to be prepared for 22.10.2013

Exercise 6. An integral domain is a commutative ring $D \neq \{0\}$ without zero divisors, that means, $rs = 0 \;\Rightarrow\; r = 0 \vee s = 0$ $(\forall r, s \in D)$. Give a proof for the following statement.

1. If D is an integral domain, then also the polynomial ring $D[x]$.
2. Derive from this that - for arbitrary fields k - the ring $k[x_1, \ldots, x_n]$ is an integral domain.
3. Give similar arguments for the ring $D[[x]]$ of formal power series.

Exercise 7. Explain why the following is or is not a Euclidean domain.

1. $\mathbb{Z}$ with degree function $\delta(n) = |n|$;
2. $\mathbb{Q}$ with $\delta(r) = |r|$;
3. $\mathbb{Z}[i]$ with $\delta(z) = |z|^2$;
4. $k[x]$ where k is a field, with $\delta(f) = \deg(f)$;
5. $k[x, y]$ where k is a field, with $\delta(f) = \deg(f)$ (the total degree of f).

Exercise 8. Let R be commutative ring, $R[x]$ the corresponding polynomial ring. Prove that $R[x]$ is a Euclidean domain if and only if R is a field.

Exercise 9. Prove that every Euclidean domain is a principal ideal domain, and that every principal ideal domain is a unique factorization domain.

Exercise 10.

1. Prove the following theorem: *If I is a unique factorization domain, then so is $I[x]$.*
2. Conclude that, for a field k, $k[x_1, \ldots, x_n]$ is a unique factorization domain.

Exercise 11. Prove the following statement:

Let R be an integral domain, $a(x), b(x) \in R[x], b \neq 0$, and $m = \deg(a) \geq n = \deg(b)$. There are uniquely defined polynomials $q(x), r(x) \in R[x]$ such that

$$\begin{array}{l} \mathrm{lc}(b)^{m-n+1} \cdot a(x) = q(x) \cdot b(x) + r(x) \quad \textit{and} \\ r(x) = 0 \textit{ or } \deg(r) < \deg(b). \end{array}$$

Exercise 12. Compute the pseudo-quotient $q(x)$ and the pseudo-remainder $r(x)$ for the two integral polynomials

$$\begin{array}{rcl} u(x) & = & x^6 + x^5 - x^4 + 2x^3 + 3x^2 - x + 2 \quad \text{and} \\ v(x) & = & 2x^3 + 2x^2 - x + 3. \end{array}$$

Exercise 13. Let R be an integral domain, $a, b \in R[x]$, $b \neq 0$, $m = \deg(a) \geq n = \deg(b)$, $\alpha, \beta \in R$. Prove:

$$\begin{array}{rcl} \mathrm{pquot}(\alpha a, \beta b) & = & \beta^{m-n} \alpha \cdot \mathrm{pquot}(a, b) \quad \text{and} \\ \mathrm{prem}(\alpha a, \beta b) & = & \beta^{m-n+1} \alpha \cdot \mathrm{prem}(a, b). \end{array}$$