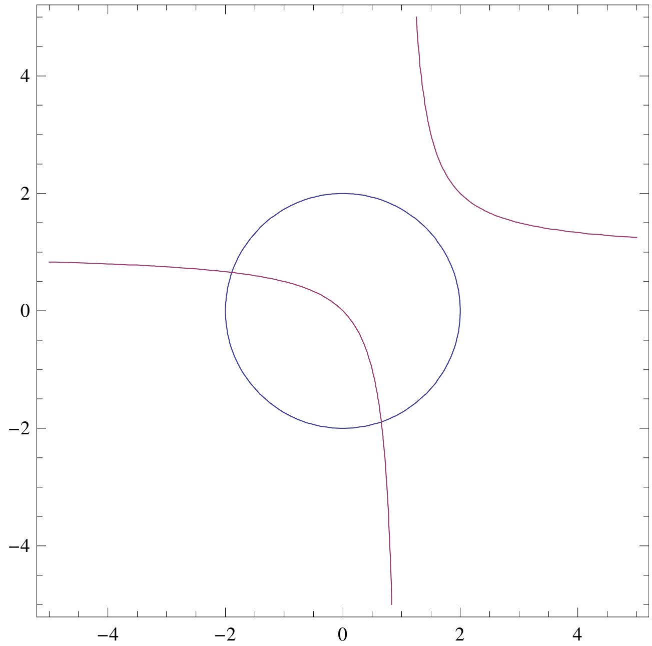
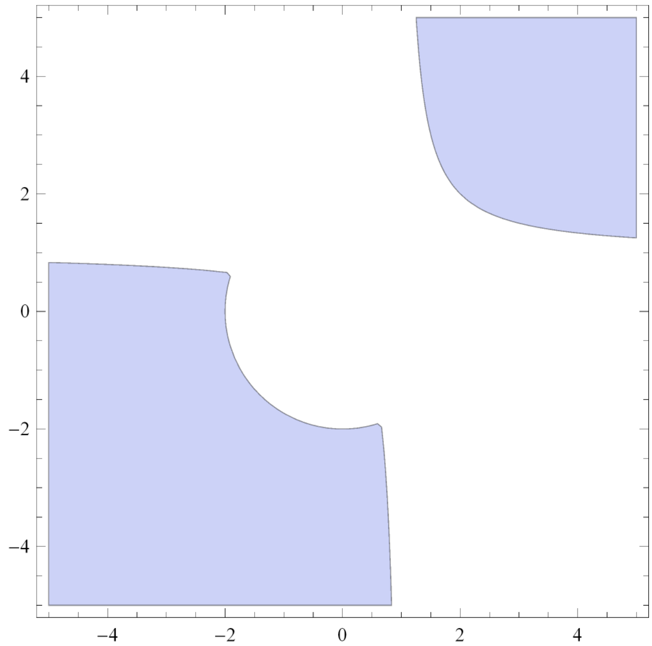
Example (Cells)

```
In[7]:= ContourPlot[{x^2 + y^2 - 4 == 0, (x - 1) (y - 1) - 1 == 0}, {x, -5, 5}, {y, -5, 5}]
        RegionPlot[x^2 + y^2 - 4 > 0 && (x - 1) (y - 1) - 1 > 0, {x, -5, 5}, {y, -5, 5}]
        RegionPlot[x^2 + y^2 - 4 < 0 && (x - 1) (y - 1) - 1 > 0, {x, -5, 5}, {y, -5, 5}]
        RegionPlot[x^2 + y^2 - 4 > 0 && (x - 1) (y - 1) - 1 < 0, {x, -5, 5}, {y, -5, 5}]
        RegionPlot[x^2 + y^2 - 4 < 0 && (x - 1) (y - 1) - 1 < 0, {x, -5, 5}, {y, -5, 5}]
```

