Theorem 2.4.18. *If the elements of $F = (f_1, \ldots, f_s)$ are a Gröbner basis, then S is a basis for $\mathrm{Syz}(F)$, where S is defined as follows.*

For $1 \leq i \leq s$ let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the i–th unit vector and for $1 \leq i < j \leq s$ let

$$t = \mathrm{lcm}(\mathrm{lpp}(f_i), \mathrm{lpp}(f_j)),$$
$$p_{ij} = \frac{1}{\mathrm{lc}(f_i)} \cdot \frac{t}{\mathrm{lpp}(f_i)}, \quad q_{ij} = \frac{1}{\mathrm{lc}(f_j)} \cdot \frac{t}{\mathrm{lpp}(f_j)},$$

and $k_{ij}^1, \ldots, k_{ij}^s$ be the polynomials extracted from a reduction of $\mathrm{spol}(f_i, f_j)$ *to 0, such that*

$$\mathrm{spol}(f_i, f_j) = p_{ij} f_i - q_{ij} f_j = \sum_{l=1}^{s} k_{ij}^l f_l.$$

Then

$$S = \{\underbrace{p_{ij} \cdot e_i - q_{ij} \cdot e_j - (k_{ij}^1, \ldots, k_{ij}^s)}_{S_{ij}} \mid 1 \leq i < j \leq s\}.$$

Proof: Obviously every element of S is a syzygy of F, since every S–polynomial reduces to 0.

On the other hand let $z = (z_1, \ldots, z_s) \neq (0, \ldots, 0)$ be an arbitrary non-trivial syzygy of F. Let p be the hightest power product occurring in

$$f_1 z_1 + \ldots + f_s z_s = 0, \qquad (*)$$

i.e.

$$p = \max_{<}\{t \in [X] \mid \mathrm{coeff}(f_i \cdot z_i, t) \neq 0 \text{ for some } i\}$$

and let $i_1 < \ldots < i_m$ be those indices such that $\mathrm{lpp}(f_{i_j} \cdot z_{i_j}) = p$. We have $m \geq 2$. Suppose that $m > 2$. By subtracting a suitable multiple of S_{i_{m-1}, i_m} from z, we can reduce the number of positions in z that contribute to the highest power product p in $(*)$. Iterating this process $m - 2$ times, we finally reach a situation, where only two positions i_1, i_2 in the syzygy contribute to the power product p. Now the highest power product in $(*)$ can be decreased by subtracting a suitable multiple of S_{i_1, i_2}. Since $<$ is Noetherian, this process terminates, leading to an expression of z as a linear combination of elements of S. □

Theorem 2.4.19. *Let $F = (f_1, \ldots, f_s)^T$ be a vector of polynomials in $K[X]$ and let the elements of $G = (g_1, \ldots, g_m)^T$ be a Gröbner basis for $\langle f_1, \ldots, f_s \rangle$. We view F and G as column vectors. Let the r rows of the matrix R be a basis for $\mathrm{Syz}(G)$ and let the matrices A, B be such that $G = A \cdot F$ and $F = B \cdot G$. Then the rows of Q are a basis for $\mathrm{Syz}(F)$, where*

$$Q = \begin{pmatrix} I_s - B \cdot A \\ \ldots\ldots\ldots\ldots\ldots \\ R \cdot A \end{pmatrix}.$$

Proof: Let $b_1, \ldots, b_{s+r}$ be polynomials, $b = (b_1, \ldots, b_{s+r})$.

$$\begin{array}{l} (b \cdot Q) \cdot F = \\ ((b_1, \ldots, b_s) \cdot (I_s - B \cdot A) + (b_{s+1}, \ldots, b_{s+r}) \cdot R \cdot A) \cdot F = \\ (b_1, \ldots, b_s) \cdot (F - \underbrace{B \cdot A \cdot F}_{=F}) + (b_{s+1}, \ldots, b_{s+r}) \cdot R \cdot \underbrace{A \cdot F}_{=G} = 0 \end{array}.$$

So every linear combination of the rows of Q is a syzygy of F.

On the other hand, let $H = (h_1, \ldots, h_s)$ be a syzygy of F. Then $H \cdot B$ is a syzygy of G. So for some H' we can write $H \cdot B = H' \cdot R$, and therefore $H \cdot B \cdot A = H' \cdot R \cdot A$. Thus,

$$H = H \cdot (I_s - B \cdot A) + H' \cdot R \cdot A = (H, H') \cdot Q,$$

i.e. H is a linear combination of the rows of Q. □