Logic 1
First-Order Logic

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Example 1 (Semantics) Let

$$F_1 \quad :\Longleftrightarrow \quad \underset{x}{\forall}\,\underset{y}{\forall}\; x \leq y$$

$$I : \begin{cases} D = \{0, 1\} \\ \leq_I \to \leq_{\mathbb{Z}} \end{cases}$$

Solution. $\langle F_1 \rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\left\langle \underset{y}{\forall}\; x \leq y \right\rangle^I_{\sigma \uplus \{x \to d\}} = \mathbb{T}$$

- Case $x \to 0$. We have

$$\left\langle \underset{y}{\forall}\; x \leq y \right\rangle^I_{\{x \to 0\}} = \mathbb{T} \text{ iff for each } d \in D : \quad \langle x \leq y \rangle^I_{\{x \to 0\} \uplus \{y \to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned} & \langle x \leq y \rangle^I_{\{x \to 0, y \to 0\}} \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x \to 0, y \to 0\}}, \langle y \rangle^I_{\{x \to 0, y \to 0\}} \right] \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} [0, 0] \\ \rightsquigarrow \quad & \mathbb{T} \end{aligned}$$

  - Case $y \to 1$. We have

$$\begin{aligned} & \langle x \leq y \rangle^I_{\{x \to 0, y \to 1\}} \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x \to 0, y \to 1\}}, \langle y \rangle^I_{\{x \to 0, y \to 1\}} \right] \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} [0, 1] \\ \rightsquigarrow \quad & \mathbb{T} \end{aligned}$$

- Case $x \to 1$. We have

$$\left\langle \underset{y}{\forall}\; x \leq y \right\rangle^I_{\{x \to 1\}} = \mathbb{T} \text{ iff for each } d \in D : \quad \langle x \leq y \rangle^I_{\{x \to 1\} \uplus \{y \to d\}} = \mathbb{T}$$

– Case $y \to 0$. We have

$$\begin{array}{ll} & \langle x \le y \rangle^I_{\{x \to 1, y \to 0\}} \\ \rightsquigarrow & \le_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x \to 1, y \to 0\}}, \langle y \rangle^I_{\{x \to 1, y \to 0\}} \right] \\ \rightsquigarrow & \le_{\mathbb{Z}} [1, 0] \\ \rightsquigarrow & \mathbb{F} \end{array}$$

Hence $\langle F_1 \rangle_I = \mathbb{F}$. ◀

Example 2 (Semantics) Let

$$F_2 \quad :\Longleftrightarrow \quad \underset{x}{\forall} \underset{y}{\exists}\, x + y > c$$

$$I : \begin{cases} D = \{0, 1\} \\ c_I = 0 \\ +_I \to +_{\mathbb{Z}} \\ >_I \to >_{\mathbb{Z}} \end{cases}$$

Solution. $\langle F_2 \rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\left\langle \underset{y}{\exists}\, x + y > c \right\rangle^I_{\sigma \uplus \{x \to d\}} = \mathbb{T}$$

- Case $x \to 0$. We have

$$\left\langle \underset{y}{\exists}\, x + y > c \right\rangle^I_{\{x \to 0\}} = \mathbb{T} \text{ iff for some } d \in D : \quad \langle x + y > c \rangle^I_{\{x \to 0\} \uplus \{y \to d\}} = \mathbb{T}$$

  – Case $y \to 0$. We have

$$\begin{array}{ll} & \langle x + y > c \rangle^I_{\{x \to 0, y \to 0\}} \\ \rightsquigarrow & >_{\mathbb{Z}} \left[\langle x + y \rangle^I_{\{x \to 0, y \to 0\}}, \langle c \rangle^I_{\{x \to 0, y \to 0\}} \right] \\ \rightsquigarrow & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x \to 0, y \to 0\}}, \langle y \rangle^I_{\{x \to 0, y \to 0\}} \right], 0 \right] \\ \rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 0], 0] \\ \rightsquigarrow & >_{\mathbb{Z}} [0, 0] \\ \rightsquigarrow & \mathbb{F} \end{array}$$

