Name: .... 25 Jan 2011

Studienkennzahl: ....

Matrikelnummer: ....

Final Exam
Computer Algebra (326.017)
(no books allowed)

(1) Consider the following polynomials in $\mathbb{Q}[x,y]$:

$$a(x,y) = xy^2 + 2y^2 - xy - 2, \qquad b(x,y) = xy + 2y + x - 4$$

(i) Compute $\mathrm{res}_y(a,b) \in \mathbb{Q}[x]$, the resultant of a and b w.r.t. y;
(ii) Which of the solutions $x = 1, 6, -2$ of $\mathrm{res}_y(a,b)$ can be extended to a solution of the system $a(x,y) = 0 = b(x,y)$? Explain this situation.

(2) Consider the polynomials $a(x,y)$ and $b(x,y)$ of Question (1). We want to determine a Gröbner basis G for the ideal generated by $\{a,b\}$ w.r.t. the lexicographic order with $x < y$:
spol(a,b) can be reduced to $c(x,y) = 4y + x - 5$, so we add c to the basis;
spol(a,c) can be reduced to $d(x,y) = x^2 - 7x + 6$, so we add d to the basis;
Complete the algorithm and determine G.

(3) Consider systems of polynomial equations over a field K.
(i) Which systems can be solved by the Gauss elimination method, Euclid's gcd algorithm, and the Gröbner basis method?
(ii) How is the gcd of two univariate polynomials related to the Gröbner basis of the ideal generated by these polynomials?
(iii) How is the triangularized version of a system of linear equations (the result of Gauss elimination) related to a Gröbner basis for this system (w.r.t. to lexicographic ordering)?

(4) (i) What is a minimal Gröbner basis, and what is a reduced Gröbner basis?
(ii) Prove: *Let G be a Gröbner basis (w.r.t. the admissibel ordering $>$) for the ideal I in $K[x_1,\ldots,x_n]$. Let g,h be different elements of G.*
If $\mathrm{lpp}(g)|\mathrm{lpp}(h)$, then $G \setminus \{h\}$ is also a Gröbner basis (w.r.t. $>$) for I.

(5) Let K be a field, and $F \subseteq K[x_1,\ldots,x_n]$. Let $>$ be an admissible ordering on $[x_1,\ldots,x_n]$, and let $\longrightarrow_F$ be the reduction relation induced by F.
(i) What do the following statements mean?
$\longrightarrow_F$ is Church-Rosser
$\longrightarrow_F$ is confluent
$\longrightarrow_F$ is locally confluent
(ii) Prove: *$\longrightarrow_F$ is Church-Rosser if and only if $\longrightarrow_F$ is confluent.*