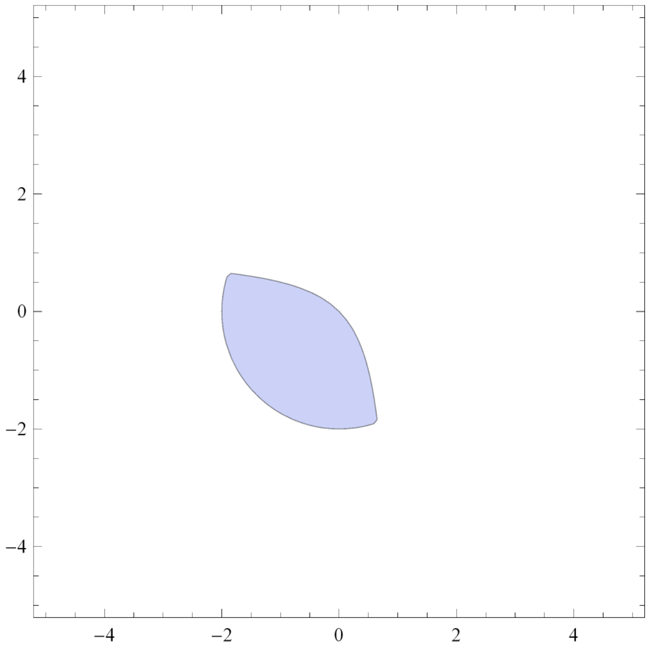
Out[7]=



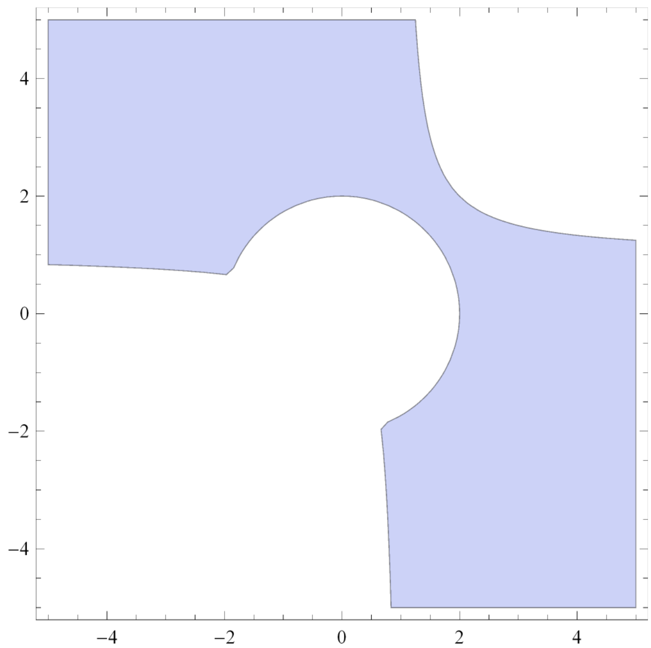
Out[8]=



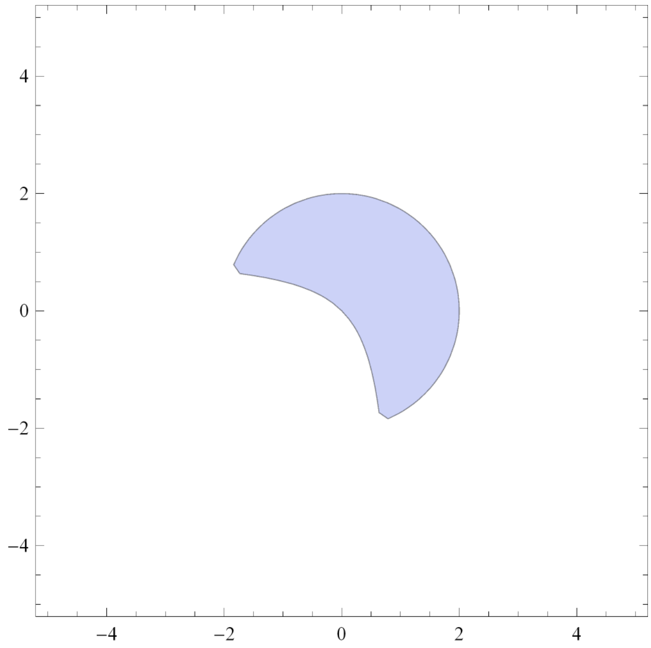
Out[9]=



Out[10]=

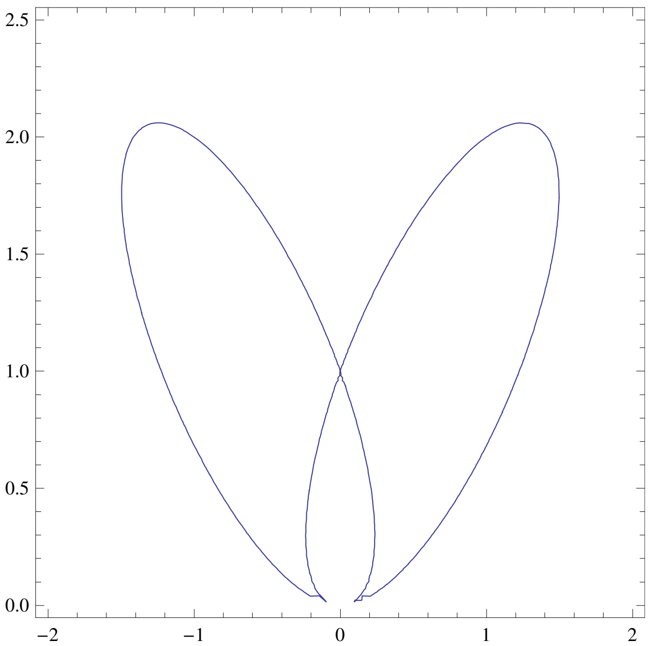


Out[11]=



The Tacnode Example

```
p[x_, y_] := 2 x^4 - 3 x^2 y + y^4 - 2 y^3 + y^2;
ContourPlot[p[x, y] == 0, {x, -2, 2}, {y, 0, 2.5}]
```

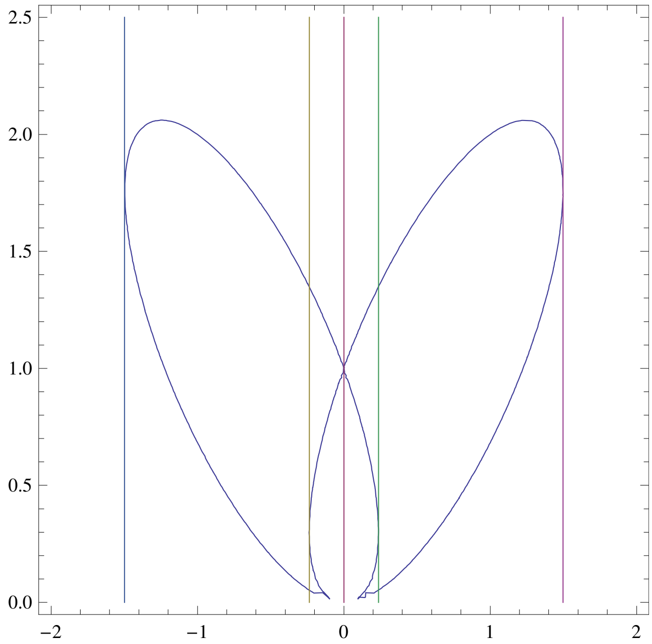


- **CAD with respect to x,y**

```
Reduce[Discriminant[p[x, y], y] == 0, x, Reals] // N
```

```
x == 0. || x == -0.236556 || x == 0.236556 || x == -1.49692 || x == 1.49692
```

```
ContourPlot[{p[x, y] == 0, x == 0, x == -0.2365557162041041`, x == 0.2365557162041041`,
  x == -1.4969203224061072`, x == 1.4969203224061072`}, {x, -2, 2}, {y, 0, 2.5}]
```