  – Case $y \to 1$. We have

$$\begin{array}{ll} & \langle x + y > c \rangle^I_{\{x \to 0, y \to 1\}} \\ \rightsquigarrow & >_{\mathbb{Z}} \left[\langle x + y \rangle^I_{\{x \to 0, y \to 1\}}, \langle c \rangle^I_{\{x \to 0, y \to 1\}} \right] \\ \rightsquigarrow & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x \to 0, y \to 1\}}, \langle y \rangle^I_{\{x \to 0, y \to 1\}} \right], 0 \right] \\ \rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 1], 0] \\ \rightsquigarrow & >_{\mathbb{Z}} [1, 0] \\ \rightsquigarrow & \mathbb{T} \end{array}$$

- Case $x \to 1$. We have

$$\left\langle \exists_{y}\, x+y>c \right\rangle^{I}_{\{x\to 1\}} = \mathbb{T} \text{ iff for some } d \in D: \quad \langle x+y>c\rangle^{I}_{\{x\to 1\}\uplus\{y\to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned}
& \langle x+y>c\rangle^{I}_{\{x\to 1, y\to 0\}} \\
\rightsquigarrow\ & >_{\mathbb{Z}} \left[\langle x+y\rangle^{I}_{\{x\to 1, y\to 0\}}, \langle c\rangle^{I}_{\{x\to 1, y\to 0\}}\right] \\
\rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}}\left[\langle x\rangle^{I}_{\{x\to 1, y\to 0\}}, \langle y\rangle^{I}_{\{x\to 1, y\to 0\}}\right], 0\right] \\
\rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}}[1,0], 0\right] \\
\rightsquigarrow\ & >_{\mathbb{Z}} [1,0] \\
\rightsquigarrow\ & \mathbb{T}
\end{aligned}$$

Hence $\langle F_2\rangle_I = \mathbb{T}$. ◀

Example 3 (Semantics) Let

$$F_3 \quad :\Longleftrightarrow \quad \forall_{x}\, (P[x] \Longrightarrow Q[f[x],a])$$

$$I: \begin{cases} D = \{1,2\} \\ a_I = 1 \\ f_I : D \to D & \begin{cases} f_I[1] = 1 \\ f_I[2] = 1 \end{cases} \\ P_I : D \to \{\mathbb{T},\mathbb{F}\} & \begin{cases} P_I[1] = \mathbb{T} \\ P_I[2] = \mathbb{F} \end{cases} \\ Q_I : D^2 \to \{\mathbb{T},\mathbb{F}\} & \begin{cases} Q_I[1,1] = \mathbb{T} \quad Q_I[1,2] = \mathbb{F} \\ Q_I[2,1] = \mathbb{F} \quad Q_I[2,2] = \mathbb{T} \end{cases} \end{cases}$$

Solution. $\langle F_3\rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\langle P[x] \Longrightarrow Q[f[x],a]\rangle^{I}_{\sigma\uplus\{x\to d\}} = \mathbb{T}$$

- Case $x \to 1$. We have

$$\begin{aligned}
& \langle P[x] \Longrightarrow Q[f[x],a]\rangle^{I}_{\{x\to 1\}} \\
\rightsquigarrow\ & \mathcal{B}_{\Longrightarrow}\left[\langle P[x]\rangle^{I}_{\{x\to 1\}}, \langle Q[f[x],a]\rangle^{I}_{\{x\to 1\}}\right] \\
\rightsquigarrow\ & \mathcal{B}_{\Longrightarrow}\left[P_I[\langle[x]\rangle^{I}_{\{x\to 1\}}],\quad Q_I[f_I[\langle x\rangle^{I}_{\{x\to 1\}}],\quad \langle a\rangle^{I}_{\{x\to 1\}}\right] \\
\rightsquigarrow\ & \mathcal{B}_{\Longrightarrow}\left[P_I[1],\quad Q_I\left[f_I[1],\ 1\right]\right] \\
\rightsquigarrow\ & \mathcal{B}_{\Longrightarrow}\left[P_I[1],\quad Q_I[1,\ 1]\right] \\
\rightsquigarrow\ & \mathcal{B}_{\Longrightarrow}[\mathbb{T},\quad \mathbb{T}] \\
\rightsquigarrow\ & \mathbb{T}
\end{aligned}$$

