Introduction

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Computer Algebra Systems
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Self-Introduction

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Computer Algebra System

Definition: A computer algebra system (CAS) is a software for doing computer algebra, where

$$\text{computer algebra} = \text{symbolic computation}.$$

More concretely: a software for solving mathematical problems with *symbolic* methods.

Symbolic Computation (1)

Problem: What is $\sqrt{72}$?

Numeric Evaluation:

$$8.48528$$

Symbolic Solution:

$$6\sqrt{2}$$

Symbolic Computation (2)

Problem: What is the 100th harmonic number

$$\sum_{n=1}^{100} \frac{1}{n}?$$

Numeric Evaluation:

$$5.18738$$

Symbolic Solution:

$$\frac{14466636279520351160221518043104131447711}{2788815009188499086581352357412492142272}$$

Symbolic Computation (3)

Problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Numeric Evaluation:

$$1.08232$$

Symbolic Solution:

$$\frac{\pi^4}{90}$$

Symbolic Computation (4)

Problem: Find all solutions of the equation $x^3 + x^2 + x - 1 = 0$.

Numeric Evaluation:

$$x_1 = 0.543689$$
$$x_2 = -0.771845 + 1.11514i$$
$$x_3 = -0.771845 - 1.11514i$$

Symbolic Solution:

$$x_1 = \frac{1}{3}\left(-1 - \frac{2}{\sqrt[3]{17+3\sqrt{33}}} + \sqrt[3]{17+3\sqrt{33}}\right)$$
$$x_2 = -\frac{1}{3} + \frac{1+i\sqrt{3}}{3\sqrt[3]{17+3\sqrt{33}}} - \frac{1}{6}\left(1-i\sqrt{3}\right)\sqrt[3]{17+3\sqrt{33}}$$
$$x_3 = -\frac{1}{3} + \frac{1-i\sqrt{3}}{3\sqrt[3]{17+3\sqrt{33}}} - \frac{1}{6}\left(1+i\sqrt{3}\right)\sqrt[3]{17+3\sqrt{33}}$$

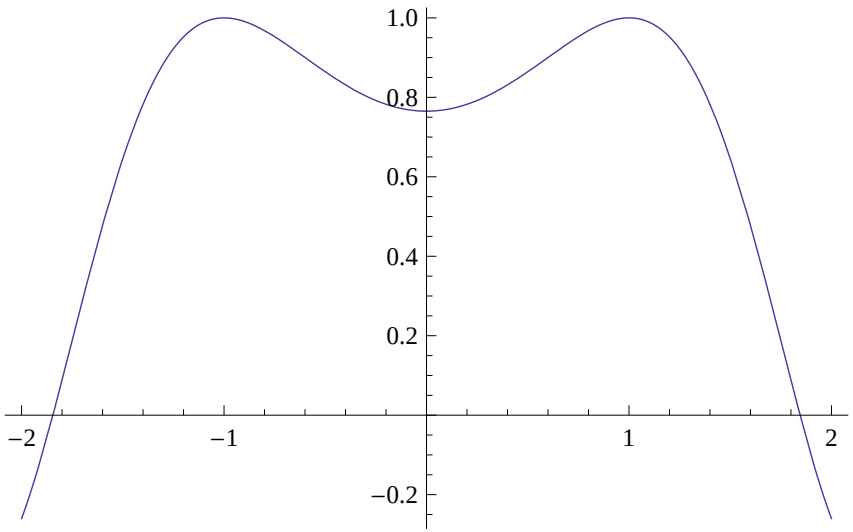
Symbolic Computation (5)

Problem: Solve the differential equation

$$x(x^2-1)f''(x)+(x^2+1)f'(x)+4x^3(x^2-1)f(x)=0$$

subject to the initial values $f(1)=1$ and $f'(0)=0$.

Numeric Solution:



Symbolic Solution:

$$f(x)=J_0(x^2-1)$$

Symbolic Computation (6)

Problem: Compute the integral

$$I(c) := \int_0^\infty e^{-cx} \log x \, \mathrm{d}x \quad (c > 0)$$

Numeric Evaluation: only possible for concrete c!

Symbolic Solution:

$$I(c) = -\frac{\log(c) + \gamma}{c}$$

General vs. Special Purpose

There are general purpose CAS and special purpose CAS.