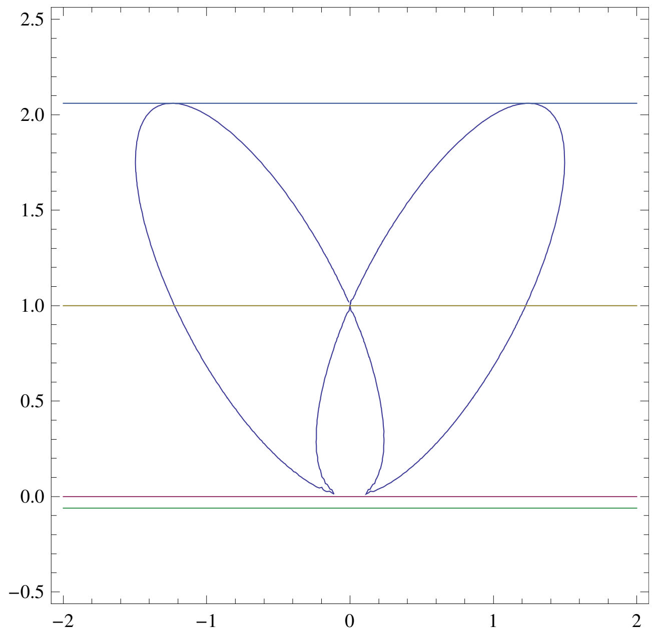


CAD with respect to y, x

```
Reduce[Discriminant[p[x, y], x] == 0, y, Reals] // N
```

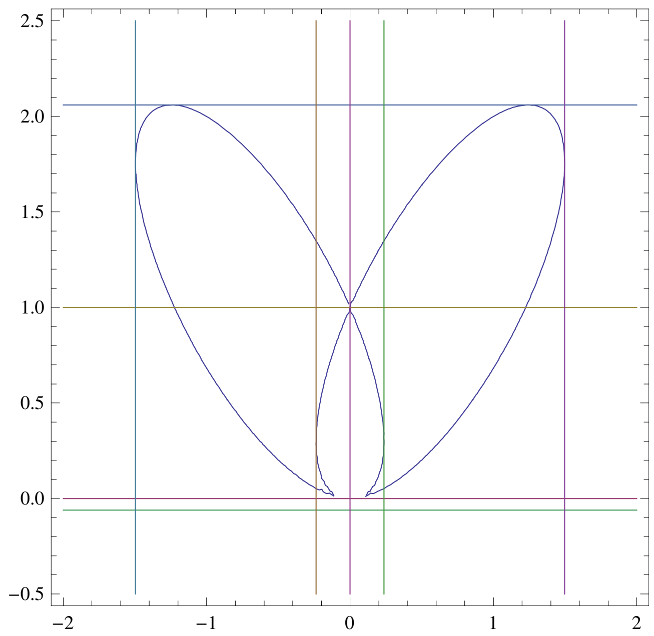
```
y == 0. || y == 1. || y == -0.0606602 || y == 2.06066
```

```
ContourPlot[{p[x, y] == 0, y == 0, y == 1, y == -0.060660171779821415`,
  y == 2.0606601717798214`}, {x, -2, 2}, {y, -0.5, 2.5}]
```



Solution (Region of truth)

```
ContourPlot[{p[x, y] == 0, y == 0, y == 1, y == -0.06066017177982141`,
  y == 2.0606601717798214`, x == 0, x == -0.2365557162041041`, x == 0.2365557162041041`,
  x == -1.4969203224061072`, x == 1.4969203224061072`}, {x, -2, 2}, {y, -0.5, 2.5}]
```