- Case $x \to 2$. We have

$$\begin{aligned}
& \langle P[x] \Longrightarrow Q[f[x], a] \rangle^I_{\{x \to 2\}} \\
\rightsquigarrow \;& \mathcal{B}_{\Longrightarrow} \left[\langle P[x] \rangle^I_{\{x \to 2\}}, \langle Q[f[x], a] \rangle^I_{\{x \to 2\}} \right] \\
\rightsquigarrow \;& \mathcal{B}_{\Longrightarrow} \left[P_I[\langle [x] \rangle^I_{\{x \to 2\}}], \quad Q_I[f_I[\langle x \rangle^I_{\{x \to 2\}}], \ \langle a \rangle^I_{\{x \to 2\}} \right] \\
\rightsquigarrow \;& \mathcal{B}_{\Longrightarrow} [P_I[2], \quad Q_I[f_I[2], \ 1]] \\
\rightsquigarrow \;& \mathcal{B}_{\Longrightarrow} [P_I[2], \quad Q_I[1, \ 1]] \\
\rightsquigarrow \;& \mathcal{B}_{\Longrightarrow} [\mathbb{F}, \quad \mathbb{T}] \\
\rightsquigarrow \;& \mathbb{T}
\end{aligned}$$

Hence $\langle F_1 \rangle_I = \mathbb{T}$. ◀

Example 4 ((Un)Satisfiability & (In)Validity) Prove that the formula

$$\underset{x}{\forall} P[x] \ \wedge \ \underset{y}{\exists} \neg P[y]$$

is inconsistent by definition.

Solution. We have to prove that the formula is $\mathbb{F}$ for all interpretations I.
Let $I = (D_I, P_I)$ be an arbitrary interpretation. Then

$$\begin{aligned}
& \left\langle \underset{x}{\forall} P[x] \ \wedge \ \underset{y}{\exists} \neg P[y] \right\rangle_I = \mathbb{F} \\
\rightsquigarrow \;& \mathcal{B}_{\wedge} \left[\left\langle \underset{x}{\forall} P[x] \right\rangle_I, \ \left\langle \underset{y}{\exists} \neg P[y] \right\rangle_I \right] = \mathbb{F}
\end{aligned}$$

- Case $\left\langle \underset{x}{\forall} P[x] \right\rangle_I = \mathbb{T}$ and $\left\langle \underset{y}{\exists} \neg P[y] \right\rangle_I = \mathbb{F}$. We have
  - For each $x \in D_I$, $\langle P[x] \rangle_I = \mathbb{T}$ and
  - For some $y \in D_I$, $\langle \neg P[y] \rangle_I = \mathbb{F}$ and further for some $y \in D_I$, $\mathcal{B}_{\neg} [\langle P[y] \rangle_I] = \mathbb{F}$ We obtain that for some $y \in D_I$, $\langle P[y] \rangle_I = \mathbb{T}$
- Case $\left\langle \underset{x}{\forall} P[x] \right\rangle_I = \mathbb{F}$ and $\left\langle \underset{y}{\exists} \neg P[y] \right\rangle_I = \mathbb{T}$. Similarly.