General Purpose CAS

$\longrightarrow$ address a wide variety of mathematical problems

The most prominent general purpose CAS are

- Maple
- Mathematica
- Macsyma, Maxima
- Axiom, FriCAS
- Derive
- Reduce
- MuPAD
- Sage

Special Purpose CAS

$\longrightarrow$ designed to solve problems in a special subfield of mathematics

Some special purpose CAS are

- Singular
- Macaulay
- GAP
- CoCoA
- PARI/GP
- Magma
- Waldmeister
- ...

Not a CAS

- MathCAD, specialized on numeric computations, only minimal symbolic processor of Maple 3.0
- Matlab ("matrix laboratory", numeric computations)

$\longrightarrow$ not here

Typical objects in a CAS

- integers and rational numbers with arbitrary many digits
- algebraic numbers
- special constants (π, e, ...)
- real numbers (= arbitrary precision floating points numbers)
- variables
- polynomials and rational functions
- standard functions of expressions (sin, exp, log, ...)
- special functions (Bessel, Γ, ...)
- matrices and vectors
- integrals, sums, products

Typical functionality of CAS

- standard exact arithmetic
- simplification to the smallest possible expression or some normal form
- manipulation of expressions: substitution, extracting parts, ...
- rewriting of expressions: factorizing, expanding, trigonometric rewriting, ...
- limit computations
- partial and total differentiation
- indefinite and definite integration, including multidimensional integrals
- solving linear and non-linear equations over various domains
- solving of differential and difference equations
- series operations such as expansion, summation and products
- matrix operations including products, inverses, eigenvalues
- plotting functions in 2D and 3D
- programming

Why?

Why Computer **Algebra**?

- many typical problems from algebra
- use of algebraic methods for solving problems e.g. in analysis
- most important data type: polynomials!

Topic 1: GAP

- GAP (Groups, Algorithms and Programming)
- CAS for computational discrete algebra, with particular emphasis on computational group theory
- studying groups and their representations, rings, vector spaces, algebras, combinatorial structures,. . .
- freely available
- http://www.gap-system.org/

Topic 2: Magma

MAGMA
COMPUTER ALGEBRA

- CAS designed to solve problems in group theory, number theory, linear algebra, geometry, representation theory, combinatorics, ...
- mathematically rigorous environment for computing with algebraic, number-theoretic, combinatorial, and geometric objects
- not free
- http://magma.maths.usyd.edu.au/magma/

Topic 3: PARI/GP

- CAS designed for fast computations in number theory
- factorizations, algebraic number theory, elliptic curves
- large number of other useful functions to compute with mathematical entities (matrices, polynomials, power series, and algebraic numbers)
- freely available
- http://pari.math.u-bordeaux.fr/

Topic 4: Sage

- free open-source mathematics software system licensed under the GPL
- combines the power of many existing open-source packages
- Python-based interface
- freely available
- http://www.sagemath.org/

Topic 5: Singular

- CAS for polynomial computations
- special emphasis on commutative and non-commutative algebra, algebraic geometry, and singularity theory
- freely available
- http://www.singular.uni-kl.de/
- short introduction to Gröbner bases

Topic 6: Axiom

- general purpose CAS
- defines a strongly typed, mathematically correct type hierarchy
- freely available
- http://www.axiom-developer.org

Topic 7: Differential Equations with Computer Algebra

- symbolic algorithms and methods of how to treat differential equations
- Abramov's algorithm, Frobenius method, Kovacic's algorithm, etc.
- comparison of different CAS with respect to their abilities in this field

Topic 8: Algebraic Numbers

- in general the solutions of a polynomial equation of degree ≥ 5 cannot be expressed with radicals.
- how can we deal with such quantities *symbolically*?
- which functionality do computer algebra systems provide to deal with such numbers?

Topic 9: Quantifier Elimination with CAD

- Cylindrical Algebraic Decomposition (CAD)
- algorithm for solving problems that are given in terms of polynomial inequalities
- for example:

$$\forall x \exists y : x^2 + y^2 > 4 \iff (x-1)(y-1) > 1$$

- some CAS (e.g., Mathematica) use it to perform quantifier elimination

Topic 10: Decidability Questions

- in symbolic computation we often encounter undecidable problems
- equality test and zero test in rather simple domains are undecidable
- present one such result
- Daniel Richardson: *Some undecidable problems involving elementary functions of a real variable*, Journal of Symbolic Logik, 1968)

More topics

Come up with own ideas!

- a CAS not mentioned above
- theoretical results and algorithms related to computer algebra and what role they play in actual CAS
- certain class of problems (integration, summation, factorization, etc.) and how different CAS compare w.r.t. this class
- ...