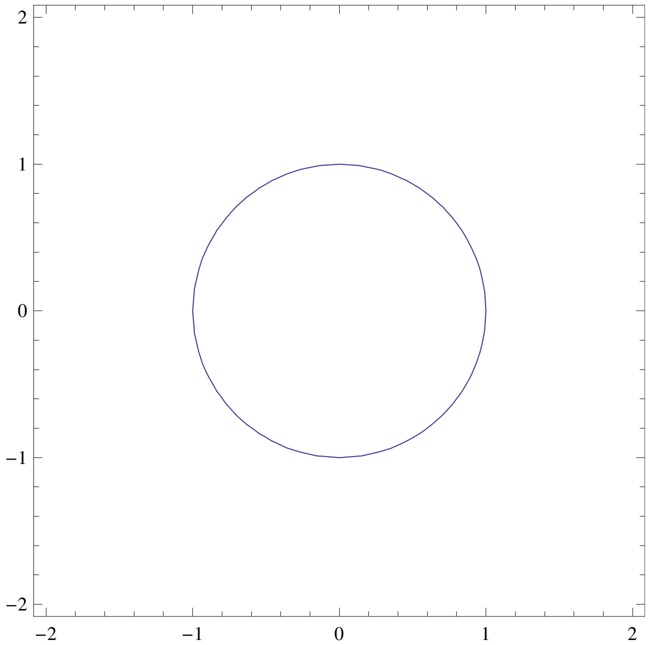
with CAD

```
CylindricalDecomposition[2 x^4 - 3 x^2 y + y^4 - 2 y^3 + y^2 == 0, {x, y}] // N
```

```
(x == -1.4969203224061072` && y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1]) ||
 (-1.4969203224061072` < x < -0.2365557162041041` &&
   (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2])) ||
 (x == -0.2365557162041041` && (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] || y == Root[
        2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2] || y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 4])) ||
 (-0.2365557162041041` < x < 0.` && (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] || y ==
      Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2] || y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 3] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 4])) || (x == 0.` && (y == 0.` || y == 1.`)) ||
 (0.` < x < 0.2365557162041041` && (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] || y ==
      Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2] || y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 3] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 4])) ||
 (x == 0.2365557162041041` && (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 4])) ||
 (0.2365557162041041` < x < 1.4969203224061072` &&
   (y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1] ||
     y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 2])) ||
 (x == 1.4969203224061072` && y == Root[2 x^4 - 3 x^2 #1 + #1^2 - 2 #1^3 + #1^4 &, 1])
```

Circle

```
p[x_, y_] := x^2 + y^2 - 1;
ContourPlot[p[x, y] == 0 , {x, -2, 2}, {y, -2, 2}]
```




```
In[1]:= CylindricalDecomposition[{x^2 - y^2 - 1 == 0, y ≥ 0}, {x, y}] // N
```

Out[1]= $\left(x \le -1.\ \&\&\ y == \sqrt{-1. + x^2}\right) \,||\, \left(x \ge 1.\ \&\&\ y == \sqrt{-1. + x^2}\right)$

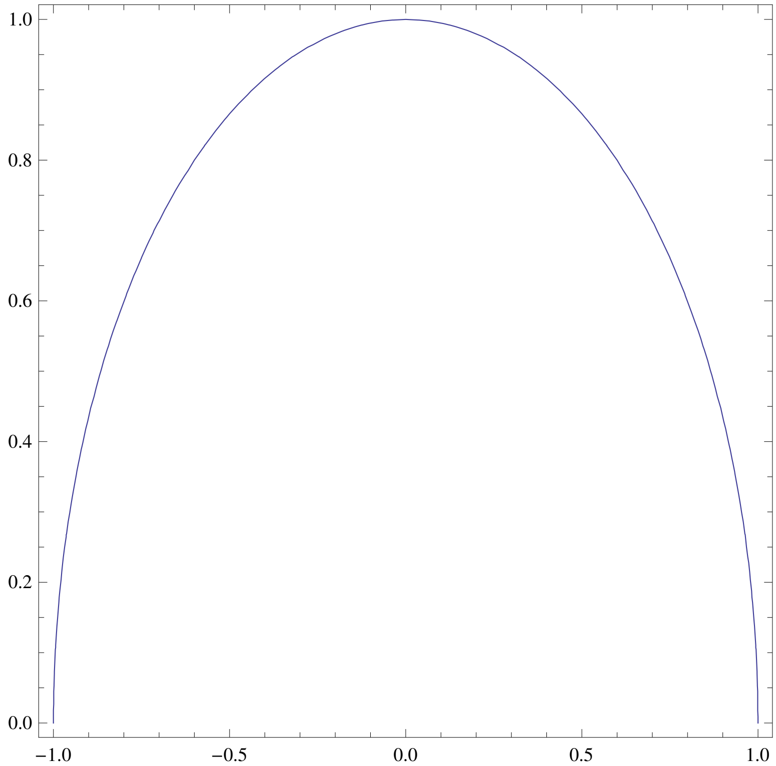
```
Discriminant[p[x, y], y]
```

$-4\left(-1 + x^2\right)$

```
Reduce[Discriminant[p[x, y], y] == 0, x, Reals] // N
```

