C^\infty[0,1]$, we can then form other boundary operators by combining them with ∂ and $\int$, for example $u \mapsto u'(0) - 3u(1)$ or even $u \mapsto \int_0^{1/2} u(\xi)\, d\xi$ as a composition of $\int = \int_0^x$ and evaluation at $1/2$.

As for the differential operators, we will have to factor out an *ideal of relations*. Of course we have now more relations than just the Leibniz rule. Let us study them systematically. We have four types of basic operators: derivation ∂, integral $\int$, multiplication operators f, and characters φ. We study them now with the model $C^\infty[a,b]$ in mind.

- Taking ∂ on the left, we have to consider $\partial\partial$ and $\partial\int$ and ∂f and $\partial\varphi$. The first cannot be reduced, the second should give unity by the section axiom (3.1), the third is regulated by the Leibniz axiom (3.2), the fourth must vanish since φ corresponds via Corollary 4.4 to an integral $\int_\varphi$ such that $\varphi = 1 - \int_\varphi \partial$.
- Taking $\int$ on the left, let us first look at $\int\partial$ and $\int\int$. The first gives $1 - \mathtt{E}$, where $\mathtt{E}$ is a character by Corollary 3.22. The second can be computed by the pure Baxter axiom (3.6); substituting $f = 1$ yields $\int\int g = x\int g - \int(xg)$ or in operator notation $\int\int \to x\int - \int x$, where as always $x = \int \cdot 1$.
- The next combination to consider would be $\int f$. But by itself, this cannot be reduced. In fact, we have just had $\int x$ in the previous relation, and we cannot improve on that. But we could have left f general instead of taking $f = 1$, which leads to the *weak Baxter rule* $\int f\int \to F\int - \int F$, where $F = \int \cdot f$. So $\int f$ is irreducible but $\int f\int$ is not. Since $\int fg$ is trivially reducible, it remains to check $\int f\partial$ and $\int f\varphi$.
- For reducing the monomial $\int f\partial$, we can directly translate the *evaluation variant* (3.5) of integration by part into operator form, which now yields the *strong Baxter rule* $\int f\partial \to f - \int(\partial \cdot f) - (\mathtt{E} f)\,\mathtt{E}$.
- The reduction of $\int f\varphi$ is simple since φ maps any function to a constant, which can be pulled out from the integral, meaning $\int f\varphi \to (\int \cdot f)\,\varphi$. Obviously this also covers the case $\int\varphi$.
- Starting with f on the left, we now turn to $f\partial$ and $f\int$ and fg and $f\varphi$. Obviously all of these cannot be reduced. Except for fg, that is: Precisely speaking, the composition of the multiplication operators corresponding to f and g yields a single multiplication operator, which corresponds to fg. We will make this explicit below.

- Finally, we consider products with φ on the left, namely $\varphi\partial$ and $\varphi\int$ and φf and $\varphi\psi$. The first two are obviously irreducible; in fact, they are used in boundary functionals like the above examples $\varphi_0\partial - 3\varphi_1$ and $\varphi_{1/2}\int$, writing φ_ξ for evaluation at ξ. Applying the third to a function u yields $\varphi \cdot fu = \varphi(f)\,\varphi(u)$ or $\varphi f = c\varphi$ with $c = \varphi \cdot f$. Similarly for the fourth, $\varphi\psi \cdot u = \varphi(c) = c\,\varphi(1) = c$ with $c = \psi \cdot u$, which just means $\varphi\psi = \psi$.

For casting this into a solid algebraic form, we must take ∂ and $\int$ as well as the f and φ as indeterminates in a *free algebra*, factoring out the relations discussed above. Note that the elements of the free algebra have an obvious action on the given integro-differential algebra. As we did above, we will write $\cdot$ for this action. In particular, $f \cdot g$ denotes the product of functions in $\mathscr{F}$. Furthermore, we have to fix a K-basis: Choose $\mathscr{F}^\#$ such that $\{1\} \cup \mathscr{F}^\#$ is a K-basis of $\mathscr{F}$.

Definition 5.1. Let $(\mathscr{F}, \partial, \int)$ be an ordinary integro-differential algebra over a field K and $\Phi \subseteq \mathscr{F}^\bullet$. The *integro-differential operators* $\mathscr{F}_\Phi[\partial, \int]$ are defined as the K-algebra generated by the symbols ∂ and $\int$, the "functions" $f \in \mathscr{F}^\#$ and the "characters" $\varphi \in \Phi \cup \{\mathsf{E}\}$, modulo the rewrite rules given in Table 5.1. If $\Phi = \mathscr{F}^\bullet$, we write simply $\mathscr{F}[\partial, \int]$.

$fg \;\to\; f \cdot g$	$\partial f \;\to\; \partial \cdot f + f\partial$	$\int f \int \;\to\; (\int \cdot f)\int - \int(\int \cdot f)$
$\varphi\psi \to \psi$	$\partial\varphi \to 0$	$\int f\partial \;\to\; f - \int(\partial \cdot f) - (\mathsf{E} \cdot f)\,\mathsf{E}$
$\varphi f \;\to\; (\varphi \cdot f)\,\varphi$	$\partial\int \to 1$	$\int f\varphi \to (\int \cdot f)\,\varphi$

Table 5.1 Rewrite Rules for Integro-Differential Operators

Note that the usage of a basis $\mathscr{F}^\#$ introduces some *technical complications* in the rewrite system given below. Whenever an action on a basis element $f \in \mathscr{F}^\#$ is involved in a rule, its result has to be expanded with respect $\mathscr{F}^\#$. For example, take the exponential polynomials $\mathscr{F} = K[x, e^{Kx}]$ with their natural K-basis $\mathscr{F}^\# = (x^i e^{\lambda x} \mid i \in \mathbb{N}, \lambda \in K)$, and consider the weak Baxter rule for $f = xe^x$. Then we have $\int \cdot f = 1 - e^x + xe^x$, and the right-hand side of the rule must be understood as $\int - e^x \int + xe^x \int - \int + \int e^x - \int xe^x$. ▼

In order to avoid this type of difficulties, it is desirable to find an alternative formulation that does not need the choice of a basis. Such a *coordinate-free formulation* could be obtained by taking as a starting point for the quotient construction not the free K-algebra in $\{\partial, \int\} \cup \mathscr{F}^\# \cup \Phi$ but a slightly finer algebra that incorporates already all relations of $\mathscr{F}$. Of course, we can then also omit the rule $fg \to f \cdot g$ of Table 5.1.

An algebra of this type is given by what we shall call the *uncommutative polynomials*. Let X be a set of indeterminates, $\mathscr{F}$ an algebra over a commutative ring K. Then $\mathscr{F}_K\langle X\rangle$ is defined as the algebra generated by $\mathscr{F}$ and X, modulo centralizing K. This means one adjoins to $\mathscr{F}$ all elements of X and factors out the centralizing relations $\lambda x = x\lambda$ for all $\lambda \in K$ and $x \in X$. The notation $\mathscr{F}_K\langle X\rangle$ for this algebra is taken from [16, p. 171]; we call its elements the uncommutative polynomials in X with coefficients in $\mathscr{F}$ over K. If $K = \mathbb{Z}$, we may also drop the qualification "over K"; this is the case considered in the appendix of [37].

Note that one can also describe $\mathscr{F}_K\langle X\rangle$ as a *universal polynomial algebra* in the sense of [30], namely as the polynomials in X with coefficients $\mathscr{F}$, taken in the variety of K-algebras (which coincides with the variety of rings if $K = \mathbb{Z}$). It is also clear that the free algebra $K\langle X\rangle$ arises as

the special case $K = \mathscr{F}$; its elements are usually referred to as "noncommutative" (as opposed to "uncommutative") polynomials.

The construction $\mathscr{F}_K\langle X\rangle$ can now be applied to ordinary integro-differential algebras $(\mathscr{F},\partial,\int)$ over a field K, using the indeterminates $X = \{\partial,\int\}\cup\Phi$. As already announced, we can now define the algebra of *integro-differential operators* $\mathscr{F}[\partial,\int]$ as the quotient modulo the ideal spanned by the rewrite rules of Table 5.1, except for $fg \to f\cdot g$. The only problem with this approach is that we cannot directly prove confluence with the machinery of Section 5.2, which is set up for noncommutative rather than uncommutative polynomials. It would be worthwhile to develop a generalization that can handle these cases (not only for our present purposes); the remarks given in Section 6 of [4] seem to be a good starting point for such an undertaking. ▲

We have now written the relations in the form of *rewrite rules*. Algebraically speaking, it is enough to consider the corresponding ideal generated by elements like $\partial f - \partial\cdot f - f\partial$. In fact, Table 5.1 provides an algorithmic realization as a rewrite system since we will see that it is Noetherian and confluent. Before proving this, let us add a few further remarks on this definition:

- We use the variables f,g,h for *functions*, meaning elements of $\mathscr{F}^{\#}$, and the variables φ,ψ,χ for *characters*, meaning elements of $\Phi\subseteq\mathscr{F}^{\bullet}$. The same applies to the decorated versions of these variables.
- Fixing ambiguous terminology, we say that an integro-differential operator is a sum of *monomials*, and a monomial a coefficient times a *term*.
- The rewrite system is understood as including *implicit rules* for $\int\int$, $\int\partial$ and $\int\varphi$ by substituting $f = 1$ in the rules for $\int f\int$, $\int f\partial$ and $\int f\varphi$, respectively. Moreover, we have the *derived rule* $\mathsf{E}\int = 0$ from the definition of the evaluation E
- It is an easy matter to check that the rewrite rules of Table 5.1 are fulfilled in $(\mathscr{F},\partial,\int)$, so we may transport $\cdot$ to an *action* of $\mathscr{F}[\partial,\int]$ on $\mathscr{F}$.