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Example 5 (CNF) Prove the following by bringing the formulas into CNF

$$\left(\underset{x}{\forall} P[x] \right) \Rightarrow Q \quad \equiv \quad \underset{x}{\exists} (P[x] \Rightarrow Q) \, .$$

Solution. We have

$$\left(\underset{x}{\forall} P[x] \right) \Rightarrow Q \quad \equiv \quad \neg \left(\underset{x}{\forall} P[x] \right) \vee Q \quad \equiv \quad \left(\underset{x}{\exists} \neg P[x] \right) \vee Q \quad \equiv \quad \underset{x}{\exists} (\neg P[x] \vee Q)$$

Further we have

$$\underset{x}{\exists} (P[x] \Rightarrow Q) \quad \equiv \quad \underset{x}{\exists} (\neg P[x] \vee Q)$$

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Example 6 (Skolem Standard Form)   Bring the following formula into Skolem Standard Form

$$\underset{x}{\forall}\ \underset{y,z}{\exists}\left((\neg P[x,y]\ \wedge\ Q[x,z])\ \vee\ R[x,y,z]\right)$$

Solution.

$$\begin{aligned}
& \underset{x}{\forall}\ \underset{y,z}{\exists}\left((\neg P[x,y]\ \wedge\ Q[x,z])\ \vee\ R[x,y,z]\right) \\
\Longleftrightarrow\ & \underset{x}{\forall}\ \underset{y,z}{\exists}\left((\neg P[x,y]\ \vee\ R[x,y,z])\ \wedge\ (Q[x,z]\ \vee\ R[x,y,z])\right) \\
\rightsquigarrow\ & \underset{x}{\forall}\left((\neg P[x,f[x]]\ \vee\ R[x,f[x],g[x]])\ \wedge\ (Q(x,g[x])\ \vee\ R[x,f[x],g[x]])\right)
\end{aligned}$$

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Example 7 (Skolem Standard Form)   Bring the following formula into Skolem Standard Form

$$\underset{x,y}{\forall}\left(\underset{z}{\exists}P[x,z] \wedge P[y,z]\right)\ \Rightarrow\ \underset{u}{\exists}Q[x,y,u]$$

Solution.

$$\begin{aligned}
& \underset{x,y}{\forall}\left(\underset{z}{\exists}P[x,z] \wedge P[y,z]\right)\ \Rightarrow\ \underset{u}{\exists}Q[x,y,u] \\
\Longleftrightarrow\ & \underset{x,y}{\forall}\neg\left(\underset{z}{\exists}P[x,z] \wedge P[y,z]\right)\ \vee\ \underset{u}{\exists}Q[x,y,u] \\
\Longleftrightarrow\ & \underset{x,y}{\forall}\left(\underset{z}{\forall}\neg P[x,z] \vee \neg P[y,z]\right)\ \vee\ \underset{u}{\exists}Q[x,y,u] \\
\Longleftrightarrow\ & \underset{x,y,z}{\forall}\ (\neg P[x,z] \vee \neg P[y,z])\ \vee\ \underset{u}{\exists}Q[x,y,u] \\
\Longleftrightarrow\ & \underset{x,y,z}{\forall}\ \underset{u}{\exists}\ (\neg P[x,z] \vee \neg P[y,z])\ \vee\ Q[x,y,u] \\
\rightsquigarrow\ & \underset{x,y,z}{\forall}\neg P[x,z] \vee \neg P[y,z]\ \vee\ Q[x,y,f[x,y,z]]
\end{aligned}$$

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Example 8   Prove that the formula

$$\underset{x}{\forall}P[x]\ \Rightarrow\ \underset{y}{\exists}P[y]$$

is valid by equivalent transformations.

Solution. We assume that the formula is invalid and derive a contradiction. Hence, it exists an interpretation I under which the formula is false. That is

$$\begin{aligned}
& \left\langle\underset{x}{\forall}P[x]\right\rangle^{I} = \mathbb{T} \\
& \wedge \\
& \left\langle\underset{y}{\exists}P[y]\right\rangle^{I} = \mathbb{F}\ \rightsquigarrow\ \left\langle\underset{y}{\forall}\neg P[y]\right\rangle^{I} = \mathbb{T}
\end{aligned}$$

From the above we obtain that

$$\left\langle\underset{x}{\forall}P[x]\right\rangle^{I} = \left\langle\underset{y}{\forall}\neg P[y]\right\rangle^{I}\ \text{which is a contradiction}$$

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Example 9 ((Un)Satisfiability & (In)Validity) Prove that the formula

$$\underset{x}{\forall} P[x] \ \wedge \ \underset{y}{\exists} \neg P[y]$$

is inconsistent by equivalent transformations.

