to be prepared for 18.11.2014

Exercise 24. Let I be an ideal in $K[x_1, \ldots, x_n]$, $F \subset K[x_1, \ldots, x_n]$ with $\langle F \rangle = I$. Prove the equivalence of the following statements.

1. F is a Gröbner basis for I.
2. $\forall f \in I$ we have that $f \longrightarrow_F^\star 0$.
3. $f \longrightarrow_F$ for every $f \in I \setminus 0$.
4. $\forall g \in I\ \forall h \in K[x_1, \ldots, x_n]$: if $g \longrightarrow_F^\star \underline{h}$ then $h = 0$.
5. $\forall g, h_1, h_2 \in K[x_1, \ldots, x_n]$: if $g \longrightarrow_F^\star \underline{h_1}$ and $g \longrightarrow_F^\star \underline{h_2}$ then $h_1 = h_2$.
6. $\langle \mathrm{in}(F) \rangle = \langle \mathrm{in}(I) \rangle$.

Exercise 25. Show that the result of applying the Euclidean Algorithm in $K[x]$ to any pair of polynomials f, g is a reduced Gröbner basis for $\langle f, g \rangle$.

Exercise 26. Consider linear polynomials in $K[x_1, \ldots, x_n]$

$$f_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \ldots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $g_1, \ldots, g_r$ form the reduced Groebner basis of $\langle f_1, \ldots, f_m \rangle$.

Exercise 27. Compute the normed reduced Gröbner basis for the ideal

$$I = \langle xz - 3x^2 + x + 6x^3 + 1,\ x^2 + y^2 - 2,\ x^5 - 6x^3 + x^2 - 1 \rangle$$

in $\mathbb{Q}[x, y, z]$ w.r.t. the lexicographic ordering with $x < y < z$. What is $\dim \mathbb{Q}[x, y, z]/I$?

Exercise 28. Let I be the ideal in $\mathbb{Q}[x, y, z]$ generated by

$$\begin{aligned} f_1 &= x^4 - 2x + z + 1 \\ f_2 &= x^2 + y^2 - 2 \\ f_3 &= x^5 - 6x^3 + x^2 - 1 \end{aligned}$$

1. How many solutions does the system $f_1 = f_2 = f_3 = 0$ have?
2. Give a basis for the vector space $\mathbb{Q}[x, y, z]/I$ over $\mathbb{Q}$.