Some words on the *notation of integral operators*. Similar to Corollary 4.4, we use the abbreviation $\int_\varphi$ for the operator $(1-\varphi)\int\in\mathscr{F}[\partial,\int]$ since it acts as an integral with evaluation φ. Of course, $\int$ itself coincides with $\int_{\mathsf{E}}$. If one works with multiple integral (and differential) structures, it is important to distinguish them by labels like x and y, writing $\int^x$ and $\int^y$ for the corresponding integrals. The standard example of combining two integro-differential structures is given by $(C^\infty(\mathbb{R}^2),\partial_x,\partial_y,\int^x,\int^y)$, with the obvious interpretation of the operations. Note while derivations are labelled below, the corresponding integrals are labelled above—remotely reminiscent of the index conventions for tensors. In conjunction with the previous conventions, the meaning of operators like $\int_\varphi^x$ should now be clear.

Even without other integrals, it is sometimes convenient to use the notation $\int_\varphi^x$ instead of $\int_\varphi$ since then one can write $\int_x^\varphi$ as an *abbreviation* for $-\int_\varphi^x$, and similarly $\int_\varphi^\psi$ for the definite integral $\psi\int_\varphi^x = \int_\varphi^x + \int_x^\psi$. In the standard example $(C^\infty[a,b],\partial_x,\int^x)$, we may furthermore identify the characters $u\mapsto u(c)$ with the real numbers $c\in[a,b]$, thus writing the familiar $\int_c^x$ for the integral initialized at c.

It is sometimes practical to specialize $\mathscr{F}_\Phi[\partial,\int]$ to the frequently occurring situation of two-point boundary problems. For that purpose, we have introduced biintegro-differential algebras $(\mathscr{F},\partial,\int^*,\int_*)$ in Section 3.2, effectively as integro-differential algebras with two distinguished initialization points. Accordingly, we define the algebra $\mathscr{F}[D,A,B]$ of *biintegro-differential operators*, previously named

"analytic polynomials" [38, p. 176], as either $\mathscr{F}_{\Phi}[\partial, \int^*]$ or $\mathscr{F}_{\Phi}[\partial, -\int_*]$ with the character set Φ consisting of the two evaluations $\mathbf{E}^*$ and $\mathbf{E}_*$ corresponding to the two integrals $\int^*$ and $\int_*$ on $\mathscr{F}$; both algebras are clearly the same (the sign in $-\int_*$ is merely conventional for making $\int^*$ and $\int_*$ adjoints of each other). In this context, we use the symbols

$$\begin{aligned} A &= \textstyle\int^* = (\mathbf{E}^* - 1)\int_*, & L &= \mathbf{E}^*, \\ B &= \textstyle\int_* = (\mathbf{E}_* - 1)\int^*, & R &= \mathbf{E}_*, \\ D &= \partial, & \lceil f \rceil &= f, \end{aligned}$$

Moreover, we use the abbreviation $F = A + B$ for the operator of definite integration. The reason for using new symbols D, A, B, L, R, $\lceil f \rceil$ is that they can be treated as distinct indeterminates, if additional rewrite rules are introduced for compensating the redundancy [38].

The case of biintegro-differential operators suggests an alternative way of defining $\mathscr{F}[\partial, \int]$, one in which all integrals like A and $-B$ are treated on an equal footing. This is not the case in Definition 5.1, where the evaluation $\mathbf{E}$ plays the role of a distinguished character. For many applications, this is indeed a convenient approach, but we will now sketch an alternative treatment based on a *collection of integrals*. ▼

Fix an ordinary integro-differential algebra $(\mathscr{F}, \partial)$ over a field K and characters $\Phi \subseteq \mathscr{F}^{\bullet}$. Every character $\varphi \in \Phi$ corresponds to the direct decomposition $\mathscr{F} = K \dotplus \mathscr{I}_{\varphi}$ with $\mathscr{I}_{\varphi} = \operatorname{Ker}(\varphi)$. By Proposition 4.1, this in turn corresponds to the unique section $\int_{\varphi}^{x} \colon \mathscr{F} \to \mathscr{F}$ of the derivation $\partial \colon \mathscr{F} \to \mathscr{F}$ that satisfies $\operatorname{Im}(\int_{\varphi}^{x}) = \mathscr{I}_{\varphi}$. Of course, every $\int_{\varphi}^{x}$ is an integral for ∂ according to Corollary 3.22. It is therefore natural to define the *equitable integro-differential operators* $\mathscr{F}[\partial, \int_{\Phi}]$ as the K-algebra generated by ∂ and $\int_{\varphi}^{x}$ and f with φ ranging over Φ and f over $\mathscr{F}^{\#}$, modulo a suitable set of rewrite rules that we will specify below.

Our guiding principle for finding the rewrite rules will be that we must eventually come up with a *translation isomorphism*

$$\iota \colon \mathscr{F}_{\Phi}[\partial, \textstyle\int] \to \mathscr{F}[\partial, \int_{\Phi}] \tag{5.1}$$

that sends ∂ and the $f \in \mathscr{F}$ to "themselves" but otherwise translates between integrals and characters. In view of Corollary 3.22, we define $\iota(\varphi) = 1 - \int_{\varphi}^{x}\partial \in \mathscr{F}[\partial, \int_{\Phi}]$, and in view of Corollary 4.4 also $\iota^{-1}(\int_{\varphi}^{x}) = (1-\varphi)\int \in \mathscr{F}_{\Phi}[\partial, \int]$ for the inverse map. Note that the latter was introduced above as an abbreviation within $\mathscr{F}_{\Phi}[\partial, \int]$; similarly the former may be regarded as an abbreviation within $\mathscr{F}[\partial, \int_{\Phi}]$. Supplementing the definition of (5.1) by $\iota(\int) = \int_{\mathbf{E}}^{x}$ with the distinguished character $\mathbf{E} = 1 - \int\partial \in \Phi$, we extend ι and ι^{-1} to homomorphisms between the free algebras underlying $\mathscr{F}_{\Phi}[\partial, \int]$ and $\mathscr{F}_{\Phi}[\partial, \int]$.

Using the rules $\varphi\psi \to \psi$ and $\partial\int \to 1$ for $\mathscr{F}_{\Phi}[\partial, \int]$, one may immediately verify $\iota^{-1}\iota = 1$. For ensuring $\iota\iota^{-1} = 1$, we need the rewrite rule $\partial\int_{\mathbf{E}}^{x} \to 1$ for $\mathscr{F}[\partial, \int_{\Phi}]$. In fact, we will add section rules $\partial\int_{\varphi}^{x} \to 1$ for all $\varphi \in \Phi$. Hence we may infer that the map ι will indeed be an *isomorphism* with inverse ι^{-1}, if only we can make sure that ι respects all relations existing in $\mathscr{F}_{\Phi}[\partial, \int]$ and ι^{-1} all the ones to be defined for $\mathscr{F}[\partial, \int_{\Phi}]$.

Since we need an isomorphic copy of $\mathscr{F}[\partial]$ within $\mathscr{F}[\partial, \int_{\Phi}]$, we clearly must retain the rewrite rules $fg \to f \cdot g$ and $\partial\int_{\varphi}^{x} \to 1$ for all $f, g \in \mathscr{F}^{\#}$ and for all $\varphi \in \Phi$. The *strong Baxter rule* generalizes immediately to

$$\textstyle\int_{\varphi}^{x} f\partial \to f - \int_{\varphi}^{x}(\partial \cdot f) - (\varphi \cdot f)\,\varphi = f - \int_{\varphi}^{x}(\partial \cdot f) - \varphi \cdot f + (\varphi \cdot f)\int_{\varphi}^{x}\partial$$

since the character $\varphi = \mathbf{E}$ is in no way intrinsically special (the expression on the left-hand side of the equality is not in $\mathscr{F}[\partial, \int_{\Phi}]$, so we view it via ι as an abbreviation). Note that—as opposed to

the strong Baxter rule of Table 5.1—the implicit rule for the special case $f = 1$ is trivial. In fact, $\int_\varphi^x \partial$ must be a normal form since it appears on the right-hand side of the new strong Baxter rule!