$x == -1. \,||\, x == 1.$

```
ContourPlot[p[x, y] == 0, {x, -1, 1}, {y, 0, 1}]
```




```
Solve[x^2 - y^2 - 1 == 0, y]
```

$$\left\{\left\{y \to -\sqrt{-1+x^2}\right\}, \left\{y \to \sqrt{-1+x^2}\right\}\right\}$$

- **Sample points in x**

```
Solve[p[x, y] == 0, x]
```

$$\left\{\left\{x \to -\sqrt{1-y^2}\right\}, \left\{x \to \sqrt{1-y^2}\right\}\right\}$$

There are 2 real roots, x=±1, therefore the sample points are {-3/2,-1,0,1,3/2}.

- **Sample points in R^2**

```
Solve[p[x, y] == 0, y]
```

$$\left\{\left\{y \to -\sqrt{1-x^2}\right\}, \left\{y \to \sqrt{1-x^2}\right\}\right\}$$

Sample points are $(-1, -1)$, $(-1, 1)$, $(-1, 0)$, $\left(0, -\sqrt{-1+x^2}-1\right)$, $\left(0, -\sqrt{-1+x^2}\right)$, $(0, 0)$, $\left(0, \sqrt{-1+x^2}\right)$, $\left(0, \sqrt{-1+x^2}+1\right)$, $(1, -1)$, $(1, 0)$, $(1, 1)$.

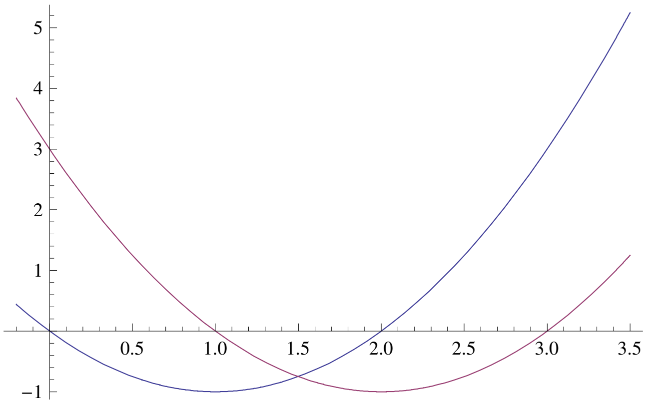
- **Truth values for x^2 - y^2 = 1 ∧ y >= 0**

True for $(-1, 0)$, $\left(0, \sqrt{-1+x^2}\right)$, $(1, 0)$

Therefore for - 1 < x < 1 => y=$\sqrt{-1 + x^2}$, the solution od CAD.

Other example

```
Plot[{x^2 - 2 x, x^2 - 4 x + 3}, {x, -0.2, 3.5}, PlotRange → Automatic]
```




```
Solve[x^2 - 2 x == 0, x]
```

```
{{x → 0}, {x → 2}}
```

```
Solve[x^2 - 4 x + 3 == 0, x]
```

```
{{x → 1}, {x → 3}}
```

Both functions has to be >= 0.
Therefore , we have following real roots {0, 1, 2, 3}.
Sample points : {-1, 0, 1/2, 1, 3/2, 2, 5/2, 3, 4}

For quantified formulas the sample points determine whether the formulas are true or false.
If we have: ∃x∈R: (x^2-2x>=0)∧(x^2-4x+3>=0).
For x=0, it is true.
For x=1, it is false.

If we have: ∀x∈R: (x^2-2x>=0)∧(x^2-4x+3>=0), therefore it is false, because we have to implement all sample points.

```
CylindricalDecomposition[{x^2 - 2 x ≥ 0, x^2 - 4 x + 3 ≥ 0}, {x, y}] // N
```

```
x ≤ 0. || x ≥ 3.
```