to be prepared for 11.11.2014

Exercise 19. Consider the polynomial ring $K[x, y]$. Can you give admissible orderings so that different terms in the polynomial $2xy^2 - xy + x^3$ become the highest term w.r.t. this ordering?

Exercise 20. Fix an admissible ordering and consider an ideal $I \subseteq K[x_1, \ldots, x_n]$. Suppose that $f \in K[x_1, \ldots, x_n]$.

1. Show that f can be written in the form $f = g + r$ with $g \in I$ and no term of r is divisible by any element of $\text{lpp}(I)$.
2. Given two expressions $f = g + r = g' + r'$ as in part 1, prove that $g = g'$ and $r = r'$.

Exercise 21. Prove the following theorem (Theorem 2.3.14):

Let $c \in K \setminus 0$, $s \in [X]$, $F \subseteq K[X]$, $g_1, g_2, h \in K[X]$.

(a) $\longrightarrow_F \subseteq >>$,

(b) $\longrightarrow_F$ is Noetherian,

(c) if $g_1 \longrightarrow_F g_2$ then $csg_1 \longrightarrow_F csg_2$,

(d) if $g_1 \longrightarrow_F g_2$ then $g_1 + h \downarrow_F^* g_2 + h$.

Exercise 22. Prove the following theorem.
Let $F \subseteq k[x_1, \ldots, x_n]$. The ideal congruence modulo $\langle F \rangle$ equals the reflexive-transitive-symmetric closure of the reduction relation $\longrightarrow_F$, i.e., $\equiv_{\langle F \rangle} = \longleftrightarrow_F^*$.

Exercise 23. Let $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ be ideals in $K[x_1, \ldots, x_n]$. Prove the following statements.

1. $I + J = \langle f_1, \ldots, f_r, g_1, \ldots, g_s \rangle$.
2. $I \cdot J = \langle f_i g_j \mid 1 \leq i \leq r,\ 1 \leq j \leq s \rangle$.