The only rule that needs proper generalization is the *weak Baxter rule* because in $\mathscr{F}[\partial, \int_\Phi]$ is must describe the interaction of two integral operators with different evaluations. We can find it by calculating in $\mathscr{F}_\Phi[\partial, \int]$ with the abbreviations mediated by ι^{-1}, which leads at first to

$$\int_\varphi^x f \int_\psi^x = \int_\varphi^x f \left(\int_\psi^\varphi + \int_\varphi^x\right) = \int_\varphi^x f\varphi \int_\psi^x + \int_\varphi^x f \int_\varphi^x.$$

Of course, we may apply the weak Baxter rule for the character φ, yielding $(\int_\varphi^x \cdot f)\int_\varphi^x - \int_\varphi^x (\int_\varphi^x \cdot f)$ for the right summand $\int_\varphi^x f \int_\varphi^x$. We can also apply the φ-version of the lower left rule in Table 5.1 to obtain $(\int_\varphi^x \cdot f)\,\varphi$ for the factor $\int_\varphi^x f\varphi$ appearing in the left summand. Hence we have

$$\int_\varphi^x f \int_\psi^x = (\int_\varphi^x \cdot f)\,\varphi \int_\psi^x + (\int_\varphi^x \cdot f)\int_\varphi^x - \int_\varphi^x (\int_\varphi^x \cdot f) = (\int_\varphi^x \cdot f)\big(\int_\psi^\varphi + \int_\varphi^x\big) - \int_\varphi^x (\int_\varphi^x \cdot f),$$

where we can finally simplify $\int_\psi^\varphi + \int_\varphi^x = \int_\psi^x$. Altogether we have thus found

$$\int_\varphi^x f \int_\psi^x \to (\int_\varphi^x \cdot f)\int_\psi^x - \int_\varphi^x (\int_\varphi^x \cdot f)$$

as the desired generalization of the weak Baxter rule (regained by equating φ and ψ), now also including a nontrivial implicit rule for $f = 1$. This rule appears in the special case $\varphi = f \mapsto f(0)$ and $\psi = f \mapsto f(1)$ in the rewrite system for biintegro-differential operators [38], where one has accordingly $\int_\varphi^x = A$ and $\int_x^\psi = B$.

We can now conclude the definition of $\mathscr{F}[\partial, \int_\Phi]$ by summarizing *all required relations* in the following rewrite system:

$fg \to f\cdot g$	$\partial f \to \partial\cdot f + f\partial$	$\int_\varphi^x f \int_\psi^x \to (\int_\varphi^x \cdot f)\int_\psi^x - \int_\varphi^x (\int_\varphi^x \cdot f)$
$\partial \int_\varphi^x \to 1$	$\int_\varphi^x f\partial \to f - \int_\varphi^x (\partial\cdot f) - \varphi\cdot f + (\varphi\cdot f)\int_\varphi^x \partial$	

Table 5.2 Rewrite Rules for Equitable Integro-Differential Operators

Note that we need only five instead of nine rules this time, so we can see that the more symmetric formulation of $\mathscr{F}[\partial, \int_\Phi]$ has also gained in economy.

It remains to prove that the translation isomorphisms ι and ι^{-1} respect the relations (such that they are well-defined). Starting with the translation *from left to right*, we must prove that the ι-translation of Table 5.1 yield valid relations of $\mathscr{F}[\partial, \int_\Phi]$. This is trivial for the rule on fg, the Leibniz rule, and the section rule. Since $\iota(\int) = \int_{\mathtt{E}}^x$, the weak Baxter rule translates into the mixed Baxter rule of $\mathscr{F}[\partial, \int_\Phi]$ with $\varphi = \psi = \mathtt{E}$. Let us now check the relation $\varphi\psi = \psi$. Indeed, the left-hand side translates correctly to

$$(1 - \int_\varphi^x \partial)(1 - \int_\psi^x \partial) = 1 - \int_\varphi^x \partial - \int_\psi^x \partial + \int_\varphi^x \partial \int_\psi^x \partial = 1 - \int_\psi^x \partial,$$

where we have applied the instance $\partial \int_\psi^x = 1$ of the section rule for $\mathscr{F}[\partial, \int_\Phi]$. The rule $\partial\varphi = 0$ follows immediately from a φ-instance of the section rule. The strong Baxter rule of $\mathscr{F}_\Phi[\partial, \int]$ is respected since it corresponds to the $\mathtt{E}$-instance of the last rule in Table 5.2 above. It remains to check the rules for φf and $\int f\varphi$. The first is respected since it translates to

$$\begin{aligned}(1 - \int_\varphi^x \partial) f &= f - \int_\varphi^x (f\partial + \partial\cdot f) = f - \Big(f - \int_\varphi^x (\partial\cdot f) - \varphi\cdot f + (\varphi\cdot f)\int_\varphi^x \partial\Big) - \int_\varphi^x (\partial\cdot f)\\ &= \varphi\cdot f - (\varphi\cdot f)\int_\varphi^x \partial = (\varphi\cdot f)(1 - \int_\varphi^x \partial)\end{aligned}$$

where we have used the Leibniz rule and the strong Baxter rule of $\mathscr{F}[\partial, \int_\Phi]$. The second involves both the weak and the strong Baxter rule as well as the implicit rule $\iota(\mathtt{E}\int) = 0$, leading to

$$\int_{\mathsf{E}}^x f(1-\int_\varphi^x\partial) = \int_{\mathsf{E}}^x f - \Big((\int_{\mathsf{E}}^x\cdot f)\int_\varphi^x - \int_{\mathsf{E}}^x(\int_{\mathsf{E}}^x\cdot f)\Big)\partial = \int_{\mathsf{E}}^x f - (\int_{\mathsf{E}}^x\cdot f)\int_\varphi^x\partial$$
$$+\Big(\int_{\mathsf{E}}^x\cdot f - \int_{\mathsf{E}}^x f - \mathsf{E}\int_{\mathsf{E}}^x\cdot f + (\mathsf{E}\int_{\mathsf{E}}^x\cdot f)\,\mathsf{E}\int_{\mathsf{E}}^x\partial\Big) = \int_{\mathsf{E}}^x\cdot f - (\int_{\mathsf{E}}^x\cdot f)\int_\varphi^x\partial = (\int_{\mathsf{E}}^x\cdot f)(1-\int_\varphi^x\partial),$$

which is indeed the proper translation of the relation $\int f\varphi = (\int\cdot f)\varphi$. Finally, note that the implicit rule $\iota(\mathsf{E}\int) = (1-\int_{\mathsf{E}}^x\partial)\int_{\mathsf{E}}^x = 0$ follows immediately from the E-instance of the section rule.

The translation *from right to left*, via the isomorphism ι^{-1}, is much easier to treat since we have only five relations to check. The rules for fg and ∂f are again trivial. The (mixed) weak Baxter rule is respected in $\mathscr{F}_\Phi[\partial,\int]$ since we have actually constructed it via its ι^{-1}-image. The section rule correctly translates to $\partial(1-\varphi)\int = \partial\int - \partial\varphi\int = 1$. It remains to check the strong Baxter rule, which clearly follows by multiplying the corresponding rule of Table 5.1 from the left by $1-\varphi$, yielding

$$\begin{aligned}(1-\varphi)\int f\partial &= \Big(f - \int(\partial\cdot f) - (\mathsf{E}\cdot f)\mathsf{E}\Big) - \Big((\varphi\cdot f)\varphi - \varphi\int(\partial\cdot f) - (\mathsf{E}\cdot f)\mathsf{E}\Big)\\ &= f - (1-\varphi)\int(\partial\cdot f) - (\varphi\cdot f)\varphi\end{aligned}$$

where we have also used the rules for φf and $\varphi\psi$ of Table 5.1. The above right-hand side translates the right-hand side of the corresponding rule of Table 5.2 since

$$\iota^{-1}(1-\int_\varphi^x\partial) = 1-(1-\varphi)(1-\mathsf{E}) = 1-(1-\varphi) = \varphi.$$

This concludes the proof that $\iota^{-1}\colon \mathscr{F}[\partial,\int_\Phi] \to \mathscr{F}_\Phi[\partial,\int]$ is also well-defined, so both ι and ι^{-1} are homomorphism of algebras. Since we have already seen that they are inverse to each other, we see that they are indeed isomorphisms so that $\mathscr{F}_\Phi[\partial,\int] \cong \mathscr{F}[\partial,\int_\Phi]$.