Solution. We have

$$\underset{x}{\forall} P[x] \ \wedge \ \underset{y}{\exists} \neg P[y] \quad \equiv \quad \underset{x}{\forall} P[x] \ \wedge \ \neg \left(\underset{y}{\forall} P[y] \right) \quad \equiv \quad \underset{x}{\forall} P[x] \ \wedge \ \neg \left(\underset{x}{\forall} P[x] \right) \quad \equiv \quad \mathbb{F}$$

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Example 10 (Clausification) Transform the formulas F_1, F_2, F_3, F_4, and $\neg G$ into a set of clauses, where

$$F_1 : \quad \underset{x,y}{\forall} \ \underset{z}{\exists} P[x, y, z]$$

$$F_2 : \quad \begin{array}{l} \underset{x,y,z,u,v,w}{\forall} (P[x, y, u] \ \wedge \ P[y, z, v] \ \wedge \ P[u, z, w] \quad \Rightarrow \quad P[x, v, w]) \\ \wedge \\ \underset{x,y,z,u,v,w}{\forall} (P[x, y, u] \ \wedge \ P[y, z, v] \ \wedge \ P[x, v, w] \quad \Rightarrow \quad P[u, z, w]) \end{array}$$

$$F_3 : \quad \underset{x}{\forall} P[x, e, x] \wedge \underset{x}{\forall} P[e, x, x]$$

$$F_4 : \quad \underset{x}{\forall} P[x, i[x], e] \wedge \underset{x}{\forall} P[i[x], x, e]$$

$$G : \quad \left(\underset{x}{\forall} P[x, x, e] \right) \ \Rightarrow \ \left(\underset{u,v,w}{\forall} (P[u, v, w] \ \Rightarrow \ P[v, u, w]) \right)$$

Solution. F_1, F_2,F_3,F_4 can almost immediately transformed into clauses. We have

$$\begin{array}{l} P[x, y, f[x, y]] \\ \neg P[x, y, u] \ \vee \ \neg P[y, z, v] \ \vee \ \neg P[u, z, w] \ \vee \ P[x, v, w] \\ \neg P[x, y, u] \ \vee \ \neg P[y, z, v] \ \vee \ \neg P[x, v, w] \ \vee \ P[u, z, w] \\ P[x, e, x] \\ P[e, x, x] \\ P[x, i[x], e] \\ P[i[x], x, e] \end{array}$$

We transform $\neg G$ into standard form

$$\begin{array}{l} \neg \left(\left(\underset{x}{\forall} P[x, x, e] \right) \ \Rightarrow \ \left(\underset{u,v,w}{\forall} (P[u, v, w] \ \Rightarrow \ P[v, u, w]) \right) \right) \\ \Longleftrightarrow \neg \left(\neg \left(\underset{x}{\forall} P[x, x, e] \right) \ \vee \ \left(\underset{u,v,w}{\forall} (\neg P[u, v, w] \ \vee \ P[v, u, w]) \right) \right) \\ \Longleftrightarrow \left(\underset{x}{\forall} P[x, x, e] \right) \ \wedge \ \left(\underset{u,v,w}{\exists} (P[u, v, w] \ \wedge \ \neg P[v, u, w]) \right) \\ \rightsquigarrow \underset{x}{\forall} P[x, x, e] \ \wedge \ P[a, b, c] \ \wedge \ \neg P[b, a, c] \end{array}$$

which gives the following clauses

$$\begin{array}{l} P[x, x, e] \\ P[a, b, c] \\ \neg P[b, a, c] \end{array}$$

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