to be prepared for 28.10.2014

Exercise 10. Let R be a commutative ring with 1. Demonstrate that the following statements are equivalent:

1. Every ideal in R is generated by a finite set.
2. There are no infinite strictly ascending chains of ideals in R.
3. Every nonempty set S of ideals contains a maximal element (i.e. an ideal $a \in S$ such that $\forall b \in S$, if $a \subseteq b$ then $a = b$.

Exercise 11. Use the Hilbert Basis Theorem (Theorem 2.3.2) for a proof of the following statement.

> Every set $A \subseteq \mathbb{N}^n$ contains a finite subset $B \subseteq A$ such that $\forall a \in A\, \exists b \in B$ with $b_i \leq a_i\ \forall i = 1, \ldots, n$.

Hint: Read integer tuples $a \in \mathbb{N}^n$ as monomials $x_1^{a_1} \cdots x_n^{a_n}$ and consider the ideal generated by them.

Exercise 12. The graduated reverse lexicographic ordering on power products of $x_1, \ldots, x_n$ $<_{\text{grlex}}$ is defined by

$$s <_{\text{grlex}} t \quad \text{iff} \quad \begin{array}{l} \deg(s) < \deg(t) \quad \text{or} \\ \deg(s) = \deg(t) \quad \text{and} \quad t <_{\text{lex},\pi} s; \end{array}$$

where π is the permutation on n letters given by $\pi(j) = n - j + 1$ and $<_{\text{lex},\pi}$ is the lexicographic order wrto. π. Prove that $<_{\text{grlex}}$ is an admissible ordering.

Exercise 13. Let $<_1$ be an admissible ordering on $X_1 = [x_1, \ldots, x_i]$ and $<_2$ an admissible ordering on $X_2 = [x_{i+1}, \ldots, x_n]$. Show that the product ordering $<_{prod,i,<_1,<_2}$ on $X = [x_1, \ldots, x_n]$ is an admissible ordering.

Exercise 14. $R[x_1, \ldots, x_n] = R[X]$ denote the polynomial ring in n indeterminates over a commutative ring with 1. Any admissible ordering $<$ on the monoid of power products $[X]$ induces a partial order $<<$ on $R[X]$ in the following way:

$$f << g \quad \text{iff} \quad \begin{array}{l} f = 0 \text{ and } g \neq 0 \text{ or} \\ f \neq 0,\ g \neq 0 \text{ and } \operatorname{lpp}(f) < \operatorname{lpp}(g) \text{ or} \\ f \neq 0,\ g \neq 0,\ \operatorname{lpp}(f) = \operatorname{lpp}(g) \text{ and } \operatorname{red}(f) << \operatorname{red}(g). \end{array}$$

Prove that $<<$ is a Noetherian partial order on $R[X]$.