The *biintegro-differential operators* over a biintegro-differential algebra $(\mathscr{F},\partial,\int^*,\int_*)$ are now seen as $\mathscr{F}[D,A,B] = \mathscr{F}[\partial,\int_\Phi]$, where Φ consists of the two evaluations E^* and E_* corresponding to the two integrals $\int^*$ and $\int_*$ on $\mathscr{F}$. ▲

5.2 Parametrized Noncommutative Gröbner Bases

The goal of this section is to prove that the rules of Table 5.1 correspond to a noncommutative Gröbner basis in the corresponding free algebra. For that purpose, we will—very briefly—review the basic facts of noncommutative *Gröbner basis theory*. Since most expositions do not allow for infinitely generated ideals, we will base our account on the somewhat dated but still highly readable Bergman paper [4]; for a short summary of it, we refer to §3.3 of [13]. Other approaches to noncommutative Gröbner bases can be found in [31, 32, 42].

We start with some general notions for *abstract reduction relations*; see the first chapter of [1] for the broader background. We consider a relation $\to \subseteq A\times A$ for a set A; typically $\to$ realizes a single step in a simplification process like the transformation of integro-differential operators according to Table 5.1. The transitive closure of $\to$ is denoted by $\xrightarrow{+}$, its reflexive-transitive closure by $\xrightarrow{*}$.

We call $a\in A$ *irreducible* if there is no $a_0\in A$ with $a\to a_0$; we write $A_\downarrow$ for the set of all irreducible elements. If $a\xrightarrow{*}a_0$ with $a_0\in A_\downarrow$, we call a_0 a *normal form* of a, denoted by $\downarrow a = a_0$ in case it is unique. If this should always be the case, we have to impose two conditions: Noetherianity for banning infinite reduction chains and confluence for bringing forks together.

More precisely, $\to$ is called *Noetherian* if there are no infinite chains $a_1 \to a_2 \to \ldots$ and *confluent* if for all $a, a_1, a_2 \in A$ the "hill" $a_1 \overset{*}{\leftarrow} a \overset{*}{\to} a_2$ has a "valley" $a_1 \overset{*}{\to} a_0 \overset{*}{\leftarrow} a_2$ for some $a_0 \in A$. If the reduction relation $\to$ is both Noetherian and confluent, it is called *convergent*. In this case, every element $a \in A$ has a unique normal form $\downarrow a \in A_{\downarrow}$; Noetherianity gives existence while confluence provides uniqueness.

Turning now to noncommutative Gröbner bases theory, we focus on reduction relations on the free algebra. Since we will apply the theory to ordinary integro-differential algebras $(\mathscr{F}, \partial, \int)$ over a field K, we will keep the same notation also for the following *general setting*. For the time being, let $\mathscr{F}$ be an algebra over a commutative ring K, and X an arbitary set of indeterminates. We write $\langle X \rangle$ for the free monoid on X and $K\langle X \rangle$ for the free K-algebra on X. Note that $K\langle X \rangle$ is the monoid algebra of $\langle X \rangle$.

A *reduction system* for $K\langle X \rangle$ is given by a set $\Sigma \subseteq \langle X \rangle \times K\langle X \rangle$, whose elements are called the *rules* of Σ. For any rule $\sigma = (W, f)$ and words $A, B \in \langle X \rangle$, let ${}_A\sigma_B$ denote the K-module endomorphism of $K\langle X \rangle$ that fixes all elements of $\langle X \rangle$ except sending AWB to AfB. We call ${}_A\sigma_B$ the *reduction* instantiating the rule σ with prefix A and postfix B, briefly the (A,B)-reduction for σ. A reduction ${}_A\sigma_B$ for $\sigma = (W, f)$ is said to *act trivially* on $a \in K\langle X \rangle$ if the coefficient of AWB in a is zero.

Every reduction system Σ induces the *step relation* $\to_\Sigma \subseteq K\langle X \rangle \times K\langle X \rangle$ defined by setting $a \to_\Sigma b$ iff $r(a) = b$ for some reduction r acting nontrivially on a. We call its reflexive-transitive closure $\overset{*}{\to}_\Sigma$ the *reduction relation* induced by Σ, and we say that a reduces to b when $a \overset{*}{\to}_\Sigma b$. Accordingly, we have the K-module $K\langle X \rangle_{\downarrow}$ of irreducible elements, we can speak of the normal form $\downarrow a \in K\langle X \rangle_{\downarrow}$ for a suitable element $a \in K\langle X \rangle$, and we call Σ respectively Noetherian or confluent if $\to_\Sigma$ is.

For ensuring Noetherianity of Σ, one usually imposes a Noetherian *semigroup order* on $\langle X \rangle$, meaning a partial order such that $B < B'$ implies $ABC < AB'C$ for all $A, B, B', C \in \langle X \rangle$. Then Σ will be Noetherian if it *respects* $<$ in the sense that $W' < W$ for every rule $(W, f) \in \Sigma$ and every nonzero monomial W' of f. If $<$ is total and K is a field, the reduction system Σ can be replaced by the set

$$S_\Sigma = \{W - f \mid (W, f) \in \Sigma\}$$

called the *ideal basis* associated with Σ. We write I_Σ for the two-sided *ideal* of $K\langle X \rangle$ generated by S_Σ. As a K-module, I_Σ is spanned by $\{AgB \mid g \in S_\Sigma; A, B \in \langle X \rangle\}$. Conversely, a set $S \subseteq K\langle X \rangle$ determines the reduction system

$$\Sigma_{S,<} = \{(\mathrm{lm}_<(g), \mathrm{lc}_<(g)^{-1} \mathrm{red}_<(g)) \mid g \in S\},$$

where $\mathrm{lm}_<$ and $\mathrm{lc}_<$ are respectively the leading monomial and the leading coefficient, while $\mathrm{red}_<(g) = g - \mathrm{lc}_<(g)\,\mathrm{lm}_<(g)$ denotes the reductum of g.

It is often more difficult to ensure confluence of a reduction system Σ. According to the definition, we would have to investigate every hill $a_1 \overset{*}{\leftarrow} a \overset{*}{\to} a_2$, which is usually much too laborious. The key for a practically useful criterion is to consider just the *minimal divergences* and see whether their difference eventually becomes zero. This idea was first described by Buchberger in [9] for the commutative case; see also [10, 12]. In the lucid account [11], Buchberger compares the idea of minimal

divergences for (commutative) polynomial reduction with Knuth-Bendix completion and Robinson's resolution principle.

An *overlap ambiguity* of Σ is given by a quintuple (σ,τ,A,B,C) with Σ-rules $\sigma=(W,f)$ $\tau=(V,g)$ and monomials $A,B,C\in\langle X\rangle\backslash\{1\}$ such that $W=AB$ and $V=BC$. Its associated S-polynomial is defined as $fC-Ag$, and the ambiguity is called *resolvable* if the S-polynomial reduces to zero.

Although overlap ambiguities are much more important, it can be necessary to study also the other type of minimial divergence: An *inclusion ambiguity* of Σ is likewise given by a quintuple (σ,τ,A,B,C) with Σ-rules $\sigma=(W,f)$ $\tau=(V,g)$ and monomials $A,B,C\in\langle X\rangle\backslash\{1\}$, but now with the condition that $W=B$ and $V=ABC$. The associated S-polynomial is then $AfC-g$, and again we speak of a resolvable ambiguity if the S-polynomial reduces to zero.

The reason why inclusion ambiguities are of lesser importance is that they are in principle not needed—every reduction system with inclusion ambiguities can be replaced by one without inclusion ambiguities that has the same notion of reducibility and—in case of confluence—induces the same canonical form. In this sense one may always work with reduction systems having *no inclusion ambiguities*; this is clearly the case for the one in Table 5.1.

Finally, we introduce the following refinement of the notion of resolvability. Note that $a\xrightarrow{*}_\Sigma 0$ is equivalent to $a\in I_\Sigma$, just as in the commutative case. Hence an (overlap or inclusion) ambiguity (σ,τ,A,B,C) is resolvable iff its S-polynomial belongs to I_Σ. It is called *resolvable relative to* $\leq$ iff all monomials of its S-polynomial are below ABC with respect to $\leq$.

We come now to the main result needed for proving convergence of Table 5.1. It is called the *Diamond Lemma for Ring Theory* in Bergman's homonymous paper [4].

Theorem 5.2. *Let Σ be a reduction system for $K\langle X\rangle$ and $\leq$ a Noetherian semigroup order that respects Σ. Then the following conditions are equivalent:*

1. *All ambiguities of Σ are resolvable.*
2. *All ambiguities of Σ are resolvable relative to $\leq$.*
3. *The reduction relation $\xrightarrow{*}_\Sigma$ is convergent.*
4. *We have the direct decomposition $K\langle X\rangle=K\langle X\rangle_{\downarrow}\dotplus I_\Sigma$ as K-modules.*

When these conditions hold, the quotient algebra $K\langle X\rangle/I_\Sigma$ may be identified with the K-module $K\langle X\rangle_{\downarrow}$, having the multiplication $a\cdot b=\downarrow ab$.

