Mathematical Logic

Propositional Logic. Syntax and Semantics

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Outline

Syntax

Semantics

Syntax

The syntax of propositional logic consists in the definition of the set of all propositional logic formulae, or the language of propositional logic formulae, which will contain formulae like:

$\neg A$

$A \wedge B$

$A \wedge \neg B$

$(\neg A \wedge B) \Leftrightarrow (A \Rightarrow B)$

$A \wedge \neg A$

The language $\mathfrak{L}$

$\mathfrak{L}$ is defined over a certain set Σ of symbols:
the parentheses, the logical connectives, the logical constants, and an infinite set Θ of propositional variables.

Set of symbols: alphabet

$\Sigma = \{(,)\} \cup \{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\} \cup \{\mathbb{T}, \mathbb{F}\} \cup \Theta$

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Syntax

Set of propositional variables Θ

For instance this could be $\{A, B, C, P, Q, \ldots, A_1, A_2, \ldots\}$.
This set Θ is infinite, but enumerable.

Generalized inductive definition of $\mathfrak{L}$

– The logical constants $\mathbb{T}, \mathbb{F}$ are formulae, i.e., $\{\mathbb{T}, \mathbb{F}\} \subset \mathfrak{L}$.

– All the variables $\vartheta \in \Theta$ are formulae, i.e.,
$\{\vartheta_1, \vartheta_2, \ldots\ A, B, C, P, Q, \ldots\} \subset \mathfrak{L}$.

– If φ and ψ are formulae, then
$\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \Rightarrow \psi), (\varphi \Leftrightarrow \psi)$ are formulae.

– These are all the formulae.

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Semantics provides the “meaning” of propositional logic formulae. It is defined very precisely in a mathematical way.

The semantics allows us to identify correct inference rules, for instance transformations of formulae which preserve the meaning.

Example:

Intuitively, the meaning of “$A \wedge B$” is that “*this is only true if both A and B are true*”.

The precise semantics of the logical connectives
NOT $\neg$
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IMPLIES $\Rightarrow$
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Semantics

Truth table for negation

$\neg A$	A
$\mathbb{F}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$

Truth table for conjunction

$A \wedge B$	A	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
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Semantics

Truth table for disjunction

$A \vee B$	A	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
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Truth table for implication

$A \Rightarrow B$	A	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
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$A \vee B$	A	B
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Semantics

Example

Consider the formula $(P \vee \neg Q) \Rightarrow R$.

Construct its truth table.

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Consider the formula $(P \wedge (Q \Rightarrow R)) \Rightarrow S$.

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Example

Consider the formula $(A \wedge (A \Rightarrow B)) \Rightarrow B$.

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Example

Consider the formula $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$.

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Interpretation

Let $Var(\varphi)$ be a set of boolean variables, e.g., $\{A, B, C\}$, and let I be a function $I : Var(\varphi) \to \{\mathbb{T}, \mathbb{F}\}$.

The function I is called an "interpretation". It assigns value to the variables.

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Interpretation

Example

Consider the formula $(A \wedge B) \ \vee \ (C \wedge B)$.

Let I_0 be an interpretation defined as follows:
$I_0[A] = \mathbb{T}$, $I_0[B] = \mathbb{F}$, and $I_0[C] = \mathbb{T}$.

Compute the evaluation of the formula under the interpretation I_0.

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Consider the formula $(A \wedge B) \ \vee \ (C \wedge B)$.

Let I_1 be an interpretation defined as follows:
$I_1[A] = \mathbb{F}$, $I_1[B] = \mathbb{T}$, and $I_1[C] = \mathbb{F}$.

Compute the evaluation of the formula under the interpretation I_1.

Model, validity, satisfiability

Let φ be a formula and *I* be an interpretation of its variables.
If φ evaluates to true under *I*, we write $\langle\varphi\rangle_I = \mathbb{T}$, and we say "*I* satisfies φ" or "*I* is a model of φ".

If for any interpretation *I*, $\langle\varphi\rangle_I = \mathbb{T}$, then we say *"$\varphi$ is valid"*, (otherwise it is *"invalid"*)

If for any interpretation *I*, $\langle\varphi\rangle_I = \mathbb{F}$, then we say *"$\varphi$ is unsatisfiable"*, (otherwise it is *"satisfiable"*)

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Model, validity, satisfiability

Valid
The formula $(A \wedge (A \Rightarrow B)) \Rightarrow B$ is valid.
Under all the interpretations it evaluates to true.

Unsatisfiable
The formula $A \wedge \neg A$ is unsatisfiable.
There is no interpretation such that it evaluates to true.

Satisfiable
The formula $(A \wedge B) \ \vee \ (C \wedge B)$ is satisfiable.
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The formula is $(A \wedge B) \ \vee \ (C \wedge B)$ is invalid.
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We say *"the formula ψ is a logical consequence of the formula φ"*
(also denoted as $\varphi \models \psi$), if and only if:
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Show that $(P \Rightarrow Q) \ \wedge \ (Q \Rightarrow R) \ \models \ (P \Rightarrow R)$.

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Show that $\models \ (A \wedge (A \Rightarrow B)) \Rightarrow B$.

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Show that $C \ \models \ C \ \wedge \ ((A \wedge (A \Rightarrow B)) \Rightarrow B)$.

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Logical consequence

The notion of logical consequence captures the essence of logical thinking.
It is basis for characterizing “correct” operations in logic.

We “transport” the truth from some facts to other facts by logical means.
If the original facts are true, than anything obtained by logical methods from them will also be true.

An inference rule is correct if the result of transformation is a logical consequence of the formulae which are transformed.

Using this principle we can construct *syntactical* methods (which can also be implemented on computer) for the systematic transformation of formulae in a correct way.

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