Proof. See Theorem 1.2 in [4] or Theorem 3.21 in [13].

Proposition 5.3. *For every ordinary integro-differential algebra $(\mathscr{F},\partial,\int)$ over a field K and for all $\Phi\subseteq\mathscr{F}^\bullet$, the rules of Table 5.1 constitute a convergent reduction system on the corresponding free K-algebra.*

Proof. We set $X=\{\partial,\int,\mathsf{E}\}\cup\mathscr{F}^\#\cup\Phi$ and write Σ for the reduction system described by Table 5.1. Using 5.2, we construct a Noetherian semigroup order $\leq$ on $\langle X\rangle$ that respects Σ, and we prove that all ambiguities of Σ are resolvable. Regarding the former, we have a lot of freedom. We put $f<\partial$ for all $f\in\mathscr{F}^\#$, extended

to words by the graded lexicographic construction. The resulting partial order is clearly Noetherian (since it is on the generators) and compatible with the monoid structure (by its grading). It respects the reduction system Σ because all rules reduce the word length except for the Leibniz rule, which is compatible because of $f < \partial$.

For proving that the ambiguities of Σ are resolvable, we must consider 14 overlap ambiguities (and no inclusion ambiguities as noted earlier). The calculation is easy in all 14 cases, using also the axioms of integro-differential algebras for $\mathscr{F}$. As a representative example, let us compute the S-polynomial of the two reduction rules $\sigma = (\int f \int, (\int \cdot f)\int - \int(\int \cdot f))$ and $\tau = (\int g\partial, g - \int(\partial \cdot g) - (\mathtt{E} \cdot g)\mathtt{E})$ as

$$\begin{aligned}
&(\textstyle\int \cdot f)\int g\partial - \int(\int \cdot f)g\partial - \int fg + \int f\int g' + \int f\,(\mathtt{E}\cdot g)\,\mathtt{E}\\
&\quad= (\textstyle\int \cdot f)g - (\int \cdot f)\int g' - (\int \cdot f)\,(\mathtt{E}\cdot g)\,\mathtt{E} - (\int \cdot f)g + \int((\int \cdot f)\cdot g)'\\
&\qquad\quad + (\mathtt{E}\cdot((\textstyle\int \cdot f)\cdot g))\,\mathtt{E} - \int(f\cdot g) + (\int \cdot f)\int g' - \int(\int \cdot f)g' + (\mathtt{E}\cdot g)(\int \cdot f)\,\mathtt{E}\\
&\quad= \textstyle\int((\int \cdot f)\cdot g)' + (\mathtt{E}\cdot((\int \cdot f)\cdot g))\,\mathtt{E} - \int(f\cdot g) - \int(\int \cdot f)g'\\
&\quad= \textstyle\int(f\cdot g) + \int(\int \cdot f)g' + 0 - \int(f\cdot g) - \int(\int \cdot f)g'\\
&\quad= 0,
\end{aligned}$$

meaning the overlap ambiguity $(\sigma, \tau, \int f, \int, g\partial)$ is resolvable. □

▼ It should also be mentioned that—in the equational theory—the *weak Baxter rule* actually follows from the strong one (as the names suggest). In fact, the weak Baxter rule is nothing else than the S-polynomial of the strong Baxter rule and the section rule: Applying the former to the overlap $\int f\partial\int$ yields $f\int - \int f'\int$ since $\mathtt{E}\int = 0$, while applying the latter gives $\int f$, so we obtain the relation $f\int - \int f'\int - \int f = 0$. Replacing f by $\int \cdot f$ leads to $\int f\int = (\int \cdot f)\int - \int(\int \cdot f)$, which is just the pure Baxter axiom (3.6). Could we possibly also infer the strong Baxter rule from the weak? No because one may easily check that the rules of Table 5.1 also form a Gröbner basis when the strong Baxter rule is removed, and its leading term $\int f\partial$ is clearly irreducible relative to the diminished Gröbner basis.

The same holds for the *equitable integro-differential operators* $\mathscr{F}[\partial, \int_\Phi]$ introduced at the end of Section 5.1, where one obtains the corresponding mixed variant of the weak Baxter rule.

Just to make sure, let us emphasize that—in the rewrite system—the weak Baxter rule is of course absolutely *indispensable*. If we left it out, it would actually be added by any completion ▲ procedure "computing" the noncommutative Gröbner basis.

5.3 Normal Forms for Integro-Differential Operators

Since the rewrite system of Table 5.1 is Noetherian and confluent, we know that every integro-differential operator can be written in a unique *normal form*, and we will now turn to a detailed description of these normal forms.

First of all, it should be clear that even a Noetherian and confluent rewrite system provides a canonical simplifier only relative to a presupposed canonical simplifier on the free algebra underlying $\mathscr{F}_\Phi[\partial, \int]$. Expansion with respect to the fixed basis $\mathscr{F}^\#$ provides such a *ground simplifier*, but there may also be others. In [38], we have

implemented a ground simplifier via basis expansion (for biintegro-differential operators over exponential polynomials). We will always assume that the free algebra is equipped with some ground simplifier, but we will not restrict it to any particular choice.

We start by describing a set of *generators*, which will subsequently be restricted to normal forms of $\mathscr{F}_\Phi[\partial, \int]$.

Lemma 5.4. *Every integro-differential operator in $\mathscr{F}_\Phi[\partial, \int]$ can be reduced to a linear combination of monomials $f\varphi\int g\psi\partial^i$, where $i \geq 0$ and each of $f, \varphi, \int, g, \psi$ may also be absent.*

Proof. Call a monomial consisting only of functions and functionals "algebraic". Using the left column of Table 5.1, it is immediately clear that all such monomials can be reduced to f or φ or $f\varphi$. Now let w be an arbitrary monomial in the generators of $\mathscr{F}_\Phi[\partial, \int]$. By using the middle column of Table 5.1, we may assume that all occurrences of ∂ are moved to the right, so that all monomials have the form $w = w_1 \cdots w_n \partial^i$ with $i \geq 0$ and each of $w_1, \ldots, w_n$ either a function, a functional or $\int$. We may further assume that there is at most one occurrence of $\int$ among the $w_1, \ldots, w_n$. Otherwise the monomials $w_1 \cdots w_n$ contain $\int \tilde{w} \int$, where each $\tilde{w} = f\varphi$ is an algebraic monomial. But then we can reduce

$$\int \tilde{w} \int = (\int f\varphi)\int = (\int \cdot f)\varphi \int$$

by using the corresponding rule of Table 5.1. Applying these rules repeatedly, we arrive at algebraic monomials left and right of $\int$ (or just a single algebraic monomial if $\int$ is absent). □

We turn now to the normal forms of *boundary functionals*, meaning those elements of $\mathscr{F}_\Phi[\partial, \int]$ that are used for describing various boundary conditions. How can we describe them? Since boundary conditions always induce mappings $\mathscr{F} \to K$, it is near at hand to select those combinations of integro-differential operators that "end" in an evaluation $\varphi \in \Phi$. If φ corresponds to evaluation at 1, composition with ∂ gives the local boundary condition $u'(1) = 0$ while composition with $\int$ yields the global condition $\int u(\xi)\, d\xi = 0$. Of course, boundary conditions will in general be linear combinations of such composites; they are known under the name "Stieltjes conditions" in the literature [7, 8].

Definition 5.5. The elements of the right ideal

$$|\Phi) = \Phi \cdot \mathscr{F}_\Phi[\partial, \int]$$

are called *Stieltjes boundary conditions* over $\mathscr{F}$; if there is no danger of ambiguity, we will henceforth just speak of "boundary conditions".

The *normal forms of boundary conditions* are exactly the linear combinations of local and global conditions that we have just brought up on an intuitive basis.

Proposition 5.6. *Every boundary condition of* $|\Phi)$ *has the normal form*

$$\sum_{\varphi\in\Phi}\left(\sum_{i\in\mathbb{N}} a_{\varphi,i}\,\varphi\partial^i + \varphi\int f_\varphi\right) \tag{5.2}$$

with $a_{\varphi,i}\in K$ *and* $f_\varphi\in\mathscr{F}$ *almost all zero.*

Proof. By Lemma 5.4, every boundary condition of $|\Phi)$ is a linear combination of monomials having the form

$$w=\chi f\varphi\textstyle\int g\psi\partial^i \quad\text{or}\quad w=\chi f\varphi\partial^i \tag{5.3}$$

where each of f,g,φ,ψ may also be missing. Using the left column of Table 5.1, the prefix $\chi f\varphi$ can be reduced to a scalar multiple of a functional, so we may as well assume that f and φ are not present; this finishes the right-hand case of (5.3). For the remaining case $w=\chi\int g\psi\partial^i$, assume first that ψ is present. Then we have

$$\chi(\textstyle\int g\psi)=\chi(\int\cdot\, g)\,\psi=(\chi\int\cdot\, g)\,\chi\psi=(\chi\int\cdot\, g)\,\psi,$$

so w is again a scalar multiple of $\psi\partial^i$, and we are done. Finally, assume we have $w=\chi\int g\partial^i$. If $i=0$, this is already a normal form. Otherwise we obtain

$$w=\chi(\textstyle\int g\partial)\,\partial^{i-1}=(\chi\cdot g)\,\chi\partial^{i-1}-\chi\int g'\partial^{i-1}-(\mathtt{E}\cdot g)\,\mathtt{E}\partial^{i-1},$$

where the first and the last summand are in the required normal form, while the middle summand is to be reduced recursively, eventually leading to a middle term in normal form $\pm\chi\int g'\partial^0=\pm\chi\int g'$. □

There are at least three *reasons* for considering Stieltjes boundary conditions of the form (5.2). First of all, they are interesting in themselves because certain boundary problems are naturally expressed in terms of global side conditions (for example, specifying the heat radiated through the boundary). This in also true for regularizing ill-posed problems and computing their generalized Green's function [38, p. 191]. A second reason for introducing Stieltjes boundary conditions will become manifest in Chapter 6.3 since factoring a boundary problem leads to factor problems with global conditions, even for a problem having only point conditions. Finally, a third advantage of Stieltjes boundary conditions is that they have the natural algebraic description as the right ideal given in Definition 5.5.

Let us now turn to the other two ingredients of integro-differential operators: We have already used the *differential operators* $\mathscr{F}[\partial]$, now seen as a subalgebra of $\mathscr{F}_\Phi[\partial,\int]$. Note that they have the usual normal forms since the Leibniz rule is part of the rewrite system. Analogously, one can introduce the subalgebra of *integral operators* generated by the functions and $\int$. Using Lemma 5.4, it is clear that the normal forms of integral operators are $\mathscr{F}$ itself and linear combinations of $f\int g$ with $f,g\in\mathscr{F}$, and the only rule applicable to them is the strong Baxter rule. Since we have already included $\mathscr{F}$ in $\mathscr{F}[\partial]$, we introduce $\mathscr{F}[\int]$ as the $\mathscr{F}$-bimodule generated by $\int$, which contains only monomials of the form $f\int g$.

Finally, we must consider the two-sided ideal (Φ) of $\mathscr{F}_\Phi[\partial,\int]$ generated by Φ; its elements are called *Stieltjes boundary operators* (briefly "boundary operators"). In fact, a more economical description of (Φ) is as the left $\mathscr{F}$-submodule generated by $|\Phi)$ because any $w\chi\tilde{w}$ with $w,\tilde{w}\in\mathscr{F}[\partial,\int]$ can be reduced to $f\varphi\int g\psi\partial^i\chi\tilde{w}$ by Lemma 5.4. Hence (Φ) includes all finite dimensional projectors P along Stieltjes boundary conditions. Using Proposition 4.7, these can be described in the following way: If $u_1,\dots,u_n\in\mathscr{F}^\#$ and $\beta_1,\dots,\beta_n\in|\Phi)$ are biorthogonal, then

$$P=\sum_{i=1}^{n}u_i\,\beta_i, \tag{5.4}$$

is the projector onto $[u_1,\dots,u_n]$ along $[\beta_1,\dots,\beta_n]^\perp$. From the representation (5.4) it is immediately clear that $P\in(\Phi)$. Note that all elements of (Φ) have the normal form (5.4), except that the (u_j) need not be biorthogonal to the (β_i).

We can now characterize the normal forms of $\mathscr{F}_\Phi[\partial,\int]$ in a very straightforward and intuitive manner: Every monomial is either a *differential operator* or an *integral operator* or a *boundary operator*. Hence every element of $\mathscr{F}_\Phi[\partial,\int]$ can be written uniquely as a sum $T+G+B$, with a differential operator $T\in\mathscr{F}[\partial]$, an integral operator $G\in\mathscr{F}[\int]$, and a boundary operator $B\in(\Phi)$.

Proposition 5.7. *For an ordinary integro-differential algebra $\mathscr{F}$ and characters $\Phi\subseteq\mathscr{F}^\bullet$, we have the direct decomposition $\mathscr{F}_\Phi[\partial,\int]=\mathscr{F}[\partial]\dotplus\mathscr{F}[\int]\dotplus(\Phi)$.*

Proof. Inspection of Table 5.1 confirms that all integro-differential operators having the described sum representation $T+G+P$ are indeed in normal form. Let us now prove that every integro-differential operator of $\mathscr{F}_\Phi[\partial,\int]$ has such a representation. It is sufficient to consider its monomials w. If w starts with a functional, we obtain a boundary condition by Proposition 5.6; so assume this is not the case. From Lemma 5.4 we know that

$$w=f\varphi\textstyle\int g\psi\partial^i\quad\text{or}\quad w=f\varphi\partial^i,$$

where each of φ,g,ψ may be absent. But $w\in(\Phi)$ unless φ is absent, so we may actually assume

$$w=f\textstyle\int g\psi\partial^i\quad\text{or}\quad w=f\partial^i.$$

The right-hand case yields $w\in\mathscr{F}[\partial]$. If ψ is present in the other case, we may reduce $\int g\psi$ to $(\int\cdot\,g)\,\psi$, and we obtain again $w\in(\Phi)$. Hence we are left with $w=f\int g\partial^i$, and we may assume $i>0$ since otherwise we have $w\in\mathscr{F}[\int]$ immediately. But then we can reduce

$$\begin{aligned}w&=f\,(\textstyle\int g\partial)\,\partial^{i-1}=f\Big(g-\int(\partial\cdot g)-(\mathsf{E}\cdot g)\,\mathsf{E}\Big)\partial^{i-1}\\&=(fg)\,\partial^{i-1}-f\textstyle\int(\partial\cdot g)\,\partial^{i-1}-(\mathsf{E}\cdot g)\,f\,\mathsf{E}\,\partial^{i-1},\end{aligned}$$

where the first term is obviously in $\mathscr{F}[\partial]$ and the last one in (Φ). The middle term may be reduced recursively until the exponent of ∂ has dropped to zero, leading to a term in $\mathscr{F}[\int]$. □

▼ For the *equitable integro-differential operators* $\mathscr{F}[\partial, \int_{\Phi}]$ introduced in Section 5.1, we have a decomposition analogous to that of Proposition 5.7. In the following, φ ranges over Φ. We write $\mathscr{F}[\int_{\varphi}^{x}]$ for the $\mathscr{F}$-bimodule generated by the integral $\int_{\varphi}^{x}$, and $\mathscr{F}[\int_{\Phi}]$ for the $\mathscr{F}$-bimodule generated by all the integrals $\int_{\varphi}^{x}$; clearly the latter is the direct sum of the former (as vector spaces). Similarly, we write $\mathscr{F}[\int_{\varphi}^{x}\partial]$ for the left $\mathscr{F}$-module generated by $\int_{\varphi}^{x}\partial, \int_{\varphi}^{x}\partial^2, \ldots$, and $\mathscr{F}[\int_{\Phi}\partial]$ for the left $\mathscr{F}$-module generated by all the $\int_{\varphi}^{x}\partial, \int_{\varphi}^{x}\partial^2, \ldots$; again the latter is clearly the direct sum of the former.

Using these blocks, we have the *direct decomposition* $\mathscr{F}[\partial, \int_{\Phi}] = \mathscr{F}[\partial] + \mathscr{F}[\int_{\Phi}] + \mathscr{F}[\int_{\Phi}\partial]$. Note that all three summands, and also the subsummands $\mathscr{F}[\int_{\varphi}^{x}]$ and $\mathscr{F}[\int_{\varphi}^{x}\partial]$, form subalgebras of $\mathscr{F}[\partial, \int_{\Phi}]$ according to the rewrite rules given in Table 5.2.

In Section 5.1 we have given special attention to the case of *biintegro-differential operators*. Starting from a biintegro-differential algebra $(\mathscr{F}, \partial, \int^{*}, \int_{*})$, we have here $\mathscr{F}[\partial, \int_{\Phi}] = \mathscr{F}[D, A, B]$, where Φ consists of the two evaluations $\mathtt{E}^{*}$ and $\mathtt{E}_{*}$ corresponding to the two integrals $\int^{*}$ and $\int_{*}$ ▲ on $\mathscr{F}$. Accordingly we will write $\mathscr{F}[A, B]$ instead of $\mathscr{F}[\int_{\Phi}]$ and $\mathscr{F}[AD, BD]$ instead of $\mathscr{F}[\int_{\Phi}\partial]$.

Chapter 6
Solving Boundary Problems in Symbolic Computation

In this chapter we combine the tools developed in the previous chapters to build an algorithm for solving linear boundary problems over an ordinary integro-differential algebra. The algorithm presupposes a constructive fundamental system for the underlying homogeneous equation but imposes no other conditions (in the literature one often restricts to self-adjoint and/or second-order boundary problems). This is always possible in the important special case of LODEs with constant coefficient (in systems theory one speaks of time-invariant systems).

6.1 The Solution Algorithm

We introduce the notion of boundary problem by specializing the definition in Section 4.3 to the setting of integro-differential algebras. Unless specified otherwise all integro-differential algebras in this section are over a fixed field K.

Definition 6.1. Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra. Then a *boundary problem* of order n is a pair $(T,\mathscr{B})$, where $T\in\mathscr{F}[\partial]$ is monic with $\deg T=n$ and $\mathscr{B}\leq|\mathscr{F}^\bullet)$ is a K-subspace with $\dim\mathscr{B}=n$.

Thus a boundary problem is specified by a differential operator T and a *boundary space* $\mathscr{B}=[\beta_1,\ldots,\beta_n]$ generated by n Stieltjes conditions $\beta_1,\ldots,\beta_n\in|\mathscr{F}^\bullet)$. In traditional notation, the boundary problem $(T,\mathscr{B})$ is then given by

$$\boxed{\begin{aligned}&Tu=f,\\&\beta_1u=\ldots=\beta_nu=0.\end{aligned}}$$

Normally, one will restrict the evaluations to a (typically finite) subset $\Phi\subseteq\mathscr{F}^\bullet$, with the consequence that all subsequent calculations can be carried out in $\mathscr{F}_\Phi[\partial,\int]$ instead of $\mathscr{F}[\partial,\int]$. We will disregard this issue here for keeping the notation simpler.

The definition of regularity also carries over from Section 4.3. As pointed out there, the condition $\deg T=\dim\mathscr{B}$ required in Definition 6.1 is only necessary but

not sufficient for ensuring regularity: the boundary conditions might collapse on $\mathrm{Ker}(T)$. A simple example of such a *singular boundary problem* is $(-D^2,[LD,RD])$, written with biintegro-differential operators $\mathscr{F}[D,A,B]$; see also [38, p. 191] for more details on this particular boundary problem.

Definition 6.2. Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra. Then a boundary problem $(T,\mathscr{B})$ is called *regular* if $\mathrm{Ker}(T)\dotplus\mathscr{B}^\perp=\mathscr{F}$.

As explained in Section 4.3, this means that the boundary problem (6.1) has a unique solution $u\in\mathscr{F}$ for every given forcing function $f\in\mathscr{F}$. Of course we may also apply the criterion of Proposition 4.16 for getting an algorithmic regularity test. Taking any fundamental system of solutions $u_1,\ldots,u_n$ for the homogeneous equation, we see that the boundary problem $(T,\mathscr{B})$ is regular iff the *evaluation matrix* $\beta(u)\in K^{n\times n}$ is regular.

For a regular boundary problem $(T,\mathscr{B})$ we can now define the *Green's operator* as the operator mapping the given forcing function $f\in\mathscr{F}$ to the unique solution $u\in\mathscr{F}$ of (6.1). As in Section 4.3, we can also express this without referring to the action of the Green's operator.

Definition 6.3. Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra. Then the *Green's operator* of a regular boundary problem $(T,\mathscr{B})$ is defined by $TG=1$ and $\mathrm{Im}(G)=\mathscr{B}^\perp$. As before we write $G=(T,\mathscr{B})^{-1}$.

We can now recast Theorem 3.25 in the language of Green's operators of initial value problems. Given a monic differential operator T of order n, the theorem implies that the boundary problem $(T,[\mathtt{E},\mathtt{E}\partial,\ldots,\mathtt{E}\partial^{n-1}])$ is regular. We call its Green's operator the *fundamental right inverse* of T and denote it by $T^\blacklozenge$.

Theorem 6.4. *Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra and let $T\in\mathscr{F}[\partial]$ be monic with regular fundamental system $u_1,\ldots,u_n$. Then its fundamental right inverse is given by*

$$T^\blacklozenge=\sum_{i=1}^n u_i\textstyle\int d^{-1}d_i\ \in\ \mathscr{F}[\partial,\int],\tag{6.1}$$

where $d,d_1,\ldots,d_n$ are as in Theorem 3.25.

Before turning to the solution algorithm for boundary problems, let us also mention the following practical formula for specializing Theorem 6.4 to the important special case of LODEs with *constant coefficients*, which could also be proved directly e.g. via the Lagrange interpolation formula. For simplicity, we restrict ourselves to the case where the characteristic polynomial is separable.

Corollary 6.5. *Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra and consider the differential operator $T=(\partial-\lambda_1)\cdots(\partial-\lambda_n)\in\mathscr{F}[\partial]$ with $\lambda_1,\ldots,\lambda_n\in K$ mutually distinct. Assume each $u'=\lambda_i u,\mathtt{E}\cdot u=1$ has a solution $u=e^{\lambda_i x}\in\mathscr{F}$ with reciprocal $u^{-1}=e^{-\lambda_i x}\in\mathscr{F}$. Then we have*

$$T^\blacklozenge=\sum_{i=1}^n\mu_i e^{\lambda_i x}\textstyle\int e^{-\lambda_i x},$$

where $\mu_i^{-1} = (\lambda_i - \lambda_1)\cdots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\cdots(\lambda_i - \lambda_n)$.

Proof. Let us write V for the $n \times n$ Vandermonde determinant in $\lambda_1, \ldots, \lambda_n$ and V_i for the $(n-1) \times (n-1)$ Vandermonde determinant in $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n$. Evaluating the quantities of (6.1), one sees immediately that

$$d = e^{(\lambda_1 + \cdots \lambda_n)x} V \quad \text{and} \quad d_i = (-1)^{n+i} e^{(\lambda_1 + \cdots + \lambda_{i-1} + \lambda_{i+1} + \cdots + \lambda_n)x} V_i.$$

Hence we have $d_i/d = (-1)^{n+i} e^{-\lambda_i x} V_i/V$. Using the well-known formula for the Vandermonde determinant, one obtains $d_i/d = \mu_i e^{-\lambda_i x}$, and now the result follows from Theorem 6.4. □

Summarizing our earlier results, we can now give a *solution algorithm* for computing $G = (T, \mathscr{B})^{-1}$, provided we have a regular fundamental system $u_1, \ldots, u_n$ for $Tu = 0$ and a K-basis $\beta_1, \ldots, \beta_n$ for $\mathscr{B}$. The algorithm proceeds in three steps:

1. Construct the fundamental right inverse $T^\blacklozenge \in \mathscr{F}[\partial, \int]$ as in Theorem 6.4.
2. Determine the projector $P = \sum_{i=1}^n u_i \tilde{\beta}_i \in \mathscr{F}[\partial, \int]$ as in Theorem 4.17.
3. Compute $G = (1-P)T^\blacklozenge \in \mathscr{F}[\partial, \int]$.

Theorem 6.6. *The above algorithm computes the Green's operator $G \in \mathscr{F}[\partial, \int]$ for any regular boundary problem $(T, \mathscr{B})$.*

Proof. Follows directly from Theorem 6.4 and Theorem 4.17. □

Since *first-order LODEs* can always be solved up to quadratures, it is worthwhile to write down the corresponding Green's operators explicitly.

Proposition 6.7. *Let $(\partial - h, [\beta])$ be boundary problem over the integro-differential algebra $C^\infty[a,b]$. Writing $H = \int \cdot\, h$, this boundary problem is regular iff $c = \beta \cdot e^H$ is nonzero. In this case, the Green's operator is $(1 - c^{-1}e^H\beta)\, e^H \int e^{-H}$.*

Proof. The regularity criterion $c = \beta(e^H) \neq 0$ follows directly from Proposition 4.16. Now we proceed according to the solution algorithm given above. One may use Theorem 6.4 or direct verification for seeing that $(\partial - h)^\blacklozenge = e^H \int e^{-H}$. Since $\tilde{\beta} = c^{-1}\beta$ we obtain $P = c^{-1}e^H\beta$, and the formula for the Green's operator follows. □

As explained in Chapter 2, the classical approach to boundary problems formulates their solution via *Green's functions* g. They induce Green's operators G of a special form, namely

$$Gf(x) = \int_a^b g(x,\xi)\, f(\xi)\, d\xi$$

for boundary problems in $\mathscr{F} = C^\infty[a,b]$. As remarked in ???, the Green's function is always patched together from two branches $g^<$ for $\xi < x$, and $g^>$ for $\xi > x$, and each of these decomposes into a sum of tensor products

$$g^<(x,\xi) = l_1^<(x)\, \lambda_1^<(\xi) + \cdots + l_n^<(x)\, \lambda_n^<(\xi),$$
$$g^>(x,\xi) = l_1^>(x)\, \lambda_1^>(\xi) + \cdots + l_n^>(x)\, \lambda_n^>(\xi)$$

so that we can write each such Green's operator as

$$G = l_1^< A\lambda_1^< + \cdots + l_n^< A\lambda_n^< + l_1^> B\lambda_1^> + \cdots + l_n^> B\lambda_n^> \in \mathscr{F}[A,B] \tag{6.2}$$

within the biintegro-differential operator ring $\mathscr{F}[D,A,B]$. With the above algorithm we get only G, how can we extract the corresponding Green's function g from it? Is this always possible?

Example 6.8. The answer is no since Green's operators are more general than Green's functions: We have just seen that the latter are included in the former, so let us now give a simple example of a regular boundary problem that has *no Green's function* in the sense above. Taking $\mathscr{F} = C^\infty[0,1]$, we consider the boundary problem $(D-1, LD^2)$ in $\mathscr{F}[D,A,B]$. In classical notation: Given $f \in \mathscr{F}$, we want to find $u \in \mathscr{F}$ such that

$$\boxed{\begin{aligned} u' - u &= f, \\ u''(0) &= 0 \end{aligned}}. \tag{6.3}$$

Since e^x is a fundamental solution, the 1×1 evaluation matrix is just $LD^2 \cdot e^x = 1$, and we conclude that we have a regular boundary problem. In fact, using Proposition 6.7 one may compute its Green's operator $G = e^x A e^{-x} - e^x L - e^x LD$, so the unique solution is

$$u(x) = \int_0^x e^{x-\xi} f(\xi)\, d\xi - (f(0) + f'(0))\, e^x.$$

Since this Green's operator is normalized, there is no way of writing it as in (6.2), which is also in normal form. □

One might object that the boundary problem (6.3) is in some sense "ill". In fact, the classical literature never studies an n-th order differential equations with boundary conditions involving derivatives of order n or beyond. Why? Obviously the problem is not with existence or uniqueness; from the algebraic viewpoint there is no problem at all. But the presence of the evaluations in L and LD leads to an unpleasant analytic behavior. If we regard $\mathscr{F}$ as a Banach space with Chebyshev norm $\|\cdot\|_\infty$, the Green's operator $G\colon \mathscr{F} \to \mathscr{F}$ of (6.3) is *not continuous* while those of the form (6.2) are always.

The case of boundary conditions involving derivatives of the same order as the differential equation constitutes a borderline case. For example, if we replace the condition $u''(0) = 0$ in (6.3) by $u'(0) = 0$, one gets the same Green's operator but without the LD term. There is no deriviative, consequently the Green's operator is continuous with respect to the Chebyshev norm $\|\cdot\|_\infty$. But for applications, one will usually work in the larger space $L^2[a,b] \supseteq C^\infty[a,b]$ with the corresponding norm $\|\cdot\|_2$. In this case, point evaluations like L are not even defined on $L^2[a,b]$, and they are *not continuous* on $C^\infty[a,b]$ with respect to $\|\cdot\|_2$.

This leads us to the following definition for excluding (or at least recognizing) cases like (6.3). Following Hadamard, a *well-posed* problem [18, p. 86] must be regular (meaning the solution u exists and is unique) as well as stable (meaning the

solution u depends continuously on the forcing function f). Otherwise, one speaks of an *ill-posed* problem. We can transfer this idea to our algebraic setting by requiring the appropriate form of boundary conditions.

Definition 6.9. Let $(\mathscr{F},\partial,\int)$ be an ordinary differential algebra. Then we call an n-th order boundary problem $(T,\mathscr{B})$ *well-posed* if it is regular and every Stieltjes condition in $\mathscr{B}$ is of the form (5.2) with $a_{\varphi,i}=0$ for all $i\geq n$. Otherwise we call the boundary problem $(T,\mathscr{B})$ *ill-posed*.

We will see that well-posed boundary problems are indeed well-posed in the analysis sense when we specialize to the standard setting $\mathscr{F}=C^\infty[a,b]$ for two-point boundary problems. The following result is a generalization of this observation since it allows to have evaluations in more than two points as well as global terms. For obtaining a convenient formulation, we move from our usual integro-differential operator ring $\mathscr{F}_\Phi[\partial,\int]$ to its *equitable* variant $\mathscr{F}[\partial,\int_\Phi]$ since this allows us to write Green's operators of the form (6.2) in a more natural way.

Lemma 6.10. *Let $(\mathscr{F},\partial)$ be a differential algebra and $\Phi\subseteq\mathscr{F}^\bullet$ a nonempty collection of characters on $\mathscr{F}$. For any well-posed boundary problem $(T,\mathscr{B})$ we have*

$$(T,\mathscr{B})^{-1}\in\mathscr{F}[\textstyle\int_\Phi]$$

when computed in the equitable ring $\mathscr{F}[\partial,\int_\Phi]$.

Proof. We designate one character of Φ as a distinguished evaluation $\mathtt{E}$ and compute $(T,\mathscr{B})^{-1}=T^\blacklozenge-PT^\blacklozenge$ according to the above algorithm. The first summand is clearly contained in $\mathscr{F}[\int_\Phi]\leq\mathscr{F}[\partial,\int_\Phi]$, hence it remains to prove $PT^\blacklozenge\in\mathscr{F}[\int_\Phi]$. Since P is a left $\mathscr{F}$-linear combination of local conditions $\varphi\partial^i$ with $i<n$ and global conditions $\varphi\int_\mathtt{E}^x f$, it suffices to ensure that $\varphi\partial^iT^\blacklozenge\in\mathscr{F}[\int_\Phi]$ and $\varphi\int_\mathtt{E}^x fT^\blacklozenge\in\mathscr{F}[\int_\Phi]$. The second claim follows immediately from the fact that $\mathscr{F}[\int_\Phi]$ is a subalgebra of $\mathscr{F}[\partial,\int_\Phi]$ because we have

$$\varphi\textstyle\int_\mathtt{E}^x=\int_\mathtt{E}^\varphi=\int_\mathtt{E}^x-\int_\varphi^x\in\mathscr{F}[\int_\Phi]\qquad\text{and}\qquad fT^\blacklozenge\in\mathscr{F}[\int_\Phi].$$

The first claim will be established by showing the identity

$$\partial^iT^\blacklozenge=\sum_{j=1}^n u_j^{(i)}\textstyle\int_\mathtt{E}^x\frac{d_j}{d}\quad(i<n),\tag{6.4}$$

which implies $\varphi\partial^iT^\blacklozenge\in\mathscr{F}[\int_\Phi]$ since we have $\varphi u_j^{(i)}\int_\mathtt{E}^x=(\varphi\partial^i\cdot u_j)\,(\int_\mathtt{E}^x-\int_\varphi^x)$ by the observations made above. For proving (6.4) we use induction on n. The base case $n=0$ is given by (6.1). Now assume the claim for $i<n-1$; we must prove it for $i+1<n$. Using (6.4) for i, we obtain

$$\partial^{i+1}T^\blacklozenge=\partial\sum_{j=1}^n u_j^{(i)}\textstyle\int_\mathtt{E}^x\frac{d_j}{d}=\sum_{j=1}^n u_j^{(i+1)}\int_\mathtt{E}^x\frac{d_j}{d}+\sum_{j=1}^n u_j^{(i)}\frac{d_j}{d},$$

which is just (6.4) for $i+1$ because the last sum vanishes by the definition of the d_j. □

The classical results reported in Chapter 2 now follow from Lemma 6.10 by considering the standard setting $(\mathscr{F},\partial,\int^*,\int_*) = (C^\infty[a,b],\frac{d}{dx},\int_a^x,\int_x^b)$. In this case, the equitable operator ring $\mathscr{F}[\partial,\int_\Phi]$ is the ring $\mathscr{F}[D,A,B]$ of biintegro-differential operators.

Corollary 6.11. *Let $(T,\mathscr{B})$ be a well-posed boundary problem over a biintegro-differential algebra $(\mathscr{F},\partial,\int^*,\int_*)$. Then we have $(T,\mathscr{B})^{-1} \in \mathscr{F}[A,B]$.*

Note that the classical case of two-point boundary problems is included in Corollary 6.11 by restricing to purely local conditions. In any case, Corollary 6.11 ensures that one can immediately *extract Green's functions* from such Green's operators since every $G \in \mathscr{F}[A,B]$ has the normal form (6.2).

6.2 Multiplicative Structure on Boundary Problems

6.3 Factorization of Boundary Problems

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